NONLOCAL LAGRANGE MULTIPLIERS AND TRANSPORT DENSITIES

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Abstract. We prove the existence of generalised solutions of the Monge-Kantorovich equations with fractional s-gradient constraint, $0 < s < 1$, associated to a general, possibly degenerate, linear fractional operator of the type,

$$\mathscr{L}^s u = -D^s \cdot (AD^s u + \boldsymbol{b}u) + \boldsymbol{d} \cdot D^s u + cu,$$

with integrable data, in the space $\Lambda_0^{s,p}(\Omega)$, which is the completion of the set of smooth functions with compact support in a bounded domain Ω for the L^p-norm of the distributional Riesz fractional gradient D^s in $\mathbb{R}^d$ (when $s = 1$, $D^1 = D$ is the classical gradient). The transport densities arise as generalised Lagrange multipliers in the dual space of $L^\infty(\mathbb{R}^d)$ and are associated to the variational inequalities of the corresponding transport potentials under the constraint $|D^s u| \leq g$. Their existence is shown by approximating the variational inequality through a penalisation of the constraint and nonlinear regularisation of the linear operator $\mathscr{L}^s u$. For this purpose, we also develop some relevant properties of the spaces $\Lambda_0^{s,p}(\Omega)$, including the limit case $p = \infty$ and the continuous embeddings $\Lambda_0^{s,q}(\Omega) \subset \Lambda_0^{s,p}(\Omega)$, for $1 \leq p \leq q \leq \infty$. We also show the localisation of the nonlocal problems $(0 < s < 1)$, to the local limit problem with classical gradient constraint when $s \to 1$, for which most results are also new for a general, possibly degenerate, partial differential operator $\mathscr{L}^1 u$ only with integrable coefficients and bounded gradient constraint.

1. Introduction

In a bounded open set Ω of $\mathbb{R}^d$, consider the model problem for the pair of functions (u, λ),

$$-D \cdot \big((\delta + \lambda) Du\big) = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega \tag{1.1}$$

$$|Du| \leq 1, \quad \lambda \geq 0, \quad \lambda(|Du| - 1) = 0 \quad \text{in } \Omega, \tag{1.2}$$

where $\delta \geq 0$ is a constant, D denotes the gradient, $D\cdot$ denotes the divergence and $f = f(x)$ is a given function.

For $\delta > 0$, the problem (1.1)–(1.2), being equivalent to minimise the functional

$$u \mapsto \tfrac{\delta}{2} \int_\Omega |Du|^2 - \int_\Omega fu \tag{1.3}$$

in the convex subset of $H_0^1(\Omega)$ subjected to the constraint $|Du| \leq 1$ in Ω, is well-known to model the elastoplastic torsion of a cylindric bar of cross section Ω, where λ is the respective Lagrange multiplier. In 1972, Brézis [10] has shown that, if $f = const > 0$ and Ω is simply connected, $\lambda \in L^\infty(\Omega)$ is unique and even continuous if Ω is convex. This was partially extended to more

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general strictly convex functionals than (1.3), by Chiadò Piat and Percivale [13], for $f \in L^p(\Omega)$, $p > d$, obtaining a solution u in $C^{1,\alpha}(\overline{\Omega})$, $\alpha = 1 - \frac{d}{p}$ and λ as a positive Radon measure (see the survey [26], for references and more results).

In the degenerate case $\delta = 0$, (1.1)–(1.2) are usually called the Monge-Kantorovich equations, as they appear in a classical mass transfer problem [18], where u and λ represent the transport potential and density, respectively. This is the dual problem of (1.3) with $\delta = 0$ over all Lipschitz continuous functions with $|Du| \leq 1$ and vanishing on $\partial\Omega$. This same problem also arises in shape optimization [7], in the equilibrium configurations [6] and in the time discretisation of the growing sandpile problem [17].

In general, and specially in the case $\delta = 0$ with more general gradients thresholds, the main difficulty in studying (1.1)–(1.2) is the non-regularity of the flux, since Du is just bounded and it can not be multiplied by λ, whenever this is a Radon measure. Several approaches have been proposed, by relaxing the Monge-Kantorovich problem (see [7], [19] or [8]).

A different and more direct approach was proposed by [4] to solve (1.1)–(1.2) with $\delta \geq 0$, $f \in L^2(\Omega)$ and a variable general constraint $|Du| \leq g \in L^\infty(\Omega)$, with $g > 0$, by proving the existence of a pair $(u, \lambda) \in W^{1,\infty}(\Omega) \times L^\infty(\Omega)'$. The generalised Lagrange multiplier λ being a charge, i.e., an element of $L^\infty(\Omega)'$, allows to interpret the equation (1.1) in a duality sense and the second and third conditions of (1.2) (with 1 replaced by g) in the dual space $L^\infty(\Omega)'$.

Recently, this charges approach was extended in [3] to a class of coercive nonlocal problems considered in [26] with fractional gradient constraint of the type

$$|D^s u| \leq g, \quad 0 < s < 1, \tag{1.4}$$

where D^s is the distributional fractional Riesz gradient. The fractional s-gradient D^s has been recently studied by several authors [28], [29], [14], [15]. It may be defined via smooth functions $C_c^\infty(\mathbb{R}^d)$ by the convolution of the classical gradient with the Riesz kernel I_{1-s}, i.e., $D^s u = I_{1-s} * Du = D(I_{1-s} * u)$, with the nice properties $(-\Delta)^s u = -D^s \cdot (D^s u)$ and

$$\int_{\mathbb{R}^d} u D^s \cdot \boldsymbol{\xi} = -\int_{\mathbb{R}^d} D^s u \cdot \boldsymbol{\xi}, \quad \forall \boldsymbol{\xi} \in C_c^\infty(\mathbb{R}^d)^d, \tag{1.5}$$

where $D^s \cdot$ denotes the s-divergence and $(-\Delta)^s$ the fractional s-Laplacian. For smooth functions with compact support D^s can also be equivalently defined by a vector-valued fractional singular integral, which satisfies elementary physical requirements, such as translational and rotational invariances, homogeneity of degree s under isotropic scaling and certain basic continuity properties [29], in order to model long-range forces and nonlocal effects in continuum mechanics.

Another important property of D^s is due to the fact that the Riesz kernel I_{1-s} approaches the identity operator as $s \to 1$, which implies that $D^s u \longrightarrow Du$ in L^p-spaces, provided $Du \in \boldsymbol{L}^p(\mathbb{R}^d) = L^p(\mathbb{R}^d)^d$ (see Section 2, for details). However it should be noted that even when u has compact support in $\mathbb{R}^d$ and $D^s u$ makes sense as a p-integrable function, in general, $D^s u$ has not compact support in contrast with $Du = D^1 u$.

Here we shall be concerned with the more general fractional Monge-Kantorovich-type problem for a function u, satisfying $u = 0$ in $\mathbb{R}^d \setminus \Omega$, and a charge λ, such that

$$(1.6)_s \qquad \mathscr{L}^s u - D^s \cdot (\lambda D^s u) = f - D^s \cdot \boldsymbol{f}$$

$$(1.7)_s \qquad |D^s u| \leq g_s, \quad \lambda \geq 0 \quad \text{and} \quad \lambda(|D^s u| - g_s) = 0.$$

For a bounded positive threshold g_s, the first condition in $(1.7)_s$ holds a.e. $x \in \mathbb{R}^d$, for $0 < s < 1$, and a.e. in Ω, for $s = 1$, while the second and third ones are interpreted in $L^\infty(\mathbb{R}^d)'$ and in $L^\infty(\Omega)'$, respectively.

The equation $(1.6)_s$ must be interpreted in an appropriate functional space duality with the bilinear form associated to a linear operator for $0 < s \leq 1$, possibly degenerate, in the general form:

$$(1.8)_s \qquad \mathscr{L}^s u = -D^s \cdot (AD^s u + \boldsymbol{b} u) + \boldsymbol{d} \cdot D^s u + cu,$$

where the nonnegative matrix $A = A(x)$ has integrable coefficients, which may degenerate or even vanish completely, the vector fields $\boldsymbol{b}$ and $\boldsymbol{d}$, as well as the function c and the given data f and $\boldsymbol{f}$ are also merely integrable in the case of bounded g_s, even in the classical local case $s = 1$.

The fractional setting for the homogeneous Dirichlet condition is considered within the functional framework of the following family of Banach spaces

$$(1.9) \qquad \Lambda_0^{s,q}(\Omega) \subset \Lambda_0^{s,p}(\Omega), \quad 1 \leq p \leq q \leq \infty, \; 0 < s < 1,$$

where $\Lambda_0^{s,2}(\Omega)$ are the usual fractional Sobolev spaces $H_0^s(\Omega)$ and the limit case $s = 1$ corresponds to the usual Sobolev spaces $H_0^1(\Omega)$ and $W_0^{1,p}(\Omega)$ if $p \neq 2$. For $1 \leq p < \infty$, $\Lambda_0^{s,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ for the norm $\|D^s u\|_{\boldsymbol{L}^p(\mathbb{R}^d)}$ (see Section 2).

We observe that, for $p \neq 2$, $0 < s < 1$, the Lions-Caldéron spaces $\Lambda_0^{s,p}(\Omega)$ are different from the Sobolev-Slobodeckij spaces $W_0^{s,p}(\Omega)$, although they are contiguous (see [1, p 219] or [12]), i.e.

$$\Lambda_0^{s+\varepsilon,p}(\Omega) \subsetneqq W_0^{s,p}(\Omega) \subsetneqq \Lambda_0^{s-\varepsilon,p}(\Omega), \quad s > \varepsilon > 0, \; 1 < p < \infty, \; p \neq 2.$$

The paper is organised as follows: in Section 2 we develop the required functional framework for the Riesz fractional derivatives and we recall and prove some interesting properties of the spaces $\Lambda_0^{s,p}(\Omega)$, including (1.9); in Section 3, we precise the assumptions on $\mathscr{L}^s$, which may be a degenerate operator, and we prove the existence of a solution to the corresponding pseudo-monotone variational inequality with the convex set of the s-gradient constraint (1.4) in $H_0^s(\Omega)$ and in $\Lambda_0^{s,\infty}(\Omega)$ for nonnegative threshold $g \in L^2_{loc}(\mathbb{R}^d)$ and $g \in L^\infty_{loc}(\mathbb{R}^d)$, respectively. We also give sufficient conditions for the operator $\mathscr{L}^s$ to be strictly coercive in $H_0^s(\Omega)$ and, as a consequence, we extend the strong continuous dependence (and the uniqueness) of the transport potential u with respect to the data, including the continuous dependence on the s-gradient thresholds.

Our main results are in Section 4, where we prove the existence of a generalised transport potential-density pair solving the Monge-Kantorovich equations $(1.6)_s$ and $(1.7)_s$ under rather general conditions on the operator $\mathscr{L}^s$, including the L^1 integrability of its coefficients. The proof is based on a new generalised weak continuous dependence on the pair (u,λ) with respect not only on the coefficients of $\mathscr{L}^s$ and on the data f, $\boldsymbol{f}$ (in L^1) but also on the threshold g (in L^∞) and on the solvability and *a priori* estimates of a suitable family of approximation problems in the space $\Lambda_0^{s,q}(\Omega)$, for a large finite q, with a penalisation of the s-gradient and with a nonlinear regularisation of q-power type of the possible degenerate operator $\mathscr{L}^s$. Finally in Section 5 we extend the weak convergence on the generalised localisation of the transport potentials and densities as the fractional parameter $s\to 1$, improving the result of [3]. In Sections 4 and 5, we work with generalised sequences, also called nets, see for instance [20].

2. The functional framework

Following [28] we recall that the fractional gradient of order $s\in(0,1)$, denoted by $D^s=\big(D_1^s,\dots,D_d^s\big)$, may be defined in the distributional sense by

$$D^s u = D(I_{1-s}u)$$

for any function $u\in L^p(\mathbb{R}^d)$, $1<p<\infty$, such that the Riesz potential $I_{1-s}u=I_{1-s}*u$ is locally integrable, i.e., for each $i=1,\dots,d$:

$$\big\langle D_i^s u,\varphi\big\rangle = -\big\langle I_{1-s}*u, D_i\varphi\big\rangle = \int_{\mathbb{R}^d}(I_{1-s}*u)D_i\varphi,\qquad \forall\varphi\in C_c^\infty(\mathbb{R}^d). \tag{2.1}$$

The Riesz kernel of order $\alpha\in(0,1)$, for $x\in\mathbb{R}^d\setminus\{0\}$, is given by

$$I_\alpha(x)=\frac{\gamma_{d,\alpha}}{|x|^{d-\alpha}},\qquad \text{with } \gamma_{d,\alpha}=\frac{\Gamma(\frac{d-\alpha}{2})}{\pi^{\frac d2}2^\alpha\Gamma(\frac\alpha2)},$$

and it satisfies the following well-known properties which proof is reproduced for completeness.

We fix the notation $B(x,r)$ for the open ball centered at $x\in\mathbb{R}^d$ and radius $r>0$.

Lemma 2.1. *Let I_α be the Riesz kernel, $0<\alpha<1$, $p\in(1,\infty)$ and $R>0$. Then, denoting by σ_{d-1} the surface area of the unit sphere in $\mathbb{R}^d$, we have:*

(i) $\|I_\alpha\|_{L^1(B(0,R))}=\sigma_{d-1}\frac{\gamma_{d,\alpha}}{\alpha}R^\alpha$;

(ii) *If $\alpha p<d$ then* $\|I_\alpha\|_{L^{p'}(\mathbb{R}^d\setminus B(0,R))}=\gamma_{d,\alpha}\Big(\sigma_{d-1}\frac{p-1}{d-\alpha p}\Big)^{\frac{1}{p'}}R^{\frac{\alpha p-d}{p}}$.

As a consequence $\lim\limits_{\alpha\to0}\|I_\alpha\|_{L^1(B(0,R))}=1$ *and* $\lim\limits_{\alpha\to0}\|I_\alpha\|_{L^{p'}(\mathbb{R}^d\setminus B(0,R))}=0$.

Proof. We start by noticing that, if $b\neq d$, $0\le R_1\le R_2<+\infty$, then

$$\int_{B(0,R_2)\setminus B(0,R_1)}\frac{1}{|x|^b}dx=\sigma_{d-1}\int_{R_1}^{R_2}r^{d-1-b}dr=\sigma_{d-1}\left[\frac{R_2^{d-b}}{d-b}-\frac{R_1^{d-b}}{d-b}\right].$$

Considering first $b=d-\alpha$, $R_2=R$ and $R_1=0$ we obtain (i). Then choosing $b=(d-a)\frac{p}{p-1}$, $R_1=R$ we obtain (ii) by letting $R_2\to\infty$ and noticing that $ap<d$ or equivalently $d-b<0$.

Since

$$\lim_{\alpha\to0}\frac{\gamma_{d,\alpha}}{\alpha}=\lim_{\alpha\longrightarrow0}\frac{\Gamma(\frac{d-\alpha}{2})}{\pi^{\frac d2}2^{\alpha+1}\frac\alpha2\Gamma(\frac\alpha2)}=\lim_{\alpha\to0}\frac{\Gamma(\frac{d-\alpha}{2})}{\pi^{\frac d2}2^{\alpha+1}\Gamma(\frac\alpha2+1)}=\frac{\Gamma(\frac d2)}{2\pi^{\frac d2}}=\frac{1}{\sigma_{d-1}},$$

the conclusions follows. □

As a consequence, the Riesz kernel is an approximation of the identity, and it was observed by Kurokawa [21], in the sense that

$$I_\alpha * f \longrightarrow f, \qquad \text{as } \alpha \to 0,$$

for instance, in $L^p(\mathbb{R}^d)$, if $f \in L^p(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$, $1 < q < p$ or pointwise at each point x of the Lebesgue set of $f \in L^p(\mathbb{R}^d)$, $1 \leq p < \infty$. In particular, if $Du \in \boldsymbol{L}^p(\mathbb{R}^d) \cap \boldsymbol{L}^q(\mathbb{R}^d)$, with $1 < q < p$, as observed in [26], we have $D^s u \underset{s \to 1}{\longrightarrow} Du$ in $\boldsymbol{L}^p(\mathbb{R}^d)$. We shall need the following stronger result which is also a consequence of this observation (see also Proposition 2.10 of [21] for the pointwise convergence).

Theorem 2.2. *If $g \in L^p(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap C(\mathbb{R}^d)$, for $p > 1$, is uniformly continuous in $\mathbb{R}^d$, then*

$$\lim_{\alpha \to 0} \|I_\alpha * g - g\|_{L^\infty(\mathbb{R}^d)} = 0.$$

Proof. Let $\varepsilon > 0$ and $0 < \delta < 1$ be such that

$$|z - x| \leq \delta \;\Rightarrow\; |g(z) - g(x)| \leq \varepsilon, \quad \forall x, z \in \mathbb{R}^d.$$

Using Lemma 2.1 consider α_0 such that, for $0 < \alpha < \alpha_0$,

$$\big| \|I_\alpha\|_{L^1(B(0,\delta))} - 1 \big| \leq \varepsilon, \quad \|I_\alpha\|_{L^{p'}(\mathbb{R}^d \setminus B(0,\delta))} \leq \varepsilon.$$

Then, for all $x \in \mathbb{R}^d$,

$$\begin{aligned}(I_\alpha * g)(x) - g(x) &= \int_{B(0,\delta)} I_\alpha(y) g(x-y)\,dy + \int_{\mathbb{R}^d \setminus B(0,\delta)} I_\alpha(y) g(x-y)\,dy - g(x) \\ &= \int_{B(0,\delta)} I_\alpha(y)(g(x-y) - g(x))\,dy + g(x)\Big(\int_{B(0,\delta)} I_\alpha(y)\,dy - 1\Big) \\ &\quad + \int_{\mathbb{R}^d \setminus B(0,\delta)} I_\alpha(y) g(x-y)\,dy.\end{aligned}$$

Hence

$$\begin{aligned}|(I_\alpha * g)(x) - g(x)| \leq &\varepsilon \|I_\alpha\|_{L^1(B(0,\delta))} + \|g\|_{L^\infty(\mathbb{R}^d)}\big(\|I_\alpha\|_{L^1(B(0,\delta))} - 1\big) \\ &+ \|I_\alpha\|_{L^{p'}(\mathbb{R}^d \setminus B(0,\delta))} \|g\|_{L^p(\mathbb{R}^d)} \\ \leq &\varepsilon^2 + \varepsilon\big(\|g\|_{L^\infty(\mathbb{R}^d)} + \|g\|_{L^p(\mathbb{R}^d)}\big)\end{aligned}$$

and the conclusion follows. □

As it was proved in [28, Theorem 1.2], the fractional gradient satisfies

$$D^s u = I_{1-s} Du = I_{1-s} * Du, \tag{2.2}$$

at least for functions $u \in C_c^\infty(\mathbb{R}^d)$, although that proof is equally valid for functions only in $C_c^1(\mathbb{R}^d)$, see [15, Proposition 2.2]. As a consequence of well-known properties of the Riesz potential, (2.2) is then also valid for functions u in the usual Sobolev space $W^{1,p}(\mathbb{R}^d)$, $1 < p < \infty$, since $Du \in \boldsymbol{L}^p(\mathbb{R}^d)$. In particular, as an immediate consequence of Theorem 2.2, we obtain the uniform approximation of continuous gradients by their fractional gradients.

Corollary 2.3. *For $w \in C_c^1(\mathbb{R}^d)$ we have*

$$D^s w \underset{s\to 1}{\longrightarrow} Dw \quad \text{in } \boldsymbol{L}^\infty(\mathbb{R}^d). \tag{2.3}$$

Remark 2.4. The convergence in (2.3) has been shown with a different proof for functions in $C_c^2(\mathbb{R}^d)$ and, if $w \in W^{1,p}(\mathbb{R}^d)$, also in $\boldsymbol{L}^p(\mathbb{R}^d)$ for $1 \leq p < \infty$, respectively in Proposition 4.4 and in Theorem 4.11 of [14]. This property can be seen as a localization of the fractional gradient. It has also been shown for functions in $W^{1,p}(\mathbb{R}^d)$ for $1 < p < \infty$, in [5, Theorem 3.2].

For smooth functions with compact support, as it was observed in [15], the distributional Riesz fractional gradient D^s can also be defined for $0 < s < 1$ by

$$D^s u(x) = \mu_s \int_{\mathbb{R}^d} \frac{u(x) - u(y)}{|x-y|^{d+s}} \frac{x-y}{|x-y|}\, dy, \tag{2.4}$$

where $\mu_s = (d+s-1)\gamma_{d,1-s}$ is bounded and $\lim_{s\to 1} \mu_s = 0$.

Let Ω be a bounded open subset of $\mathbb{R}^d$ and set

$$\Omega_R = \{x \in \mathbb{R}^d : d(x,\Omega) < R\}, \quad \text{for } R > 0.$$

In this work, for a function u defined in Ω, we still denote its extension by zero to $\mathbb{R}^d$ by u.

From (2.2) or (2.4), we see that for a function $u \in C_c^1(\Omega)$, while $Du = 0$ in $\mathbb{R}^d \setminus \Omega$, $D^s u$ is in general different from zero in the whole $\mathbb{R}^d$. Nevertheless the following remark holds.

Remark 2.5. For $u \in C_c^1(\Omega)$, from (2.4) we easily obtain

$$|D^s u(x)| \leq \frac{\mu_s}{d(x,\Omega)^{d+s}} \|u\|_{L^1(\Omega)}, \quad \forall x \in \mathbb{R}^d \setminus \overline{\Omega},$$

and, consequently, for all $R > 0$

$$\lim_{s\to 1} \|D^s u\|_{L^\infty(\mathbb{R}^d \setminus \Omega_R)} = 0.$$

Now, for $u \in C_c^\infty(\mathbb{R}^d)$ and $1 \leq p < \infty$, $0 < s \leq 1$, we introduce the norms

$$\|u\|_{\Lambda^{s,p}} = \left(\|u\|^p_{L^p(\mathbb{R}^d)} + \|D^s u\|^p_{\boldsymbol{L}^p(\mathbb{R}^d)} \right)^{\frac{1}{p}}$$

and we define the Banach spaces

$$\Lambda^{s,p}(\mathbb{R}^d) = \overline{C_c^\infty(\mathbb{R}^d)}^{\|\cdot\|_{\Lambda^{s,p}}},$$

where we recognize $\Lambda^{1,p}(\mathbb{R}^d) = W^{1,p}(\mathbb{R}^d)$, as the usual Sobolev spaces.

For $1 < p < \infty$, in [28] it was proved that $\Lambda^{s,p}(\mathbb{R}^d)$ (denoted there as $X^{s,p}(\mathbb{R}^d)$) is equal to $\{u \in L^p(\mathbb{R}^d) : u = g_s * f, \text{ for some } f \in L^p(\mathbb{R}^d)\}$, where g_s are the Bessel potentials, for $s \in \mathbb{R}$, which were introduced in 1960 by A. Caldéron and J. L. Lions. They are also called Bessel potential spaces or generalised Sobolev spaces (see [1, p. 219] or [12]). It is worth to recall that we have $\Lambda^{s+\varepsilon,p}(\mathbb{R}^d) \hookrightarrow W^{s,p}(\mathbb{R}^d) \hookrightarrow \Lambda^{s-\varepsilon,p}(\mathbb{R}^d)$, if $1 < p < \infty$ and $s > \varepsilon > 0$, where $W^{s,p}(\mathbb{R}^d)$

denotes the fractional Sobolev-Slobodeckij spaces. In fact, $\Lambda^{k,p}(\mathbb{R}^d) = W^{k,p}(\mathbb{R}^d)$ for nonnegative integers k or when $p = 2$ and $s > 0$, being $\Lambda^{s,2}(\mathbb{R}^d) = W^{s,2}(\mathbb{R}^d) = H^s(\mathbb{R}^d)$ Hilbert spaces.

For an open bounded set $\Omega \subset \mathbb{R}^d$, we define the subspace, for $0 < s \leq 1$,

$$\Lambda_0^{s,p}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{\Lambda^{s,p}}}, \quad 1 < p < \infty. \tag{2.5}$$

Clearly, considering the smooth functions with compact support trivially extended by zero outside their support, we have $\Lambda_0^{s,p}(\Omega) \subset \Lambda^{s,p}(\mathbb{R}^d)$. We observe that, by definition, for $u \in \Lambda_0^{s,p}(\Omega)$ the $D^s u$ is the limit in $\boldsymbol{L}^p(\mathbb{R}^d)$ of $D^s u_n$, for some sequence $u_n \in C_c^\infty(\Omega)$. Observing that, for $\varphi \in C_c^\infty(\Omega)$, we have

$$\int_{\mathbb{R}^d} \varphi\, D^s u_n = -\int_{\mathbb{R}^d} u_n\, D^s \varphi = -\int_{\mathbb{R}^d} u_n (I_{1-s} D\varphi) = -\int_{\mathbb{R}^d} (I_{1-s} u_n) D\varphi,$$

by using Fubini's Theorem. Letting $n \to \infty$, by Hardy-Littlewood-Sobolev's Theorem (see [30, Theorem 1, p. 119]), we conclude $I_{1-s}u_n \to I_{1-s}u$ in $L^q(\mathbb{R}^d)$, for $1/q = 1/p - (1-s)/d$, and consequently $D^s u = D(I_{1-s}u)$, i.e. $D^s u$ is the distributional Riesz fractional gradient of u. Moreover, in the limit, we may also conclude that $u \in \Lambda_0^{s,p}(\Omega)$ also satisfies

$$\int_{\mathbb{R}^d} \varphi D^s u = -\int_{\mathbb{R}^d} u D^s \varphi, \qquad \forall \varphi \in C_c^\infty(\Omega), \tag{2.6}$$

giving the distributional nature of D^s and corresponding to the definition of weak s-gradient of [15]. It is also possible to introduce the subspace $S^{s,p}(\mathbb{R}^d)$ of the $L^p(\mathbb{R}^d)$ functions with fractional s-gradient in $L^p(\mathbb{R}^d)^d$, which corresponds to the distributional approach of [29] and [15]. As proved in Appendix A of [11], $C_c^\infty(\mathbb{R}^d)$ is dense in $S^{s,p}(\mathbb{R}^d)$ for $1 \leq p < \infty$, and therefore we have $S^{s,p}(\mathbb{R}^d) = \Lambda^{s,p}(\mathbb{R}^d)$, for $0 < s < 1$.

Also in [28] it was shown the fractional Sobolev inequality for $1 < p < \infty$ and $0 < s < 1$,

$$\|u\|_{L^{p^*}(\mathbb{R}^d)} \leq C_* \|D^s u\|_{\boldsymbol{L}^p(\mathbb{R}^d)}, \quad \forall u \in C_c^\infty(\mathbb{R}^d), \tag{2.7}$$

for a constant $C_* > 0$, where $p^* = \frac{dp}{d-sp}$, if $sp < d$, as well as the fractional Trudinger ($p^* < \infty$, if $sp = d$) and Morrey ($p^* = \infty$, if $sp > d$) inequalities. If $sp > d$, in the left side of (2.7), we may take the semi-norm of β-Hölder continuous functions, $0 < \beta = s - \frac{d}{p}$.

From (2.7) we obtain a Poincaré inequality

$$\|u\|_{L^p(\Omega)} \leq C_p \|D^s u\|_{\boldsymbol{L}^p(\mathbb{R}^d)}, \quad \forall u \in \Lambda_0^{s,p}(\Omega), \tag{2.8}$$

for some $C_p > 0$, and in $\Lambda_0^{s,p}(\Omega)$ we shall use the equivalent norm

$$\|u\|_{\Lambda_0^{s,p}(\Omega)} = \|D^s u\|_{\boldsymbol{L}^p(\mathbb{R}^d)}. \tag{2.9}$$

We can extend the definition (2.5) for $p = \infty$ and define

$$\Lambda_0^{s,\infty}(\Omega) = \Big\{ u \in \bigcap_{1<p<\infty} \Lambda_0^{s,p}(\Omega) : D^s u \in \boldsymbol{L}^\infty(\mathbb{R}^d) \Big\}.$$

The fractional Poincaré inequality (2.8) can be made more precise with respect to s, $0 < s < 1$, to also include the limit cases $p = 1$ and $p = \infty$.

Proposition 2.6. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set. Then there exists a constant $C_0 = C_0(\Omega, d) > 0$ such that, for all $0 < s < 1$ and $1 \leq p \leq \infty$,*

$$\|u\|_{L^p(\Omega)} \leq \tfrac{C_0}{s}\|D^s u\|_{\boldsymbol{L}^p(\mathbb{R}^d)}, \quad \forall u \in \Lambda_0^{s,p}(\Omega). \tag{2.10}$$

Proof. For $1 < p < \infty$, this is Theorem 2.9 of [5], but the same proof is still valid for $p = 1$. Since C_0 is independent of p, the case $p = \infty$ is obtained by letting $p \to \infty$ in (2.10). □

In addition, in a bounded open set $\Omega \subset \mathbb{R}^d$ satisfying the extension property, it is well known that $\Lambda_0^{s,2}(\Omega) = W_0^{s,2}(\Omega) = H_0^s(\Omega)$ (see, for instance, [23]).

Although there is no monotone inclusions in p of $L^p(\mathbb{R}^d)$ the following result holds.

Theorem 2.7. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set, $p \in [1, \infty)$ and $0 < s < 1$. Then there exists a positive constant $C = \big(1 + \frac{1}{(p-1)d+ps}\big)C(d, \Omega)$, such that, for $R \geq 1$,*

$$\int_{\mathbb{R}^d \setminus \Omega_R} |D^s u(x)|^p dx \leq \frac{\mu_s^p C}{R^{(p-1)d+ps}}\|u\|_{L^1(\Omega)}^p, \quad \forall u \in \Lambda_0^{s,p}(\Omega). \tag{2.11}$$

As a consequence, the following inclusions hold

$$\Lambda_0^{s,q}(\Omega) \subseteq \Lambda_0^{s,p}(\Omega), \quad 1 \leq p < q < \infty, \tag{2.12}$$

and are continuous, since there exists $C_{p,q} > 0$ such that

$$\|D^s u\|_{\boldsymbol{L}^p(\mathbb{R}^d)} \leq C_{p,q}\|D^s u\|_{\boldsymbol{L}^q(\mathbb{R}^d)}, \quad u \in \Lambda_0^{s,q}(\Omega). \tag{2.13}$$

In addition, $C_{1,q} = \frac{E}{s}$, where E is independent of s, and $C_{p,q}$ is independent of s, if $p > 1$.

Proof. It is enough to consider $u \in C_c^\infty(\Omega)$. If $\delta(\Omega)$ denotes the diameter of Ω, consider $S = \frac{1}{2}\delta(\Omega) + R$ and z such that $\Omega_R \subseteq B(z, S)$. Consider the annulus $A_n = B(z, S+n+1) \setminus B(z, S+n)$, for each $n \in \mathbb{N}_0$.

Letting $\omega_d = \big|B(0, 1)\big|$ and μ_s be as in (2.4), we have

$$\begin{aligned}
\frac{1}{\mu_s^p}\int_{\mathbb{R}^d \setminus B(z,S)} |D^s u(x)|^p dx &\leq \int_{\mathbb{R}^d \setminus B(z,S)} \left|\int_\Omega \tfrac{|u(y)|}{|x-y|^{d+s}} dy\right|^p dx \\
&= \sum_{n=0}^{\infty} \int_{A_n} \left(\int_\Omega \tfrac{|u(y)|}{|x-y|^{d+s}}\, dy\right)^p dx \\
&\leq \sum_{n=0}^{\infty} \int_{A_n} \left(\int_\Omega \tfrac{|u(y)|}{(n+R)^{d+s}}\, dy\right)^p dx \\
&= \sum_{n=0}^{\infty} \frac{\omega_d\big[(S+n+1)^d - (S+n)^d\big]}{(n+R)^{p(d+s)}}\|u\|_{L^1(\Omega)}^p.
\end{aligned}$$

By the Lagrange theorem, there exists $\nu \in (n, n+1)$ such that $(n+1+S)^d - (n+S)^d = d(\nu+S)^{d-1} \leq d(n+1+S)^{d-1}$ and then, as $S \geq 1$ and $n + R \geq \frac{R}{S}(n+S)$,

$$\begin{aligned}
\tfrac{1}{\mu_s^p}\int_{\mathbb{R}^d\setminus B(z,S)} |D^s u(x)|^p dx &\leq d\left(\tfrac{S}{R}\right)^{p(d+s)} \omega_d \sum_{n=0}^{\infty} \left(\tfrac{n+1+S}{n+S}\right)^{d-1} \frac{1}{(n+S)^{(p-1)d+ps+1}} \|u\|^p_{L^1(\Omega)} \\
&\leq d\left(\tfrac{S}{R}\right)^{p(d+s)} \omega_d 2^{d-1} \sum_{n=0}^{\infty} \frac{1}{(n+S)^{(p-1)d+ps+1}} \|u\|^p_{L^1(\Omega)} \\
&\leq d\left(\tfrac{S}{R}\right)^{p(d+s)} \omega_d 2^{d-1} \left[\frac{1}{S^{(p-1)d+ps+1}} + \int_S^{\infty} \frac{1}{x^{(p-1)d+ps+1}}\, dx\right] \|u\|^p_{L^1(\Omega)} \\
&= d\left(\tfrac{S}{R}\right)^{p(d+s)} \omega_d 2^{d-1} \left[\frac{1}{S^{(p-1)d+ps+1}} + \frac{1}{(p-1)d+ps}\frac{1}{S^{(p-1)d+ps}}\right] \|u\|^p_{L^1(\Omega)} \\
&\leq d\left(\tfrac{S}{R}\right)^{p(d+s)} \omega_d 2^{d-1} \left(1 + \frac{1}{(p-1)d+ps}\right) \frac{1}{S^{(p-1)d+ps}} \|u\|^p_{L^1(\Omega)} \\
&= d\left(\tfrac{S}{R}\right)^{d} \omega_d 2^{d-1} \left(1 + \frac{1}{(p-1)d+ps}\right) \frac{1}{R^{(p-1)d+ps}} \|u\|^p_{L^1(\Omega)}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\tfrac{1}{\mu_s^p}\int_{B(z,S)\setminus\Omega_R} |D^s u(x)|^p dx &\leq \int_{B(z,S)\setminus\Omega_R} \left(\int_\Omega \tfrac{|u(y)|}{|x-y|^{d+s}}\, dy\right)^p dx \\
&\leq \int_{B(z,S)\setminus\Omega_R} \left(\int_\Omega \tfrac{|u(y)|}{R^{d+s}}\, dy\right)^p dx \\
&\leq \omega_d \left(\tfrac{S}{R}\right)^d \tfrac{1}{R^{(p-1)d+ps}} \|u\|^p_{L^1(\Omega)}.
\end{aligned}$$

As $\frac{S}{R} \leq 1 + \frac{1}{2}\delta(\Omega)$ we have

$$\tfrac{1}{\mu_s^p}\int_{\mathbb{R}^d\setminus\Omega_R} |D^s u(x)|^p dx \leq \omega_d\big(1+\tfrac{1}{2}\delta(\Omega)\big)^d \left[d2^{d-1}\left(1 + \frac{1}{(p-1)d+ps}\right) + 1\right] \frac{1}{R^{(p-1)d+ps}} \|u\|^p_{L^1(\Omega)},$$

from where we obtain (2.11).

For the inclusion (2.12), by considering $R = 1$, there exists C_1 such that

$$\begin{aligned}
\int_{\mathbb{R}^d} |D^s u(x)|^p dx &= \int_{\mathbb{R}^d\setminus\Omega_1} |D^s u(x)|^p dx + \int_{\Omega_1} |D^s u(x)|^p dx \\
&\leq C\mu_s^p \|u\|^p_{L^1(\Omega)} + |\Omega_1|^{\frac{1}{p}-\frac{1}{q}} \|D^s u\|^p_{\boldsymbol{L}^q(\Omega_1)} \\
&\leq C\big(\max_{s\in[0,1)} \mu_s^p\big)\, |\Omega|^{\frac{1}{q'}} \|u\|^p_{L^q(\Omega)} + |\Omega_1|^{\frac{1}{p}-\frac{1}{q}} \|D^s u\|^p_{\boldsymbol{L}^q(\Omega_1)} \\
&\leq C_1 \|u\|^p_{\Lambda_0^{s,q}}
\end{aligned}$$

by using Poincaré inequality (2.10), yielding the conclusion. □

As in (2.9) we define in $\Lambda_0^{s,\infty}(\Omega)$ the topology induced by $\|u\|_{\Lambda_0^{s,\infty}(\Omega)} = \|D^s u\|_{\boldsymbol{L}^\infty(\mathbb{R}^d)}$, which is a norm by Poincaré inequality.

Proposition 2.8. *There exists a constant $C_{p,\infty} > 0$, which is independent of $s \in [\sigma, 1)$ for each $\sigma > 0$, such that* (2.13) *holds for $q = \infty$. In particular the inclusion*

$$\Lambda_0^{s,\infty}(\Omega) \subset \Lambda_0^{s,p}(\Omega)$$

is continuous for all $p \geq 1$, and $\Lambda_0^{s,\infty}(\Omega)$ is a Banach space.

Proof. From Theorem 2.7, for $R \geq 1$ there exists $C > 0$ independent of R, such that for all $u \in \Lambda_0^{s,\infty}(\Omega)$,

$$\begin{aligned}\int_{\mathbb{R}^d} |D^s u(x)|^p dx &\leq \int_{\Omega_R} |D^s u(x)|^p dx + \frac{C\mu_s^p}{R^{(p-1)d+ps}} \|u\|_{L^1(\Omega)}^p \\ &\leq |\Omega_R| \|D^s u\|_{\boldsymbol{L}^\infty(\mathbb{R}^d)}^p + \frac{C\mu_s^p |\Omega|^{p-1}}{R^{(p-1)d+ps}} \|u\|_{L^p(\Omega)}^p \\ &\leq |\Omega_R| \|D^s u\|_{\boldsymbol{L}^\infty(\mathbb{R}^d)}^p + \frac{C_0^p}{s^p} \frac{C\mu_s^p |\Omega|^{p-1}}{R^{(p-1)d+ps}} \|D^s u\|_{L^p(\mathbb{R}^n)}^p\end{aligned}$$

by (2.10). Choosing

$$R = \max\left\{1, \left(\frac{2C_0^p C\mu_s^p |\Omega|^{p-1}}{s^p}\right)^{\frac{1}{(p-1)d+ps}}\right\}$$

we obtain

$$\|D^s u\|_{L^p(\Omega)} \leq 2^{\frac{1}{p}} |\Omega_R|^{\frac{1}{p}} \|D^s u\|_{\boldsymbol{L}^\infty(\mathbb{R}^d)},$$

which yields the continuity of the embedding $\Lambda_0^{s,\infty}(\Omega) \subset \Lambda_0^{s,p}(\Omega)$.

Finally, since a Cauchy sequence (u_n) in $\Lambda_0^{s,\infty}(\Omega)$ is also, for all $1 < p < \infty$, a Cauchy sequence in the nested Banach spaces $\Lambda_0^{s,p}(\Omega)$, its common limit $u \in \bigcap_{1<p<\infty} \Lambda_0^{s,p}(\Omega)$. As $D^s u_n$ are uniformly bounded, then $D^s u$ is bounded, and therefore $u \in \Lambda_0^{s,\infty}(\Omega)$. ☐

Remark 2.9. The inclusion (2.12) was also obtained independently in [12, Corollary 2.4.1], as a consequence of an interesting variant of the Poincaré inequality, see [12, Theorem 2.4.3], for some constant $C_1 = C_1(\Omega, \Omega_1, d) > 0$,

$$\|u\|_{L^p(\Omega)} \leq \tfrac{C_1}{s} \|D^s u\|_{\boldsymbol{L}^p(\Omega_1)}, \quad \forall u \in \Lambda_0^{s,p}(\Omega),$$

for an open set $\Omega_1 \supset B(0, 2R) \supset \Omega$, with $R > 1$, $1 < p < \infty$ and $0 < s < 1$.

Remark 2.10. We note that, for $p \in [1, \infty)$ we have the inclusions $W_0^{1,p}(\Omega) \subset \Lambda_0^{s,p}(\Omega) \subset \Lambda_0^{\sigma,p}(\Omega)$, for $0 < \sigma < s < 1$. We may conclude that, as a consequence of (2.11), for $R \geq 1$, as $\lim_{s\to1} \mu_s = 0$,

$$\lim_{s\to1} \|D^s u\|_{L^p(\mathbb{R}^d\setminus\Omega_R)} = 0, \quad \forall u \in W_0^{1,p}(\Omega).$$

Remark 2.11. Also from Theorem 2.7, for $R \geq 1$ we can take the limit as $p \to \infty$ in (2.11), to conclude (compare with Remark 2.5):

$$\|D^s u\|_{\boldsymbol{L}^\infty(\mathbb{R}^d\setminus\Omega_R)} \leq \tfrac{\mu_s}{R^{d+s}} \|u\|_{L^1(\Omega)}, \quad \forall u \in \Lambda_0^{s,\infty}(\Omega),\ 0 < s < 1.$$

Remark 2.12. We denote by $W_0^{1,\infty}(\Omega)$ the space of Lipschitz functions vanishing on the boundary of Ω. Extending a function $u \in W_0^{1,\infty}(\Omega)$ by zero outside Ω and using definition (2.4), we have, for each $x \in \mathbb{R}^d$,

$$\begin{aligned} |D^s u(x)| &\leq \mu_s \int_{\mathbb{R}^d} \frac{|u(x)-u(y)|}{|x-y|^{d+s}}\, dy \\ &= \mu_s \|Du\|_{\boldsymbol{L}^\infty(\Omega)} \int_{\{|x-y|\leq 1\}} \frac{dy}{|x-y|^{d+s-1}} + \mu_s\, 2\|u\|_{L^\infty(\Omega)} \int_{\{|x-y|>1\}} \frac{dy}{|x-y|^{d+s}} \\ &\leq \mu_s\, C_s \|Du\|_{\boldsymbol{L}^\infty(\Omega)}, \end{aligned}$$

for a finite $C_s > 0$, by the Poincaré inequality and since both integrals are finite for $s \in (0,1)$. Consequently,

$$W_0^{1,\infty}(\Omega) \subset \Lambda_0^{s,\infty}(\Omega), \quad \forall s \in (0,1). \tag{2.14}$$

The Lions-Calderón spaces $\Lambda_0^{s,p}(\Omega)$, $0 < s < 1$, $1 < p < \infty$, similarly to the Sobolev-Slobodeckij spaces $W_0^{s,p}(\Omega)$, have continuous and compact embeddings of Sobolev and Rellich-Kondrachov-type for $\Omega \subset \mathbb{R}^d$ open and bounded,

$$\Lambda_0^{s,p}(\Omega) \subset L^q(\Omega), \tag{2.15}$$

for $q \in \left[1, \frac{dp}{d-sp}\right]$ if $sp < d$, for all $q \geq 1$ if $sp = d$, and for $q = \infty$ if $sp > d$, the embeddings being compact in the case $sp < d$ only for $q < \frac{dp}{d-sp} = p^*$. Also the embeddings

$$\Lambda_0^{s,p}(\Omega) \subset C^{0,\beta}(\overline{\Omega}), \quad \text{for } sp > d \tag{2.16}$$

are continuous for $0 < \beta \leq s - \frac{d}{p}$ and compact for $0 < \beta < s - \frac{d}{p}$, where $C^{0,\beta}(\overline{\Omega})$ denotes the space of Hölder continuous functions in $\overline{\Omega}$ of exponent β. Consequently, by Proposition 2.8, we have the compact embeddings

$$\Lambda_0^{s,\infty}(\Omega) \subset C^{0,\beta}(\overline{\Omega}) \subset L^\infty(\Omega), \quad \text{for } 0 < \beta < s < 1. \tag{2.17}$$

We also have the non-trivial compact embeddings

$$\Lambda_0^{s,p}(\Omega) \subset \Lambda_0^{\sigma,p}(\Omega), \quad 0 < \sigma < s < 1, \quad 1 < p < \infty, \tag{2.18}$$

which proof can be found in [12, pg. 65] and is well known for $p = 2$.

We denote the dual space of $\Lambda_0^{s,p}(\Omega)$ by $\Lambda^{-s,p'}(\Omega)$, $0 < s < 1$, $1 < p < \infty$, and we have a similar characterization in terms of the fractional s-gradient as it was shown in [12, Theorem 2.4.4, p. 66] for bounded and unbounded open domains $\Omega \subset \mathbb{R}^d$.

Proposition 2.13. *Let $0 < s < 1$, $1 < p < \infty$ and $F \in \Lambda^{-s,p'}(\Omega)$. Then there exist functions $f_0 \in L^{p'}(\Omega)$ and $f_1, \ldots, f_d \in L^{p'}(\mathbb{R}^d)$ such that*

$$[F, v]_{s,p} = \int_\Omega f_0 v + \sum_{j=1}^d \int_{\mathbb{R}^d} f_j D_j^s v, \quad \forall v \in \Lambda_0^{s,p}(\Omega). \tag{2.19}$$

When $p=\infty$ and $f_0 \in L^1(\Omega)$ and $f_1,\dots,f_d \in L^1(\mathbb{R}^d)$, (2.19) defines a linear form in $\Lambda^{s,\infty}_0(\Omega)$. However, these forms do not exhaust $\Lambda^{s,\infty}_0(\Omega)'$.

We shall also work with the dual of $L^\infty(\mathbb{R}^d)$, which is also denoted as $ba(\mathbb{R}^d)$ (see, for instance, [24] and [2]) and their elements are sometimes called charges. We recall (see Example 5, Section 9, Chapter IV of [31]) that an element $\lambda \in L^\infty(\mathbb{R}^d)'$ can be represented by a Radon integral

$$\langle \lambda, \varphi\rangle = \int_{\mathbb{R}^d} \varphi d\lambda^*, \quad \forall \varphi \in L^\infty(\mathbb{R}^d), \tag{2.20}$$

for a finitely additive measure λ^*, which is of bounded variation and absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^d$.

We say that a charge λ is positive, or simply $\lambda \geq 0$, if $\langle \lambda, \varphi\rangle \geq 0$ for any $\varphi \in L^\infty(\Omega)$, $\varphi \geq 0$.

Exactly as for the Lebesgue integral, we have the Hölder inequality for positive charges (see [24, p.122]).

Proposition 2.14. *Let $p>1$ and $\lambda \in L^\infty(\mathbb{R}^d)'$ be positive. Then*

$$\big|\langle \lambda, \varphi\psi\rangle\big| \leq \langle \lambda, |\varphi|^p\rangle^{\frac{1}{p}} \langle \lambda, |\psi|^{p'}\rangle^{\frac{1}{p'}}, \quad \forall \varphi,\psi \in L^\infty(\Omega).$$

3. Variational inequalities with s-gradient constraints

Let $\Omega \subset \mathbb{R}^d$ be open and bounded, with the extension property, i.e., the extension of $u \in H^s_0(\Omega)$ by zero in $\mathbb{R}^d \setminus \Omega$ is in $H^s(\mathbb{R}^d)$, $0 < s \leq 1$. This holds, in particular, for domains with Lipschitz boundaries (see for instance [16, Section 5]). To consider s-gradient constrained problems, we define the following closed convex sets

$$\mathbb{K}^s_g = \big\{v \in H^s_0(\Omega) : |D^s v| \leq g \text{ a.e. in } \mathbb{R}^d\big\}, \quad 0 < s \leq 1, \tag{3.1}$$

for prescribed thresholds satisfying

$$g \in L^2_{loc}(\mathbb{R}^d), \quad g \geq 0 \text{ a.e. in } \mathbb{R}^d, \tag{3.2}$$

or, in the bounded case,

$$g \in L^\infty_{loc}(\mathbb{R}^d), \quad g \geq 0 \text{ a.e. in } \mathbb{R}^d. \tag{3.3}$$

In Lemma 3.2, we will see that these assumptions on g are enough for $\mathbb{K}^s_g$ to be bounded in $H^s_0(\Omega)$ and $\Lambda^{s,\infty}_0(\Omega)$, respectively.

For $0 < s \leq 1$, we define a bilinear form by letting

$$\mathscr{L}^s(u,v) = \int_{\mathbb{R}^d} AD^s u \cdot D^s v + \int_\Omega \boldsymbol{d} u \cdot D^s v + \int_\Omega (\boldsymbol{b}\cdot D^s u + cu)v. \tag{3.4}$$

Here the principal part may be degenerate, under the assumption on the matrix $A = A(x)$:

$$A(x)\boldsymbol{\xi}\cdot\boldsymbol{\xi} \geq 0, \quad \forall \boldsymbol{\xi} \in \mathbb{R}^d, \text{ a.e. } x \in \mathbb{R}^d. \tag{3.5}$$

In addition, we shall assume that the coefficients of the bilinear form satisfy, when $u, v \in H^s_0(\Omega)$,

$$A \in L^\infty(\mathbb{R}^d)^{d^2}, \boldsymbol{b}, \boldsymbol{d} \in \boldsymbol{L}^r(\Omega) \text{ and } c \in L^{\frac{r}{2}}(\Omega), \quad r > \tfrac{d}{s} \tag{3.6}$$

or, in the bounded case, when $u, v \in \Lambda_0^{s,\infty}(\Omega)$,

$$A \in L^{p_1}(\mathbb{R}^d)^{d^2},\ p_1 \in [1,\infty),\ \boldsymbol{b}, \boldsymbol{d} \in \boldsymbol{L}^1(\Omega) \text{ and } c \in L^1(\Omega). \tag{3.7}$$

Similarly, for $0 < s \leq 1$, we may define the linear form

$$F_s(v) = [F, v]_s = \int_\Omega f_\# v + \int_{\mathbb{R}^d} \boldsymbol{f} \cdot D^s v \tag{3.8}$$

for any $v \in H_0^s(\Omega)$, with

$$f_\# \in L^{2^\#}(\Omega), \quad \boldsymbol{f} \in \boldsymbol{L}^2(\mathbb{R}^d), \tag{3.9}$$

where, by the Sobolev embedding (2.7), $2^\# = \frac{2d}{d+2s}$ if $0 < s < \frac{d}{2}$, or $2^\# = q$ for any $q > 1$ when $s = \frac{1}{2}$, and $2^\# = 1$ when $\frac{1}{2} < s \leq 1$ or, in the bounded case, for any $v \in \Lambda_0^{s,\infty}(\Omega)$, with

$$f_\# \in L^1(\Omega), \quad \boldsymbol{f} \in \boldsymbol{L}^{q_1}(\mathbb{R}^d),\ q_1 \in [1,\infty). \tag{3.10}$$

Notice that in the case $s = 1$, since u, v and Du, Dv are zero in $\mathbb{R}^d \setminus \Omega$, all the integration domains in (3.4) and (3.8) reduce to Ω.

Theorem 3.1. *Assume* (3.5)*, and suppose that*

i) *either assumptions* (3.2)*,* (3.6)*, and* (3.9) *hold,*
ii) *or assumptions* (3.3)*,* (3.7)*, and* (3.10) *hold.*

Then, for $0 < s \leq 1$, there exists a solution of the s-gradient constraint variational inequality

$$u \in \mathbb{K}_g^s : \qquad \mathscr{L}^s(u, v-u) \geq [F, v-u]_s, \quad \forall v \in \mathbb{K}_g^s. \tag{3.11}$$

We will use the following lemma in the proof of this theorem.

Lemma 3.2. *For $1 \leq p \leq \infty$ and $g \in L^p_{loc}(\mathbb{R}^d)$, with $g \geq 0$, the set $\mathbb{K}_g^s$ is bounded in $\Lambda_0^{s,p}(\Omega)$. More precisely, there exists $R = R(p,s)$ such that, for $u \in \mathbb{K}_g^s$,*

$$\|D^s u\|_{\boldsymbol{L}^p(\mathbb{R}^d)} \leq 2^{\frac{1}{p}} \|g\|_{L^p(\Omega_R)},\ \text{if } p < \infty, \quad \|D^s u\|_{\boldsymbol{L}^\infty(\mathbb{R}^d)} \leq \|g\|_{L^\infty(\Omega_R)}. \tag{3.12}$$

Proof. By the Theorem 2.7, when $p < \infty$, choosing R such that $\frac{C\mu_s^p}{R^{(p-1)d+ps}} \frac{C_0^p}{s^p} |\Omega|^{p-1} \leq \frac{1}{2}$, and using (2.11), we have

$$\begin{aligned}\|D^s u\|^p_{\boldsymbol{L}^p(\mathbb{R}^d)} = \|D^s u\|^p_{\boldsymbol{L}^p(\mathbb{R}^d \setminus \Omega_R)} + \|D^s u\|^p_{\boldsymbol{L}^p(\Omega_R)} \ &\leq \ \frac{C\mu_s^p}{R^{(p-1)d+ps}} \|u\|^p_{L^1(\Omega)} + \int_{\Omega_R} |D^s u|^p \\ &\leq \tfrac{1}{2} \|D^s u\|^p_{\boldsymbol{L}^p(\mathbb{R}^d)} + \int_{\Omega_R} g^p,\end{aligned}$$

from where we obtain the first inequality. Letting $p \to \infty$, the second inequality follows. □

Proof. (of Theorem 3.1) This existence result in the Hilbertian case i) is a consequence of a theorem of H. Brézis (see [9] or [22, Theorem 8.1, p. 245]), since $\mathbb{K}_g^s$ is a nonempty, closed and bounded convex set of $H_0^s(\Omega)$ and the operator $P : H_0^s(\Omega) \longrightarrow H^{-s}(\Omega)$ defined by

$$[Pu, v]_s = \mathscr{L}^s(u, v) - [F, v]_s, \quad u, v \in \mathbb{K}_g^s \tag{3.13}$$

is pseudo-monotone in $\mathbb{K}^s_g$, i.e., if $u_n \underset{n}{\rightharpoonup} u$ in $H^s_0(\Omega)$, for $u_n, u \in \mathbb{K}^s_g$ with $\overline{\lim_n}[Pu_n, u_n - u]_s \leq 0$ then

$$\underline{\lim_n}[Pu_n, u_n - v]_s \geq [Pu, u - v]_s, \quad \forall v \in \mathbb{K}^s_g. \tag{3.14}$$

Indeed, taking $u_n \underset{n}{\rightharpoonup} u$ in $H^s_0(\Omega)$, which by compactness of the embedding (2.15) (for $p = 2$ and $1 \leq q < 2^* = \frac{2d}{d-2s}$ if $2s < d$, for all $q \geq 1$ if $2s = d$ and for $q = \infty$ if $2s > d$), we may assume also that $u_n \underset{n}{\to} u$ in $L^q(\Omega)$. Write P in the form

$$[Pu, w]_s = \int_{\mathbb{R}^d} AD^s u \cdot D^s w + [Bu, w]_s$$

with

$$[Bu, w]_s = \int_{\mathbb{R}^d} (\boldsymbol{d}u - \boldsymbol{f}) \cdot D^s w + \int_\Omega (\boldsymbol{b} \cdot D^s u + cu - f_\#)w.$$

It is then clear that the assumptions (3.7) and (3.10) imply

$$[Bu_n, u_n - v]_s \underset{n}{\to} [Bu, u - v]_s,$$

as $D^s u_n \underset{n}{\rightharpoonup} D^s u$ in $\boldsymbol{L}^2(\mathbb{R}^d)$, and we have

$$\int_\Omega (\boldsymbol{b} + \boldsymbol{d})u_n \cdot D^s u_n \underset{n}{\longrightarrow} \int_\Omega (\boldsymbol{b} + \boldsymbol{d})u \cdot D^s u \quad \text{and} \quad \int_\Omega cu_n^2 \underset{n}{\longrightarrow} \int_\Omega cu^2.$$

Hence (3.14) follows easily by noting that the assumption (3.5) implies $\int_{\mathbb{R}^d} AD^s(u_n - u) \cdot D^s(u_n - u) \geq 0$, and hence it suffices to take the limit inferior in

$$\int_{\mathbb{R}^d} AD^s u_n \cdot D^s u_n \geq \int_{\mathbb{R}^d} AD^s u_n \cdot D^s u + \int_{\mathbb{R}^d} AD^s u \cdot D^s u_n - \int_{\mathbb{R}^d} AD^s u \cdot D^s u. \tag{3.15}$$

In the non-Hilbertian case, we start by approximating the data, in the respective spaces, by smooth functions with compact support A_m, $\boldsymbol{b}_m$, $\boldsymbol{d}_m$, c_m, $f_{\#m}$ and $\boldsymbol{f}_m$ and we let u_m be a solution of the variational inequality (3.11) with these data, which exists by the previous case.

As $(D^s u_m)_m$ is bounded in $\boldsymbol{L}^\infty(\mathbb{R}^d)$ by Lemma 3.2, using Proposition 2.8 and the compact embedding (2.16), there exist a $u \in \Lambda^{s,\infty}_0(\Omega)$ and a $G \in \boldsymbol{L}^\infty(\mathbb{R}^d)$, such that, for some subsequence, $u_m \underset{m}{\longrightarrow} u$ strongly in $L^\infty(\Omega)$ and $D^s u_m \underset{m}{\rightharpoonup} G$ in $\boldsymbol{L}^\infty(\mathbb{R}^d)$-weak*. Using this limit in (2.6), we easily see that $G = D^s u$. Thus, as $\Lambda^{s,\infty}_0(\Omega)$ is continuously included in $\Lambda^{s,p}_0(\Omega)$, we have $D^s u_m \underset{m}{\rightharpoonup} D^s u$ in $\boldsymbol{L}^p(\mathbb{R}^d)$-weak, for any $p < \infty$, in particular for $p = p_1'$ and $p = q_1'$.

Using the above convergences, we immediately have, for any $v \in \mathbb{K}^s_g$,

$$\begin{aligned}\int_{\mathbb{R}^d} (\boldsymbol{d}_m u_m - \boldsymbol{f}_m) \cdot D^s(v - u_m) + \int_\Omega (\boldsymbol{b}_m \cdot D^s u_m + cu_m - f_\# m)(v - u_m)& \\ \underset{m}{\longrightarrow} \int_{\mathbb{R}^d} (\boldsymbol{d}u - \boldsymbol{f}) \cdot D^s(v - u_m) + \int_\Omega (\boldsymbol{b} \cdot D^s u + cu - f_\#)(v - u_m)&.\end{aligned}$$

On the other hand, using the monotonicity of A_m, for any $v \in \mathbb{K}^s_g$ we have

$$\int_{\mathbb{R}^d} A_m D^s u_m \cdot D^s(v - u_m) \leq \int_{\mathbb{R}^d} A_m D^s v \cdot D^s(v - u_m).$$

Since $\displaystyle\int_{\mathbb{R}^d} A_m D^s v\cdot D^s(v-u_m)\underset{m}{\longrightarrow}\int_{\mathbb{R}^d} AD^s v\cdot D^s(v-u)$ then we get that, for any $v\in\mathbb{K}^s_g$,

$$\int_{\mathbb{R}^d} AD^s v\cdot D^s(v-u)+\int_\Omega \boldsymbol{d}u\cdot D^s(v-u)+\int_\Omega(\boldsymbol{b}\cdot D^s u+cu)(v-u)\geq [F,v-u].$$

Choosing $v=u+t(w-u)\in\mathbb{K}^s_g$, for $t\in(0,1)$, as test function, we obtain

$$\mathscr{L}^s(u,w-u)\geq [F,w-u]_s,\quad \forall w\in\mathbb{K}^s_g,$$

after letting $t\longrightarrow 0^+$. Therefore u solves (3.11). □

Remark 3.3. To obtain the uniqueness to (3.11) it suffices to require the strict positivity of the bilinear form

$$\mathscr{L}^s(u-\widehat{u},u-\widehat{u})>0,\quad \forall u,\widehat{u}\in\mathbb{K}^s_g : u\neq\widehat{u},$$

which needs stronger assumptions on its coefficients.

Remark 3.4. The constrained problem (3.11) for $u\in\mathbb{K}^s_g$ determines the existence of an element $\Gamma=\Gamma(u)\in H^{-s}(\Omega)$ belonging to the sub-differential of the indicatrix function $I_{\mathbb{K}^s_g}$ of the convex set $\mathbb{K}^s_g$, i.e. $I_{\mathbb{K}^s_g}(v)=0$ if $v\in\mathbb{K}^s_g$, $I_{\mathbb{K}^s_g}(v)=+\infty$ if $v\in H^s_0(\Omega)\setminus\mathbb{K}^s_g$, (see [22, p.203]), which is given by

$$\Gamma\equiv F-\mathscr{L}^s u\in\partial I_{\mathbb{K}^s_g}\quad \text{ in } H^s_0(\Omega),$$

where $\mathscr{L}^s:\mathbb{K}^s_g\longrightarrow H^{-s}(\Omega)$ is the linear operator defined by the bilinear form as in (3.4). A main question is to relate Γ to the solution u, for instance trough the existence of a Lagrange multiplier λ such that $\Gamma=\lambda D^s u$. This has been shown only in very special cases with the classical gradient ($s=1$) (see [10] and [25], for more references).

The existence result of Theorem 3.1 includes the degenerate case $A\equiv 0$ in (3.11). On the other, when the matrix A is strictly elliptic, i.e., if we replace (3.5) by assuming the existence of $a_*>0$, such that

$$A(x)\boldsymbol{\xi}\cdot\boldsymbol{\xi}\geq a_*|\boldsymbol{\xi}|^2,\quad \forall\boldsymbol{\xi}\in\mathbb{R}^d,\ \text{ a.e. } x\in\mathbb{R}^d, \tag{3.16}$$

we may give the following sufficient condition for the bilinear form (3.4) to be strictly coercive, by imposing

$$\delta\equiv a_*-C_*\Big(\|\boldsymbol{b}+\boldsymbol{d}\|_{\boldsymbol{L}^{\frac{d}{s}}(\Omega)}+C_*\|c^-\|_{L^{\frac{d}{2s}}(\Omega)}\Big)>0. \tag{3.17}$$

Here C_* is the Sobolev constant of the embedding $H^s_0(\Omega)\hookrightarrow L^{2^*}(\Omega)$ ($2^*=\frac{2d}{d-2s}$, $2s<d$) and $c^-=\max\{0,-c\}$.

Theorem 3.5. *Let $A\in L^\infty(\mathbb{R}^d)^{d^2}$, $\boldsymbol{b},\boldsymbol{d}\in\boldsymbol{L}^{\frac{d}{s}}(\Omega)$, $c\in L^{\frac{d}{2s}}(\Omega)$ satisfy* (3.16) *and* (3.17) *for $2s<d$, and $g\in L^2_{loc}(\Omega)$. Then, for any $f_\#$ and $\boldsymbol{f}$ satisfying* (3.9), *there exists a unique solution to* (3.11). *If $\widehat{u}$ denotes the solution to* (3.11) *for $\widehat{f}_\#$ and $\widehat{\boldsymbol{f}}$, we have*

$$\|u-\widehat{u}\|_{H^s_0(\Omega)}\leq\tfrac{C_*}{\delta}\|f_\#-\widehat{f}_\#\|_{L^{2\#}(\Omega)}+\tfrac{1}{\delta}\|\boldsymbol{f}-\widehat{\boldsymbol{f}}\|_{\boldsymbol{L}^2(\mathbb{R}^d)}. \tag{3.18}$$

Proof. Using Hölder and Sobolev inequalities, we have that, for $v \in H^s_0(\Omega)$,

$$\Big|\int_\Omega (\boldsymbol{b}+\boldsymbol{d}) v D^s v\Big| \leq \|\boldsymbol{b}+\boldsymbol{d}\|_{\boldsymbol{L}^{\frac{d}{s}}(\Omega)}\|v\|_{L^{2^*}(\Omega)}\|D^s v\|_{\boldsymbol{L}^2(\mathbb{R}^d)} \leq C_*\|\boldsymbol{b}+\boldsymbol{d}\|_{\boldsymbol{L}^{\frac{d}{s}}(\Omega)}\|D^s v\|^2_{\boldsymbol{L}^2(\mathbb{R}^d)},$$

and

$$-\int_\Omega cv^2 \leq \int_\Omega c^- v^2 \leq C_*^2\|c^-\|_{L^{\frac{d}{2s}}(\Omega)}\|D^s v\|^2_{\boldsymbol{L}^2(\mathbb{R}^d)}.$$

Therefore, using (3.16) and (3.17), we obtain

$$\begin{aligned}\mathscr{L}^s(v,v) &= \int_{\mathbb{R}^d} AD^s v\cdot D^s v + \int_\Omega (\boldsymbol{b}+\boldsymbol{d})v\cdot D^s v + \int_\Omega cv^2 \\ &\geq a_*\int_{\mathbb{R}^d}|D^s v|^2 - \Big(C_*\|\boldsymbol{b}+\boldsymbol{d}\|_{\boldsymbol{L}^{\frac{d}{s}}(\Omega)} + C_*^2\|c^-\|_{L^{\frac{d}{2s}}(\Omega)}\Big)\|D^s v\|^2_{\boldsymbol{L}^2(\mathbb{R}^d)} \\ &= \delta\|D^s v\|^2_{\boldsymbol{L}^2(\mathbb{R}^d)} = \delta\|v\|^2_{H^s_0(\Omega)}.\end{aligned} \tag{3.19}$$

As $L^{2^*}(\Omega) = L^{2^\#}(\Omega)'$, (3.9) implies that $F \in H^{-s}(\Omega)$, with $\|F\|_{H^{-s}(\Omega)} \leq C_*\|f_\#\|_{L^{2^\#}(\Omega)} + \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\mathbb{R}^d)}$ and Stampacchia's theorem immediately yields the existence and uniqueness of the solution to (3.11).

Taking $v = \widehat{u}$ in (3.11) for u and $v = u$ in (3.11) for $\widehat{u}$, and using (3.19), we obtain

$$\delta\|u-\widehat{u}\|^2_{H^s_0(\Omega)} \leq \mathscr{L}^s(u-\widehat{u}, u-\widehat{u}) \leq [F-\widehat{F}, u-\widehat{u}]_s \leq \|F-\widehat{F}\|_{H^{-s}(\Omega)}\|u-\widehat{u}\|_{H^s_0(\Omega)}$$

and (3.18) easily follows. □

Remark 3.6. We observe that the assumption (3.6) is slightly stronger than the integrability conditions in Theorem 3.5 for the case $2s < d$, including $s = 1$. However, for $s \geq \frac{d}{2}$, we may have the assumption (3.6) with any $r > 1$, when $s = 1$, $d = 2$, and even $\boldsymbol{b}$, $\boldsymbol{d}$, $c \in L^1$ when $s \geq \frac{1}{2}$, $d = 1$, with the respective norms in the assumption (3.17). Note that Theorem 3.5 extends Theorem 2.1 of [3], in which the coefficients $\boldsymbol{b}$, $\boldsymbol{d}$ and c are zero.

The coercivity assumption (3.17) in the case of a bounded threshold of the s-gradient, under the stronger assumptions

$$0 < g_* \leq g(x) \leq g^* \quad \text{for a.e. } x \in \mathbb{R}^d, \tag{3.20}$$

also yields strong continuous dependence of the solutions of (3.11) with respect to the variation of the coefficients of $\mathscr{L}^s$, of the data and of the threshold g. In fact, the assumption (3.20) can be weakened as follows

$(3.20)_{loc}$ $\quad g \in L^\infty_{loc}(\mathbb{R}^d)$ with positive lower bound in any compact and $\displaystyle\lim_{|x|\to\infty} g(x)|x|^{d+s} = \infty$,

as it will be shown in Proposition 3.8.

Theorem 3.7. *Let u_i denote the solution of* (3.11) *corresponding to the data A_i, $\boldsymbol{b}_i$, $\boldsymbol{d}_i$, c_i, $f_{\#i}$, $\boldsymbol{f}_i$, g_i, for $i = 1, 2$, satisfying* (3.16), (3.17), (3.7), (3.10) *and* (3.20). *Then the following* *estimate holds with* $p > \frac{d}{s}$ *and* $0 < \gamma = s - \frac{d}{p} < s \leq 1$,

$$\begin{aligned} \|u_1 - u_2\|^p_{C^{0,\gamma}(\overline{\Omega})} + \|u_1 - u_2\|^2_{H^s_0(\Omega)} &\leq C_1 \|g_1 - g_2\|_{L^\infty(\mathbb{R}^d)} \\ &+ C_1' \Big(\|A_1 - A_2\|_{L^{p_1}(\mathbb{R}^d)^{d^2}} + \|\boldsymbol{b}_1 - \boldsymbol{b}_2\|_{\boldsymbol{L}^1(\Omega)} + \|\boldsymbol{d}_1 - \boldsymbol{d}_2\|_{\boldsymbol{L}^1(\Omega)} \\ &+ \|c_1 - c_2\|_{L^1(\Omega)} + \|f_{\#1} - f_{\#2}\|_{L^1(\Omega)} + \|\boldsymbol{f}_1 - \boldsymbol{f}_2\|_{\boldsymbol{L}^{q_1}(\mathbb{R}^d)} \Big), \end{aligned} \tag{3.21}$$

where the positive constants C_1 *and* C_1' *depend on* δ, g_*, g^*, d, s, Ω *and linearly on the* L^1*-norms of* A_i, $\boldsymbol{b}_i$, $\boldsymbol{d}_i$, c_i, $f_{\#i}$, $\boldsymbol{f}_i$.

Proof. Since $u_i \in \mathbb{K}^s_{g_i} \subset \Lambda^{s,\infty}_0(\Omega)$ and g_i satisfies (3.20) for each $i = 1, 2$, we have

$$\tfrac{s}{C_0} \|u_i\|_{L^\infty(\Omega)} \leq \|D^s u_i\|_{\boldsymbol{L}^\infty(\mathbb{R}^d)} \leq g^*,$$

where $C_0 > 0$ is the Poincaré constant in (2.10).

Set $\eta = \|g_1 - g_2\|_{L^\infty(\mathbb{R}^d)}$ and $\mu = \frac{g_*}{g_* + \eta}$. Observe that $u_{i_j} = \mu u_j \in \mathbb{K}^s_{g_i}$ $(i \neq j,\ i, j = 1, 2)$ and so it can be used as test function in (3.11) for $\mathscr{L}^s_i$ and $f_{\#i}$, $\boldsymbol{f}_i$. For $i = 1, 2$, we obtain

$$\mathscr{L}^s_i(u_i, u_{i_j} - u_i) \geq [F_i, u_{i_j} - u_i]_s,$$

or equivalently,

$$\mathscr{L}^s_i(u_i, u_i - u_j) \leq [F_i, u_i - u_j]_s + \mathscr{L}^s_i(u_i, u_{i_j} - u_i) + [F_i, u_j - u_{i_j}]_s. \tag{3.22}$$

Since $u_{i_j} - u_j = (\mu - 1) u_j$ and $0 \leq 1 - \mu \leq \frac{\eta}{g_*}$, setting $M = \max\{g^*, \frac{C_0}{s} g^*\}$, we may estimate the middle term of (3.22) by

$$\begin{aligned} \Big| \mathscr{L}^s_i(u_i, u_{i_j} - u_j) \Big| &= \Big| (\mu - 1) \mathscr{L}^s_i(u_i, u_j) \Big| \leq \tfrac{\eta}{g_*} \Big| \mathscr{L}^s_i(u_i, u_j) \Big| \\ &\leq \tfrac{\eta}{g_*} M^2 \Big(\|A_i\|_{L^1(\mathbb{R}^d)^{d^2}} + \|\boldsymbol{b}_i\|_{\boldsymbol{L}^1(\Omega)} + \|\boldsymbol{d}_i\|_{\boldsymbol{L}^1(\Omega)} + \|c_i\|_{L^1(\Omega)} \Big) = \eta \kappa_i \end{aligned}$$

and the last one by

$$\big| [F_i, u_j - u_{i_j}]_s \big| = (1 - \mu) \big| [F_i, u_j]_s \big| \leq \tfrac{\eta}{g_*} M \big(\|f_{\#i}\|_{L^1(\Omega)} + \|\boldsymbol{f}_i\|_{\boldsymbol{L}^1(\mathbb{R}^d)} \big) = \eta \nu_i.$$

Setting $w = u_1 - u_2$ and

$$\begin{aligned} E_{21} = \int_{\mathbb{R}^d} (A_2 - A_1) D^s u_2 \cdot D^s w + \int_\Omega (\boldsymbol{d}_2 - \boldsymbol{d}_1) u_2 \cdot D^s w \\ + \int_\Omega \big((\boldsymbol{b}_2 - \boldsymbol{b}_1) \cdot D^s u_2 + (c_2 - c_1) u_2 \big) w, \end{aligned}$$

by using (3.22) for $i = 2$, we have

$$-\mathscr{L}^s_1(u_2, w) = \mathscr{L}^s_2(u_2, -w) + E_{21} \leq [F_2, -w]_s + \eta(\kappa_2 + \nu_2) + e_{21}, \tag{3.23}$$

where

$$\begin{aligned} \big| E_{21} \big| \leq 2M^2 \Big(\|A_1 - A_2\|_{L^1(\mathbb{R}^d)^{d^2}} + \|\boldsymbol{d}_1 - \boldsymbol{d}_2\|_{\boldsymbol{L}^1(\Omega)} \\ + \|\boldsymbol{b}_1 - \boldsymbol{b}_2\|_{\boldsymbol{L}^1(\Omega)} + \|c_1 - c_2\|_{L^1(\Omega)} \Big) \equiv e_{21}. \end{aligned}$$

Summing (3.22) for $i = 1$ with (3.23) and using the coercivity (3.19), we obtain

$$\begin{aligned}\delta\|w\|^2_{H^s_0(\Omega)} &\leq \mathscr{L}^s_1(w,w) = \mathscr{L}^s_1(u_1,w) - \mathscr{L}^s_1(u_2,w)\\ &\leq [F_1,w]_s + \eta(\kappa_1+\nu_1) + [F_2,-w]_s + \eta(\kappa_2+\nu_2) + e_{21}\\ &\leq \eta(\kappa_1+\nu_1+\kappa_2+\nu_2) + e_{21} + \varphi_{12},\end{aligned}$$

where $\big|[F_1 - F_2, w]_s\big| \leq \varphi_{12}$, with $\varphi_{12} = 2M\big(\|f_{\#1} - f_{\#2}\|_{L^1(\Omega)} + \|\boldsymbol{f}_1 - \boldsymbol{f}_2\|_{\boldsymbol{L}^1(\mathbb{R}^d)}\big)$.

To conclude (3.21) it suffices to use the continuous embedding (2.16), which guarantees the existence of a constant $C_\beta > 0$ and $0 < \beta = s - \frac{d}{p} < s \leq 1$, with $p > \max\{2, \frac{d}{s}\}$ in

$$\big(\tfrac{1}{C_\beta}\big)^p\|w\|^p_{C^{0,\beta}(\overline{\Omega})} \leq \int_{\mathbb{R}^d}|D^s w|^p \leq \|D^s w\|^{p-2}_{\boldsymbol{L}^\infty(\mathbb{R}^d)}\int_{\mathbb{R}^d}|D^s w|^2 \leq \big(g^*\big)^{p-2}\|w\|^2_{H^s_0(\Omega)}.$$

□

The following proposition shows that we can replace the assumption (3.20) by $(3.20)_{loc}$ in the above theorem.

Proposition 3.8. *Let $g \in L^\infty_{loc}(\mathbb{R}^d)$ be positively lower bounded in any compact set and such that*

$$\lim_{|x|\to+\infty} g(x)|x|^{d+s} = \infty. \tag{3.24}$$

Then there exists $h \in L^\infty(\mathbb{R}^d)$, with positive lower bound in $\mathbb{R}^d$ and such that $\mathbb{K}^s_h = \mathbb{K}^s_g$. More precisely, we can choose $h = g\,\chi_{\Omega_R} + k\chi_{\mathbb{R}^d\setminus\Omega_R}$ for a certain $R > 0$ and $k \geq \|g\|_{L^\infty(\Omega_R)}$.

Proof. Using remarks 2.5, 2.9 and the inequality (3.12) for $p = 1$, there exists $R_0 = R_0(s) > 0$, independent of g, such that, for $R \geq R_0$, and $u \in \mathbb{K}^s_g$, we have

$$|D^s u(x)| \leq \tfrac{C_0}{s}\frac{2\mu_s|\Omega_R|}{d(x,\Omega)^{d+s}}\|g\|_{L^\infty(\Omega_R)}, \qquad \forall x \notin \Omega_R.$$

Using (3.24), fix $R \geq R_0$ such that

$$g(x)d(x,\Omega)^{d+s} \geq \tfrac{C_0}{s}2\mu_s|\Omega_{R_0}|\|g\|_{L^\infty(\Omega_{R_0})}, \qquad \forall x \notin \Omega_R \tag{3.25}$$

and

$$\tfrac{C_0}{s}\frac{2\mu_s|\Omega_R|}{d(x,\Omega)^{d+s}} \leq 1.$$

Let $k \geq \|g\|_{L^\infty(\Omega_R)}$ and consider $h : \mathbb{R}^d \longrightarrow \mathbb{R}$

$$h(x) = \begin{cases} g(x) & \text{if } x \in \Omega_R,\\ k & \text{otherwise.}\end{cases} \tag{3.26}$$

Then h satisfies assumption (3.20) and $\mathbb{K}^s_g = \mathbb{K}^s_h$. Indeed, if $u \in \mathbb{K}^s_g$ then, for $x \notin \Omega_R$,

$$|D^s u(x)| \leq \tfrac{C_0}{s}\frac{2\mu_s|\Omega_R|}{d(x,\Omega)^{d+s}}\|g\|_{L^\infty(\Omega_R)} \leq \tfrac{C_0}{s}\frac{2\mu_s|\Omega_R|}{R^{d+s}}\|g\|_{L^\infty(\Omega_R)} \leq k.$$

Reciprocally, if $u \in \mathbb{K}_h$ then, for $x \notin \Omega_R$, we have $x \notin \Omega_{R_0}$ and then, as $\|h\|_{L^\infty(\Omega_{R_0})} = \|g\|_{L^\infty(\Omega_{R_0})}$,

$$|D^s u(x)| \le \tfrac{C_0}{s}\frac{2\mu_s|\Omega_{R_0}|}{d(x,\Omega)^{d+s}}\|g\|_{L^\infty(\Omega_{R_0})}$$

and then, using (3.25), $|D^s u(x)| \le g(x)$. □

Remark 3.9. Assuming only $(3.20)_{loc}$, Theorem 3.7 remains valid by taking $R > 0$ sufficiently large and replacing $\|g_1 - g_2\|_{L^\infty(\mathbb{R}^d)}$ by $\|g_1 - g_2\|_{L^\infty(\Omega_R)}$ in (3.21).

This is true because in the proof of the last proposition for g_1 and g_2, we can define h_1 and h_2 as in (3.26) with the same R and h_1 equal to h_2 outside Ω_R.

Remark 3.10. If we assume (3.5), $\boldsymbol{b} = \boldsymbol{d} = 0$ and $c \ge c_* > 0$ instead (3.17), keeping the other assumptions in Theorem 3.7, we still have a weaker continuous dependence result, replacing $\|u_1-u_2\|^p_{C^{0,\gamma}(\overline{\Omega})}+\|u_1-u_2\|^2_{H^s_0(\Omega)}$ by $\|u_1-u_2\|^2_{L^2(\Omega)}$ in (3.21). In particular, with these assumptions we also have uniqueness of solution for the variational inequality.

4. Transport potentials and densities

In this section we consider the Lagrange multiplier problem for $0 < s \le 1$, associated with bounded s-gradient constraints: find the generalised transport potential-density pair $(u, \lambda) \in \Lambda^{s,\infty}_0(\Omega) \times L^\infty(\mathbb{R}^d)'$, such that

$$\mathscr{L}^s(u, v) + \langle\!\langle \lambda D^s u, D^s v\rangle\!\rangle = [F, v]_s, \quad \forall v \in \Lambda^{s,\infty}_0(\Omega) \tag{4.1a}$$

$$|D^s u| \le g \text{ a.e. in } \mathbb{R}^d, \quad \lambda \ge 0 \quad \text{and} \quad \lambda(|D^s u| - g) = 0 \quad \text{in } L^\infty(\mathbb{R}^d)'. \tag{4.1b}$$

In the case $s = 1$ the solution (u, λ) is to be found in $W^{1,\infty}_0(\Omega)\times L^\infty(\Omega)'$ and the test functions v in $W^{1,\infty}_0(\Omega)$, since $D^1 = D$ is the classical gradient.

Here $\langle\!\langle\cdot\,,\cdot\rangle\!\rangle$ denotes the duality pairing between $\boldsymbol{L}^\infty(\mathbb{R}^d)'$ and $\boldsymbol{L}^\infty(\mathbb{R}^d)$ and we set, for $\boldsymbol{\varphi} \in \boldsymbol{L}^\infty(\mathbb{R}^d)$,

$$\langle\!\langle \lambda\boldsymbol{\varphi}, \boldsymbol{\xi}\rangle\!\rangle = \langle \lambda, \boldsymbol{\varphi}\cdot\boldsymbol{\xi}\rangle, \quad \forall \boldsymbol{\xi} \in \boldsymbol{L}^\infty(\mathbb{R}^d),$$

where $\langle\cdot\,,\cdot\rangle$ is the duality pairing between $L^\infty(\mathbb{R}^d)'$ and $L^\infty(\mathbb{R}^d)$.

Theorem 4.1. *Assume g satisfies* (3.20)*, $\mathscr{L}^s$ is given by* (3.4) *with the assumptions* (3.5)*,* (3.7) *and $F \in \Lambda^{s,\infty}_0(\Omega)'$ is given by* (3.8) *if $0 < s < 1$ (respectively $F \in W^{1,\infty}_0(\Omega)'$, if $s = 1$) with the assumption* (3.10)*. Then problem* (4.1) *has a solution $(u, \lambda) \in \Lambda^{s,\infty}_0(\Omega) \times L^\infty(\mathbb{R}^d)'$, for any $0 < s < 1$ (respectively in $W^{1,\infty}_0(\Omega) \times L^\infty(\Omega)'$ if $s = 1$), and u solves the variational inequality* (3.11)*.*

The proof of this existence theorem is obtained by a suitable penalisation of the s-gradient constraint, combined with an elliptic nonlinear regularisation, and by a weak stability property of the generalised formulation (4.1) given by the following theorem.

Consider for $0 < \nu < 1$ the family of solutions $(u_\nu, \lambda_\nu) \in \Lambda_0^{s,\infty}(\Omega) \times L^\infty(\mathbb{R}^d)'$, if $0 < s < 1$ (respectively, $W_0^{1,\infty}(\Omega) \times L^\infty(\Omega)'$ if $s = 1$)

$$(4.1\text{a})_\nu \qquad \mathscr{L}_\nu^s(u_\nu, v) + \langle\!\langle \lambda_\nu D^s u_\nu, D^s v \rangle\!\rangle = [F_\nu, v]_s, \quad \forall v \in \Lambda_0^{s,\infty}(\Omega)$$

$$(4.1\text{b})_\nu \qquad |D^s u_\nu| \leq g_\nu \text{ a.e. in } \mathbb{R}^d, \quad \lambda_\nu \geq 0 \text{ and } \lambda_\nu(|D^s u_\nu| - g_\nu) = 0 \text{ in } L^\infty(\mathbb{R}^d)',$$

where $\mathscr{L}_\nu^s$ and F_ν are defined by the (3.4) and (3.6) with data A_ν, $\boldsymbol{b}_\nu$, $\boldsymbol{d}_\nu$, c_ν and $f_{\#,\nu}$, $\boldsymbol{f}_\nu$, respectively.

Theorem 4.2. *Suppose the functions A_ν, $\boldsymbol{b}_\nu$, $\boldsymbol{d}_\nu$, c_ν, $f_{\#,\nu}$, $\boldsymbol{f}_\nu$ and g_ν, for each ν, $0 < \nu < 1$, satisfy* (3.3) (3.5), (3.7), (3.10) *and* (3.20) *and have limit functions as $\nu \longrightarrow 0$:*

$$(4.2\text{a}) \qquad A_\nu \longrightarrow A \text{ in } L^{p_1}(\mathbb{R}^d)^{d^2}, \quad c_\nu \longrightarrow c \text{ in } L^1(\Omega),$$

$$(4.2\text{b}) \qquad \boldsymbol{b}_\nu \longrightarrow \boldsymbol{b} \text{ in } \boldsymbol{L}^1(\Omega), \quad \boldsymbol{d}_\nu \longrightarrow \boldsymbol{d} \text{ in } \boldsymbol{L}^1(\Omega),$$

$$(4.2\text{c}) \qquad f_{\#\nu} \longrightarrow f_\# \text{ in } L^1(\Omega), \quad \boldsymbol{f}_\nu \longrightarrow \boldsymbol{f} \text{ in } \boldsymbol{L}^{q_1}(\mathbb{R}^d),$$

$$(4.2\text{d}) \qquad g_\nu \longrightarrow g \text{ in } L^\infty(\mathbb{R}^d).$$

Then, if (u_ν, λ_ν) solves $(4.1\text{a})_\nu(4.1\text{b})_\nu$*, there is a generalised sequence, still denoted by $\nu \longrightarrow 0$, and a solution (u, λ) to* (4.1a)(4.1b) *such that*

$$(4.3\text{a}) \qquad u_\nu \longrightarrow u \text{ in } C^{0,\alpha}(\overline{\Omega})\text{-strong}$$

$$(4.3\text{b}) \qquad D^s u_\nu \rightharpoonup D^s u \text{ in } L^\infty(\Omega)\text{-weak}^*,$$

$$(4.3\text{c}) \qquad \lambda_\nu \rightharpoonup \lambda \text{ in } L^\infty(\mathbb{R}^d)'\text{-weak}^*,$$

where $0 < \alpha < s \leq 1$, with the convention $\Lambda_0^{s,\infty}(\Omega) = W^{1,\infty}(\Omega)$ in the case $s = 1$.

Proof. Since $|D^s u_\nu| \leq g_\nu \leq g^*$ a.e. in $\mathbb{R}^d$, for all $0 < \nu < 1$, by recalling (2.17) and (2.16), respectively, we have the *a priory* estimates

$$(4.4) \qquad \|D^s u_\nu\|_{\boldsymbol{L}^\infty(\mathbb{R}^d)} \leq g^*, \quad \text{and} \quad \|u_\nu\|_{C^{0,\beta}(\overline{\Omega})} \leq C_\beta, \quad \text{for all } 0 < \beta < s \leq 1.$$

Taking $v = u_\nu$ in $(4.1\text{a})_\nu$ we get

$$(4.5) \qquad \begin{aligned} \langle \lambda_\nu, |D^s u_\nu|^2 \rangle = \langle\!\langle \lambda_\nu D^s u_\nu, D^s u_\nu \rangle\!\rangle &= [F_\nu, u_\nu]_s - \mathscr{L}_\nu^s(u_\nu, u_\nu) \\ &\leq C_1, \quad \text{for all } \sigma < s \leq 1, \end{aligned}$$

where $C_1 > 0$ is a constant dependent on g^*, C_β and a common bound of the L^{p_1}, L^1, and L^{q_1} norms of $(A_\nu)_\nu$, $(\boldsymbol{b}_\nu)_\nu$, $(\boldsymbol{d}_\nu)_\nu$, $(c_\nu)_\nu$, $(f_{\#\nu})_\nu$ and $(\boldsymbol{f}_\nu)_\nu$, respectively, independent of ν.

Observing that, as (u_ν, λ_ν) solves problem $(4.1\text{a})_\nu(4.1\text{b})_\nu$, we have $\lambda_\nu(|D^s u_\nu| - g_\nu) = 0$ in $L^\infty(\mathbb{R}^d)'$, which, multiplying by $|D^s u_\nu| + g_\nu$, implies

$$(4.6) \qquad \langle \lambda_\nu, |D^s u_\nu|^2 \rangle = \langle \lambda_\nu, g_\nu^2 \rangle.$$

Using the assumption $g_\nu \geq g_*$ and $\lambda_\nu \geq 0$ we have

$$\begin{aligned}\|\lambda_\nu\|_{L^\infty(\mathbb{R}^d)'} &= \sup_{\|\zeta\|_{L^\infty(\mathbb{R}^d)}\leq 1} |\langle\lambda_\nu,\zeta\rangle| \leq \sup_{\|\zeta\|_{L^\infty(\mathbb{R}^d)}\leq 1}\langle\lambda_\nu,|\zeta|\rangle \leq \langle\lambda_\nu,1\rangle \leq \langle\lambda_\nu,\tfrac{g_\nu^2}{g_*^2}\rangle \\ &= \tfrac{1}{g_*^2}\langle\lambda_\nu,|D^s u_\nu|^2\rangle \leq \tfrac{C_1}{g_*^2}, \qquad \text{by (4.6) and (4.5).}\end{aligned} \tag{4.7}$$

As a consequence, letting $\boldsymbol{\Psi}_\nu = \lambda_\nu D^s u_\nu$, we also have

$$\|\boldsymbol{\Psi}_\nu\|_{\boldsymbol{L}^\infty(\mathbb{R}^d)'} = \sup_{\|\boldsymbol{\xi}\|_{\boldsymbol{L}^\infty(\mathbb{R}^d)}\leq 1} |\langle\lambda_\nu, D^s u_\nu\cdot\boldsymbol{\xi}\rangle| \leq \|\lambda_\nu\|_{L^\infty(\mathbb{R}^d)'}\|D^s u_\nu\|_{\boldsymbol{L}^\infty(\mathbb{R}^d)} \leq \tfrac{C_1 g^*}{g_*^2}. \tag{4.8}$$

Then, by the above estimates, we may choose some generalised sequence $\nu \longrightarrow 0$, such that: *i)* (4.3a) holds with $\alpha < \beta$ for some $u \in C^{0,\alpha}(\overline{\Omega})$, by estimate (4.4), and (4.3b) follows using (2.6); *ii)* (4.3c) holds for some $\lambda \in L^\infty(\mathbb{R}^d)'$, by estimate (4.7) and Banach-Alaoglu-Bourbaki theorem; as well as, by (4.8), *iii)* there exists also a $\boldsymbol{\Psi} \in \boldsymbol{L}^\infty(\mathbb{R}^d)'$ with

$$\boldsymbol{\Psi}_\nu \underset{\nu}{\rightharpoonup} \boldsymbol{\Psi} \quad \text{in } \boldsymbol{L}^\infty(\mathbb{R}^d)'. \tag{4.9}$$

Letting $\nu \longrightarrow 0$ in $(4.1\text{a})_\nu$, by the assumptions (4.2a), (4.2b), (4.2c) and the convergences (4.3a) and (4.3b), we conclude that $(u,\boldsymbol{\Psi})$ solves:

$$\mathscr{L}^s(u,v) + \langle\!\langle \boldsymbol{\Psi}, D^s v\rangle\!\rangle = [F,v]_s, \qquad \forall v \in \Lambda^{s,\infty}_0(\Omega).$$

Note that $\lambda_\nu \geq 0$ implies $\lambda \geq 0$. Given any measurable set $\omega \subset \mathbb{R}^d$ with finite measure, taking $\boldsymbol{\xi} \in \boldsymbol{L}^1(\mathbb{R}^d)$, defined by $\boldsymbol{\xi} = \frac{D^s u}{|D^s u|}$ if $x \in \omega \cap \{|D^s u| \neq 0\}$ and $\boldsymbol{\xi} = 0$ elsewhere, since $D^s u_\nu \underset{\nu}{\rightharpoonup} D^s u$ in $\boldsymbol{L}^\infty(\mathbb{R}^d)$-weak*, we have

$$\int_\omega g_\nu \geq \int_\omega |D^s u_\nu| \geq \int_\omega D^s u_\nu \cdot \boldsymbol{\xi} \underset{\nu}{\longrightarrow} \int_{\mathbb{R}^d} D^s u\cdot\boldsymbol{\xi} = \int_\omega |D^s u|, \tag{4.10}$$

and so $|D^s u| \leq g$ a.e. in $\mathbb{R}^d$, by (4.2d) and the arbitrariness of $\omega \subset \mathbb{R}^d$. Then, in order to complete the proof, it remains to show that

$$\boldsymbol{\Psi} = \lambda D^s u \qquad \text{in } \boldsymbol{L}^\infty(\mathbb{R}^d)' \tag{4.11}$$

and

$$\lambda|D^s u| = \lambda g \qquad \text{in } L^\infty(\mathbb{R}^d)', \tag{4.12}$$

or equivalently, using the assumption (3.20),

$$\langle\lambda(g-|D^s u|),\varphi\rangle = \langle\lambda,(g^2-|D^s u|^2)\tfrac{\varphi}{g+|D^s u|}\rangle = 0, \quad \forall\varphi\in L^\infty(\mathbb{R}^d).$$

Then we observe that, recalling $|D^s u| \leq g$ and using (4.7),

$$\begin{aligned}\tfrac12\langle\lambda_\nu,|D^s(u_\nu-u)|^2\rangle &= \tfrac12\Big(\langle\lambda_\nu,|D^s u_\nu|^2\rangle - 2\langle\lambda_\nu, D^s u_\nu\cdot D^s u\rangle + \langle\lambda_\nu,|D^s u|^2\rangle\Big) \\ &\leq \langle\lambda_\nu,|D^s u_\nu|^2\rangle - \langle\lambda_\nu, D^s u_\nu\cdot D^s u\rangle + \tfrac12\langle\lambda_\nu, g^2-g_\nu^2\rangle \\ &\leq \langle\lambda_\nu D^s u_\nu, D^s(u_\nu-u)\rangle + \tfrac12\|\lambda_\nu\|_{L^\infty(\mathbb{R}^d)'}\|g^2-g_\nu^2\|_{L^\infty(\mathbb{R}^d)} \\ &\leq [F_\nu,u_\nu-u]_s - \mathscr{L}^s_\nu(u_\nu,u_\nu-u) + \tfrac12\tfrac{C_1}{g_*^2}\|g^2-g_\nu^2\|_{L^\infty(\mathbb{R}^d)}.\end{aligned} \tag{4.13}$$

We have $[F_\nu, u_\nu - u]_s \underset{\nu}{\longrightarrow} 0$ and $\mathscr{L}^s_\nu(u_\nu, u) \underset{\nu}{\longrightarrow} \mathscr{L}^s(u,u)$, while, on the other hand,

$$\underline{\lim}_{\nu} \mathscr{L}^s_\nu(u_\nu, u_\nu) \geq \mathscr{L}^s(u,u) \tag{4.14}$$

since, arguing as in (3.15) we have that, using (4.2a),

$$\begin{aligned}\underline{\lim}_{\nu} \int_{\mathbb{R}^d} A_\nu D^s u_\nu \cdot D^s u_\nu &\geq \underline{\lim}_{\nu} \int_{\mathbb{R}^d} A D^s u_\nu \cdot D^s u_\nu + \lim_{\nu} \int_{\mathbb{R}^d} (A_\nu - A) D^s u_\nu \cdot D^s u_\nu \\ &\geq \int_{\mathbb{R}^d} A D^s u \cdot D^s u\end{aligned}$$

as

$$\Big| \int_{\mathbb{R}^d} (A_\nu - A) D^s u_\nu \cdot D^s u_\nu \Big| \leq (g^*)^2 \|A_\nu - A\|_{L^1(\mathbb{R}^d)^{d^2}} \underset{\nu}{\longrightarrow} 0$$

and, using the convergences (4.3a) with (4.2a) and (4.2b),

$$\int_\Omega \boldsymbol{b}_\nu \cdot D^s u_\nu (u_\nu - u) + \boldsymbol{d}_\nu u_\nu \cdot D^s(u_\nu - u) + c_\nu u_\nu (u_\nu - u) \underset{\nu}{\longrightarrow} 0.$$

Hence we conclude that

$$0 \leq \lim_{\nu} \langle \lambda_\nu, |D^s(u_\nu - u)|^2 \rangle \leq \overline{\lim_{\nu}} \langle \lambda_\nu, |D^s(u_\nu - u)|^2 \rangle \leq 0.$$

By the Hölder inequality (see Proposition 2.14),

$$\begin{aligned}|\langle \lambda_\nu, D^s(u_\nu - u) \cdot \boldsymbol{\xi} \rangle| &\leq \langle \lambda_\nu, |D^s(u_\nu - u)||\boldsymbol{\xi}| \rangle \leq \langle \lambda_\nu, |D^s(u_\nu - u)|^2 \rangle^{\frac{1}{2}} \langle \lambda_\nu, |\boldsymbol{\xi}|^2 \rangle^{\frac{1}{2}} \\ &\leq \langle \lambda_\nu, |D^s(u_\nu - u)|^2 \rangle^{\frac{1}{2}} \|\lambda_\nu\|^{\frac{1}{2}}_{L^\infty(\Omega)'} \|\boldsymbol{\xi}\|_{\boldsymbol{L}^\infty(\mathbb{R}^d)}\end{aligned}$$

which by (4.7) yields

$$\lim_{\nu} \langle \lambda_\nu, D^s(u_\nu - u) \cdot \boldsymbol{\xi} \rangle = 0, \qquad \forall \boldsymbol{\xi} \in \boldsymbol{L}^\infty(\mathbb{R}^d). \tag{4.15}$$

Now, recalling (4.9), (4.11) follows now from (4.15), since

$$\begin{aligned}\langle\!\langle \boldsymbol{\Psi}, \boldsymbol{\xi} \rangle\!\rangle &= \lim_{\nu} \langle\!\langle \boldsymbol{\Psi}_\nu, \boldsymbol{\xi} \rangle\!\rangle = \lim_{\nu} \langle \lambda_\nu, D^s u_\nu \cdot \boldsymbol{\xi} \rangle = \lim_{\nu} \langle \lambda_\nu, D^s u \cdot \boldsymbol{\xi} \rangle \\ &= \langle \lambda, D^s u \cdot \boldsymbol{\xi} \rangle = \langle\!\langle \lambda D^s u, \boldsymbol{\xi} \rangle\!\rangle \qquad \forall \boldsymbol{\xi} \in \boldsymbol{L}^\infty(\mathbb{R}^d).\end{aligned} \tag{4.16}$$

Using (4.15) and (4.16) with $\boldsymbol{\xi} = D^s u_\nu$ we obtain $\lim_\nu \langle\!\langle \lambda_\nu D^s u_\nu, D^s(u_\nu - u) \rangle\!\rangle = 0$ and

$$\langle \lambda, g^2 \rangle = \lim_{\nu} \langle \lambda_\nu, g_\nu^2 \rangle = \lim_{\nu} \langle\!\langle \lambda_\nu D^s u_\nu, D^s u_\nu \rangle\!\rangle = \lim_{\nu} \langle\!\langle \lambda_\nu D^s u_\nu, D^s u \rangle\!\rangle = \langle \lambda, |D^s u|^2 \rangle, \tag{4.17}$$

which implies $\langle \lambda(g^2 - |D^s u|^2), 1 \rangle = 0$.

Finally, (4.12) follows again by the Hölder inequality for charges, with an arbitrary $\varphi \in L^\infty(\mathbb{R}^d)$,

$$\begin{aligned}|\langle \lambda(g - |D^s u|), \varphi \rangle| &= \big| \langle \lambda(g^2 - |D^s u|^2), \tfrac{\varphi}{g + |D^s u|} \rangle \big| \\ &\leq \langle \lambda(g^2 - |D^s u|^2), 1 \rangle^{\frac{1}{2}} \langle \lambda(g^2 - |D^s u|^2), \tfrac{|\varphi|^2}{(g + |D^s u|)^2} \rangle^{\frac{1}{2}} = 0.\end{aligned} \tag{4.18}$$

□

Proof. (of Theorem 4.1) It can be obtained with the following approximating problem. Let $0<\varepsilon<1$ and fix $q>1+\frac{d}{s}>2$, so that $\Lambda^{s,r}_0(\Omega)\subset C(\overline{\Omega})$, for $0<s\le 1$ and $r=q-1>\frac{d}{s}$, recalling the convention $\Lambda^{s,r}_0(\Omega)=W^{1,r}_0(\Omega)$ if $s=1$. We firstly consider $\mathscr{L}^s$ and F defined by (3.4) and (3.8) under the assumption (3.5) and, letting $r'=\frac{r}{r-1}$,

$$A\in L^\infty(\mathbb{R}^d)^{d^2},\quad \boldsymbol{b},\boldsymbol{d}\in \boldsymbol{L}^{r'}(\Omega),\quad \boldsymbol{f}\in \boldsymbol{L}^{q'}(\mathbb{R}^d),\quad c,\ f_\#\in L^1(\Omega), \tag{4.19}$$

with which we shall prove the existence of a solution $(u,\lambda)\in\Lambda^{s,\infty}_0(\Omega)\times L^\infty(\mathbb{R}^d)'$, $0<s\le 1$ of (4.1).

The approximating problem is given, for $\varepsilon>0$, by

$$u_\varepsilon\in\Lambda^{s,q}_0(\Omega):\quad \mathscr{L}^s(u_\varepsilon,v)+[\widetilde{k}_\varepsilon(u_\varepsilon)+\varepsilon D^s_q u_\varepsilon,v]=[F,v],\qquad \forall v\in\Lambda^{s,q}_0(\Omega), \tag{4.20}$$

where $\varepsilon>0$ and the penalisation operator

$$[\widetilde{k}_\varepsilon(w),v]=\int_{\mathbb{R}^d}k_\varepsilon(|D^sw|-g)D^sw\cdot D^sv,\qquad \forall v,w\in\Lambda^{s,q}_0(\Omega),$$

is defined with the continuous monotone function

$$k_\varepsilon(t)=0\ \text{ for } t\le 0,\quad k_\varepsilon(t)=e^{\frac{t}{\varepsilon}}-1\ \text{ for } 0<t\le\tfrac{1}{\varepsilon}\quad\text{ and }\quad k_\varepsilon(t)=e^{\frac{1}{\varepsilon^2}}-1\ \text{ for } t\ge\tfrac{1}{\varepsilon}$$

and the nonlinear elliptic regularisation is given by

$$[\varepsilon D^s_q w,v]=\varepsilon\int_{\mathbb{R}^d}|D^sw|^{q-2}D^sw\cdot D^sv,\qquad \forall v,w\in\Lambda^{s,q}_0(\Omega).$$

As in Lemma 3.2, the nonlinear operator $[B_\varepsilon u,v]=\mathscr{L}^s(u,v)+[\widetilde{k}_\varepsilon(u)+\varepsilon D^s_q u_\varepsilon,v]$ is easily seen to be pseudo-monotone in $\Lambda^{s,q}_0(\Omega)$ and also coercive (see [22] or [27]), since, setting $\|v\|=\|D^sv\|_{\boldsymbol{L}^q(\mathbb{R}^d)}$,

$$\begin{aligned}\frac{[B_\varepsilon v,v]}{\|v\|}&=\frac{1}{\|v\|}\Big(\int_{\mathbb{R}^d}\big(\varepsilon|D^s_qv|^q+AD^sv\cdot D^sv+k_\varepsilon(|D^sv|-g)|D^sv|^2\big)\\&\qquad+\int_\Omega\big(v(\boldsymbol{d}+\boldsymbol{b})\cdot D^sv+cv^2\big)\Big)\\&\ge\varepsilon\|D^sv\|^{q-1}_{\boldsymbol{L}^q(\Omega)}-\tfrac{\|v\|_{L^\infty(\Omega)}}{\|v\|}\Big(\|\boldsymbol{d}+\boldsymbol{b}\|_{\boldsymbol{L}^{r'}(\Omega)}\|D^sv\|_{\boldsymbol{L}^r(\Omega)}+\|c\|_{L^1(\Omega)}\|v\|_{L^\infty(\Omega)}\Big)\\&\ge\|v\|\big(\varepsilon\|v\|^{q-1}-\widetilde{C}_q\big)\longrightarrow\infty\quad\text{ as }\|v\|\longrightarrow\infty,\end{aligned}$$

with $\widetilde{C}_q=C_{r,q}C_q\big(\|\boldsymbol{d}+\boldsymbol{b}\|_{\boldsymbol{L}^{r'}(\Omega)}+C_q\|c\|_{L^1(\Omega)}\big)$, where $C_{r,q}$ is given by (2.13) and $C_q>0$ is given by the continuous embedding (2.15), i.e., such that $\|v\|_{L^\infty(\Omega)}\le C_q\|D^sv\|_{\boldsymbol{L}^q(\mathbb{R}^d)}$, $v\in\Lambda^{s,q}_0(\Omega)$.

Then, by the theory of pseudo-monotone and coercive operators (see [22] or [27], for instance), since $F\in\Lambda^{s,q}_0(\Omega)'$, there exists $u_\varepsilon\in\Lambda^{s,q}_0(\Omega)$ solving (4.20). Taking $v=u_\varepsilon$ in (4.20) and setting $\widehat{k}_\varepsilon=k_\varepsilon(|D^su_\varepsilon|-g)$,

$$\begin{aligned}\int_{\mathbb{R}^d}\widehat{k}_\varepsilon|D^su_\varepsilon|^2+\varepsilon\int_{\mathbb{R}^d}|D^su_\varepsilon|^q&\le\widetilde{C}_r\|D^su_\varepsilon\|^2_{\boldsymbol{L}^r(\mathbb{R}^d)}\\&\quad+\Big(C_r\|f_\#\|_{L^1(\Omega)}+\|\boldsymbol{f}\|_{\boldsymbol{L}^{q'}(\mathbb{R}^d)}\Big)\|D^su_\varepsilon\|_{\boldsymbol{L}^q(\mathbb{R}^d)}\\&\le\widetilde{C}'_r\|D^su_\varepsilon\|^2_{\boldsymbol{L}^r(\mathbb{R}^d)}\le\widetilde{C}'_rC^2_{r,q}\|D^su_\varepsilon\|^2_{\boldsymbol{L}^q(\mathbb{R}^d)}\end{aligned}\tag{4.21}$$

with $\widetilde{C}_r = C_r\big(\|\boldsymbol{d}+\boldsymbol{b}\|_{\boldsymbol{L}^{q'}(\Omega)} + C_r\|c\|_{L^1(\Omega)}\big)$ by assuming, without loss of generality, that $\|D^s u_\varepsilon\|_{\boldsymbol{L}^r(\mathbb{R}^d)} \geq 1$, which will allow the proof of the following *a priori* estimates independent of ε, ε sufficiently small,

$$\|D^s u_\varepsilon\|_{\boldsymbol{L}^r(\mathbb{R}^d)} \leq C, \tag{4.22}$$

$$\|\widehat{k}_\varepsilon |D^s u_\varepsilon|^2\|_{L^1(\mathbb{R}^d)} \leq C \quad \text{and} \quad \|k_\varepsilon\|_{L^1(\mathbb{R}^d)} \leq C, \tag{4.23}$$

$$\|\widehat{k}_\varepsilon D^s u_\varepsilon\|_{\boldsymbol{L}^\infty(\mathbb{R}^d)'} \leq C. \tag{4.24}$$

From (4.21), there exists $C > 0$, independent of ε, such that $\|D^s u_\varepsilon\|_{\boldsymbol{L}^q(\mathbb{R}^d)} \leq C\varepsilon^{-1/(q-2)}$ and consequently also

$$g_*^2\|\widehat{k}_\varepsilon\|_{L^1(\mathbb{R}^d)} \leq \|\widehat{k}_\varepsilon |D^s u_\varepsilon|^2\|_{L^1(\mathbb{R}^d)} \leq C\varepsilon^{-\frac{2}{q-2}}, \tag{4.25}$$

since, as $\widehat{k}_\varepsilon = 0$ if $|D^s u_\varepsilon| < g$, we have $\widehat{k}_\varepsilon |D^s u_\varepsilon|^2 \geq g^2 \widehat{k}_\varepsilon \geq g_*^2 \widehat{k}_\varepsilon$.

Now we split $\mathbb{R}^d$ in two subsets,

$$U_\varepsilon = \{x \in \mathbb{R}^d : |D^s u_\varepsilon| - g \leq \sqrt{\varepsilon}\} \quad \text{and} \quad V_\varepsilon = \mathbb{R}^d \setminus U\varepsilon \tag{4.26}$$

and we observe that, as k_ε is a monotone function, in V_ε we have $\widehat{k}_\varepsilon = k_\varepsilon(|D^s u_\varepsilon| - g) \geq k_\varepsilon(\sqrt{\varepsilon}) = e^{\frac{1}{\sqrt{\varepsilon}}} - 1$ and

$$|V_\varepsilon| = \int_{V_\varepsilon} 1 \leq \int_{V_\varepsilon} \frac{\widehat{k}_\varepsilon}{e^{\frac{1}{\sqrt{\varepsilon}}} - 1} \leq \frac{1}{e^{\frac{1}{\sqrt{\varepsilon}}}-1}\int_{\mathbb{R}^d} \widehat{k}_\varepsilon \leq \frac{C}{\varepsilon^{\frac{2}{q-2}}\big(e^{\frac{1}{\sqrt{\varepsilon}}}-1\big)} \underset{\varepsilon\to 0}{\longrightarrow} 0, \tag{4.27}$$

$$\int_{V_\varepsilon} |D^s u_\varepsilon|^r \leq \Big(\int_{\mathbb{R}^d} |D^s u_\varepsilon|^q\Big)^{\frac{r}{q}} |V_\varepsilon|^{\frac{q-r}{q}} \leq C\Big(\varepsilon^{-\frac{1}{q-2}}\Big)^{\frac{r}{q}} \Bigg(\frac{1}{\varepsilon^{\frac{2}{q-2}}\big(e^{\frac{1}{\sqrt{\varepsilon}}}-1\big)}\Bigg)^{\frac{q-r}{q}} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Given $R > 0$,

$$\int_{U_\varepsilon\cap\Omega_R} |D^s u_\varepsilon|^r \leq \int_{U_\varepsilon\cap\Omega_R} (g+\sqrt{\varepsilon})^r \leq \big(g^*+1\big)^r |\Omega_R|,$$

$$\int_{U_\varepsilon\setminus\Omega_R} |D^s u_\varepsilon|^r \leq C(R)\|u_\varepsilon\|^r_{L^1(\Omega)} \leq C(R) C^r_{r,1}\|D^s u_\varepsilon\|^r_{L^r(\mathbb{R}^d)},$$

using (2.7) and (2.11), with C representing different constants. Then, for ε small enough,

$$\int_{\mathbb{R}^d} |D^s u_\varepsilon|^r \leq \big(g^*+1\big)^r |\Omega_R| + C(R)C^r_{r,1}\|D^s u_\varepsilon\|^r_{L^r(\mathbb{R}^d)} + 1.$$

Choosing $R_0 > 0$ such that $C(R_0)C^r_{r,1} \leq \frac{1}{2}$ we get (4.22) from

$$\|D^s u_\varepsilon\|_{\boldsymbol{L}^r(\mathbb{R}^d)} \leq 2^{\frac{1}{r}}\big((g^*+1)^r|\Omega_{R_0})| + 1\big)^{\frac{1}{r}}.$$

Hence, also from (4.21) we immediately obtain that $\|\widehat{k}_\varepsilon |D^s u_\varepsilon|^2\|_{L^1(\mathbb{R}^d)} \leq C$, and (4.23) follows from the first inequality in (4.25).

As a consequence, (4.24) now easily follows from (4.23):

$$\|\widehat{k}_\varepsilon D^s u_\varepsilon\|_{\boldsymbol{L}^\infty(\mathbb{R}^d)'} = \sup_{\|\boldsymbol{\xi}\|_{\boldsymbol{L}^\infty(\mathbb{R}^d)}\leq 1}\left|\int_{\mathbb{R}^d}\widehat{k}_\varepsilon D^s u_\varepsilon\cdot\boldsymbol{\xi}\right|$$
$$\leq \int_{\mathbb{R}^d}\widehat{k}_\varepsilon^{\frac{1}{2}}\widehat{k}_\varepsilon^{\frac{1}{2}}|D^s u_\varepsilon| \leq \|\widehat{k}_\varepsilon\|^{\frac{1}{2}}_{L^1(\mathbb{R}^d)}\|\widehat{k}_\varepsilon|D^s u_\varepsilon|^2\|^{\frac{1}{2}}_{L^1(\mathbb{R}^d)}.$$

By compactness, from the estimates (4.22), (4.23) and (4.24), using the Rellich-Kondrachov and the Banach-Alaoglu-Bourbaki theorems, there exist $u\in\Lambda^{s,r}_0(\Omega)\cap C^{0,\gamma}(\overline{\Omega})$, $\lambda\in L^\infty(\mathbb{R}^d)'$ and $\boldsymbol{\Psi}\in\boldsymbol{L}^\infty(\mathbb{R}^d)'$ and a generalised sequence $\varepsilon\to 0$ such that

$$D^s u_\varepsilon \underset{\varepsilon}{\rightharpoonup} D^s u \text{ in } \boldsymbol{L}^r(\mathbb{R}^d)\text{-weak},\quad u_\varepsilon \underset{\varepsilon}{\longrightarrow} u \text{ in } C^{0,\gamma}(\overline{\Omega}) \text{ strong},$$
$$\widehat{k}_\varepsilon \underset{\varepsilon}{\rightharpoonup} \lambda \text{ in } L^\infty(\mathbb{R}^d)'\text{-weak}^*,\quad \widehat{k}_\varepsilon D^s u_\varepsilon \underset{s}{\rightharpoonup} \boldsymbol{\Psi} \text{ in } \boldsymbol{L}^\infty(\mathbb{R}^d)'\text{-weak}^*.$$

Now, arguing as in the proof of Theorem 4.2, we show that $u\in\mathbb{K}^s_g$, $\boldsymbol{\Psi}=\lambda D^s u$ and (u,λ) satisfies (4.1).

Firstly we conclude that $|D^s u|\leq g$ a.e. in $\mathbb{R}^d$, from

$$\int_{\mathbb{R}^d}(|D^s u|-g)^+ \leq \varliminf_{\varepsilon\to 0}\int_{\mathbb{R}^d}(|D^s u_\varepsilon|-g-\sqrt{\varepsilon})^+ = \varliminf_{\varepsilon\to 0}\int_{V\varepsilon}(|D^s u_\varepsilon|-g-\sqrt{\varepsilon})$$
$$\leq \varliminf_{\varepsilon\to 0}\int_{V\varepsilon}|D^s u_\varepsilon| \leq \varliminf_{\varepsilon\to 0}\|D^s u_\varepsilon\|_{L^r(\mathbb{R}^d)}|V_\varepsilon|^{\frac{1}{r'}} = 0,$$

since $\xi\mapsto(|\xi|-g)^+$ is a convex lower semicontinuous function and V_ε, defined in (4.26), has vanishing measure as $\varepsilon\to 0$ by (4.27).

Observing that

$$\left|\int_{\mathbb{R}^d}|D^s u_\varepsilon|^{q-2}D^s u_\varepsilon\cdot D^s v\right| \leq \|D^s u_\varepsilon\|^r_{\boldsymbol{L}^r(\mathbb{R}^d)}\|D^s v\|_{\boldsymbol{L}^\infty(\mathbb{R}^r)},$$

taking the generalised limit in (4.16) with an arbitrarily $v\in\Lambda^{s,\infty}_0(\Omega)\subset\Lambda^{s,q}_0(\Omega)$, we obtain

$$\langle\boldsymbol{\Psi}, D^s v\rangle = [F,v]-\mathscr{L}^s(u,v),\quad \forall v\in\Lambda^{s,\infty}_0(\Omega), \tag{4.28}$$

and, since $\widehat{k}_\varepsilon\geq 0$ implies $\lambda\geq 0$, it remains to show that $\boldsymbol{\Psi}=\lambda D^s u$ and $\lambda|D^s u|=\lambda g$.

Taking $v=u_\varepsilon$ in (4.20) and using (4.28) and the semicontinuity (4.14) for $u_\varepsilon \underset{\varepsilon}{\rightharpoonup} u$ in $\Lambda^{s,r}_0(\Omega)$, we easily obtain

$$\varlimsup_{\varepsilon\to 0}\int_{\mathbb{R}^d}\widehat{k}_\varepsilon|D^s u_\varepsilon|^2 \leq \langle\boldsymbol{\Psi}, D^s u\rangle.$$

Comparing the inequality

$$\begin{aligned}
0 &\leq \varlimsup_{\varepsilon\to 0}\int_{\mathbb{R}^d}\widehat{k}_\varepsilon|D^s(u_\varepsilon-u)|^2\\
&= \varlimsup_{\varepsilon\to 0}\left(\int_{\mathbb{R}^d}\widehat{k}_\varepsilon|D^s u_\varepsilon|^2 - 2\int_{\mathbb{R}^d}\widehat{k}_\varepsilon D^s u_\varepsilon\cdot D^s u + \int_{\mathbb{R}^d}\widehat{k}_\varepsilon|D^s u|^2\right)\\
&\leq \langle\boldsymbol{\Psi}, D^s u\rangle - 2\lim_{\varepsilon\to 0}\langle\widehat{k}_\varepsilon D^s u_\varepsilon, D^s u\rangle + \lim_{\varepsilon\to 0}\langle\widehat{k}_\varepsilon, |D^s u|^2\rangle\\
&\leq \langle\lambda, |D^s u|^2\rangle - \langle\boldsymbol{\Psi}, D^s u\rangle,
\end{aligned} \tag{4.29}$$

with (recall that $\widehat{k}_\varepsilon g^2 \le \widehat{k}_\varepsilon |D^s u_\varepsilon|^2$ by definition of $\widehat{k}_\varepsilon$)

$$\langle \lambda, |D^s u|^2\rangle \le \langle \lambda, g^2\rangle = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^d}\widehat{k}_\varepsilon\, g^2 \le \overline{\lim_{\varepsilon\to 0}}\int_{\mathbb{R}^d}\widehat{k}_\varepsilon |D^s u_\varepsilon|^2 \le \langle\!\langle \boldsymbol{\Psi}, D^s u\rangle\!\rangle,$$

we conclude that

$$\langle \lambda, |D^s u|^2\rangle = \langle \lambda, g^2\rangle = \overline{\lim_{\varepsilon\to 0}}\int_{\mathbb{R}^d}\widehat{k}_\varepsilon |D^s u_\varepsilon|^2 = \langle\!\langle \boldsymbol{\Psi}, D^s u\rangle\!\rangle$$

and, afterwards from (4.29), also

$$\overline{\lim_{\varepsilon\to 0}}\int_{\mathbb{R}^d}\widehat{k}_\varepsilon |D^s (u_\varepsilon - u)|^2 = 0.$$

Then $\boldsymbol{\Psi} = \lambda D^s u$ since, for an arbitrary $\boldsymbol{\xi}\in \boldsymbol{L}^\infty(\mathbb{R}^d)$,

$$\begin{aligned}|\langle\!\langle \boldsymbol{\Psi} - \lambda D^s u, \boldsymbol{\xi}\rangle\!\rangle| &= \lim_{\varepsilon\to 0}\left|\int_{\mathbb{R}^d}\widehat{k}_\varepsilon D^s(u_\varepsilon - u)\cdot\boldsymbol{\xi}\right| \\ &\le \overline{\lim_{\varepsilon\to 0}}\left(\int_{\mathbb{R}^d}\widehat{k}_\varepsilon |D^s(u_\varepsilon - u)|^2\right)^{\frac12}\|\widehat{k}_\varepsilon\|^{\frac12}_{L^1(\mathbb{R}^d)}\|\xi\|_{\boldsymbol{L}^\infty(\mathbb{R}^d)} = 0.\end{aligned}$$

From (4.24) it follows $\langle \lambda, |D^s u|^2 - g^2\rangle = 0$ and we conclude, as in (4.18), that

$$\lambda(|D^s u| - g) = 0 \quad \text{in } L^\infty(\mathbb{R}^d)'.$$

The general case follows by Theorem 4.2, by approximating with solutions of $(4.1\text{a})_\nu$, $(4.1\text{b})_\nu$ with data A_ν, $\boldsymbol{b}_\nu$, $\boldsymbol{d}_\nu$, c, $f_\#$, $\boldsymbol{f}_\nu$ and g satisfying (3.5), (4.19) and (3.20) and converging strongly in L^{p_1}, L^1 and L^{q_1}, for instance by using $\boldsymbol{f}_\nu = \sup(-\frac{1}{\nu}, \inf(\frac{1}{\nu}, \boldsymbol{f}))\chi_{B(0,\frac{1}{\nu})}$, where $\chi_{B(0,\frac{1}{\nu})}$ denotes the characteristic function of $B(0,\frac{1}{\nu})$.

Finally, since $u\in\mathbb{K}^s_g$, given $v\in\mathbb{K}^s_g$, taking $v-u$ as test function in (4.1a) and noting that

$$\begin{aligned}\langle\!\langle \lambda D^s u, D^s(v-u)\rangle\!\rangle &= \langle \lambda, D^s u\cdot D^s(v-u)\rangle = \langle \lambda, D^s u\cdot D^s v - |D^s u|^2\rangle \\ &\le \langle \lambda, |D^s u|(|D^s v| - |D^s u|)\rangle \le \langle \lambda, |D^s u|(g - |D^s u|)\rangle \\ &= \langle \lambda(g - |D^s u|), |D^s u|\rangle = 0,\end{aligned}$$

it is clear that u solves the variational inequality (3.11), which concludes the proof of Theorem 4.1. □

Remark 4.3. Theorem 4.2, as a weak continuous dependence result, generalises Theorem 3.5 to the case of degenerate operators, including the case $A\equiv 0$, with L^1-data. In fact, if A satisfies (3.5) and (3.6) holds with the strictly coercive assumption (3.17), it is clear that u solving problem (4.1) is unique. On the other hand, the uniqueness of λ is an open problem, even in the local case of $s=1$, which was considered first for the Laplacian in [4] in the special case $f_\#\in L^2(\Omega)$ and $\boldsymbol{f}=0$.

Remark 4.4. As in Section 3, we may assume in Theorems 4.1 and 4.2 that g and g_ν satisfy $((3.20)_{loc})$ instead (3.20), with uniform limit in ν.

5. Localisation of transport densities as $s \to 1$

In order to consider the generalised convergence of the fractional problem to the local one as $s \to 1$, for $0 < \sigma < s \leq 1$, with σ fixed, we consider $(u_s, \lambda_s) \in \Lambda_0^{s,\infty}(\Omega) \times \boldsymbol{L}^\infty(\mathbb{R}^d)'$ such that

$$(5.1)_s \qquad \mathscr{L}^s(u_s, v) + \langle \lambda_s D^s u_s, D^s v \rangle = [F, v]_s, \quad \forall v \in \Lambda_0^{s,\infty}(\Omega)$$

$$(5.2)_s \qquad |D^s u_s| \leq g_s \text{ a.e. in } \mathbb{R}^d, \quad \lambda_s \geq 0 \text{ and } \lambda_s(|D^s u_s| - g_s) = 0 \text{ in } L^\infty(\mathbb{R}^d)',$$

with the convention $s = 1$ corresponds to the local problem $(u, \lambda) \in W_0^{1,\infty}(\Omega) \times L^\infty(\Omega)$, where $D^1 = D$ is the classical gradient in the definitions of $\mathscr{L}$ and $[F, \cdot]_s$ given by (3.4) and (3.8), respectively, and $(5.2)_1$ holds only in Ω.

Here we can also allow a variable threshold g_s under the assumption

$$(5.3) \qquad 0 < g_* \leq g_s(x) \leq g^*, \quad \text{a.e. } x \in \mathbb{R}^d,\ \sigma < s \leq 1,$$

and such that

$$(5.4) \qquad g_s \longrightarrow g_1 \quad \text{in } L^\infty(\mathbb{R}^d) \text{ as } s \to 1.$$

The corresponding variational inequality (3.11) now reads as follows

$$(5.5)_s \qquad u_s \in \mathbb{K}^s_{g_s} : \mathscr{L}^s(u_s, v - u_s) \geq [F, v - u_s]_s, \quad \forall v \in \mathbb{K}^s_{g_s},$$

where the convex set $\mathbb{K}_{g_s}$ is defined by (3.1) with g_s, $\sigma < s \leq 1$.

Now we can state the localisation theorem for $(5.1)_s$-$(5.2)_s$ as $s \to 1$, which is essentially a variant of the generalised continuous dependence property of Theorem 4.2 with the additional difficulty on the variable spaces $\Lambda_0^{s,\infty}(\Omega)$, with s, $\sigma < s < 1$. For $\zeta \in L^\infty(\mathbb{R}^d)'$, we denote its restriction to $\Omega \subset \mathbb{R}^d$ by $\zeta_\Omega \in L^\infty(\Omega)'$, defined by

$$\langle \zeta_\Omega, \varphi \rangle = \langle \zeta, \widetilde{\varphi} \rangle, \quad \forall \varphi \in L^\infty(\Omega),$$

where $\widetilde{\varphi}$ is the extension of φ by zero to $\mathbb{R}^d \setminus \Omega$.

Theorem 5.1. *For any $0 < \sigma < 1$, let $(u_s, \lambda_s) \in \Lambda_0^{s,\infty}(\Omega) \times L^\infty(\mathbb{R}^d)'$ solve $(5.1)_s$-$(5.2)_s$ for any s, $0 < \sigma < s < 1$, under the assumptions (3.5), (3.7), (3.10) and (5.3), (5.4). Then, there is a generalised sequence denoted by $s \to 1$, such that, for any $0 < \alpha < \sigma$,*

$$u_s \underset{s}{\rightarrow} u \quad \text{in} \quad \Lambda_0^{\sigma,p}(\Omega) \cap C^{0,\alpha}(\overline{\Omega}),$$
$$D^s u_s \underset{s}{\rightharpoonup} Du \quad \text{in} \quad \boldsymbol{L}^\infty(\mathbb{R}^d)\text{-weak}^*,$$
$$(\lambda_s)_\Omega \underset{s}{\rightharpoonup} \lambda \quad \text{in} \quad L^\infty(\Omega)'\text{-weak}^*,$$

where $(u, \lambda) \in W_0^{1,\infty}(\Omega) \cap L^\infty(\Omega)'$ is a solution to the local problem $(5.1)_1$-$(5.2)_1$, in Ω.

Proof. We adapt the steps of the proof of Theorem 4.2: i) *a priori* estimates with respect to s; ii) existence of limits of generalised sequences, by compactness, and iii) characterization of those limits as solutions of the local problem $(5.1)_1$- $(5.2)_1$. For $0 < \sigma < s < 1$, using the Poincaré inequality (2.8), we have $\frac{\sigma}{C_0}\|u_s\|_{L^\infty(\Omega)} \le \|D^s u_s\|_{\boldsymbol{L}^\infty(\mathbb{R}^d)}$. Then by the assumption (5.3) we obtain for any $u_s \in \mathbb{K}^s_{g_s}$ solution of $(5.1)_s$-$(5.2)_s$, we get that

$$
\begin{aligned}
&C^{-1}_{p,\infty}\|D^s u_s\|_{\boldsymbol{L}^p(\mathbb{R}^d)} \le \|D^s u_s\|_{\boldsymbol{L}^\infty(\mathbb{R}^d)} \le g^*, \ 1 \le p < \infty, \\
&\|u_s\|_{C^{0,\beta}(\overline{\Omega})} \le C_\beta, \text{ for any } \beta,\ 0<\beta<s<1,
\end{aligned} \tag{5.6}
$$

where the constant C_β is independent of s.

Letting $\boldsymbol{\Psi}_s = \lambda_s D^s u_s$, arguing exactly as in (4.5)-(4.7) and (4.8), by replacing the label ν by s, we obtain that

$$
\|\lambda_s\|'_{L^\infty(\mathbb{R}^d)} \le \frac{C_1}{g_*^2} \quad \text{and} \quad \|\boldsymbol{\Psi}_s\|'_{\boldsymbol{L}^\infty(\mathbb{R}^d)} \le \frac{C_1 g^*}{g_*^2},
$$

where $C_1 > 0$ is a constant independent of s, $\sigma < s < 1$

Therefore, by compactness, in particular, by (5.6) and (2.18), there are $u \in C^{0,\alpha}(\overline{\Omega}) \cap \Lambda^{\sigma,p}(\Omega)$, for $0<\alpha<\beta$, $0<\sigma<1$ and $1<p<\infty$, $\boldsymbol{\chi} \in \boldsymbol{L}^\infty(\mathbb{R}^d)$, $\widetilde{\lambda} \in L^\infty(\mathbb{R}^d)'$, $\boldsymbol{\Psi} \in \boldsymbol{L}^\infty(\mathbb{R}^d)'$ and a generalised sequence $s \to 1$, such that

$$
\begin{aligned}
&u_s \underset{s}{\longrightarrow} u \ \text{ in } \ \Lambda^{\sigma,p}_0(\Omega) \cap C^{0,\alpha}(\overline{\Omega}), \\
&D^s u_s \underset{s}{\rightharpoonup} \boldsymbol{\chi} \ \text{ in } \ \boldsymbol{L}^\infty(\mathbb{R}^d)\text{-weak}^*, \\
&(\lambda_s)_\Omega \underset{s}{\rightharpoonup} \lambda \text{ in } \ L^\infty(\Omega)'\text{-weak}^*, \\
&\boldsymbol{\Psi}_s \underset{s}{\rightharpoonup} \boldsymbol{\Psi} \ \text{ in } \ \boldsymbol{L}^\infty(\mathbb{R}^d)\text{-weak}^*.
\end{aligned}
$$

Letting by $\widetilde{u}$ be the extension of u by zero to $\mathbb{R}^d$, and applying Corollary 2.3 componentwise to an arbitrarily $\boldsymbol{\varphi} \in C^\infty_c(\mathbb{R}^d)^d$, we have, by (2.6) and recalling $D^s \cdot \boldsymbol{\varphi} \underset{s}{\longrightarrow} D \cdot \boldsymbol{\varphi}$ as $s \to 1$,

$$
\int_{\mathbb{R}^d} \boldsymbol{\chi} \cdot \boldsymbol{\varphi} = \lim_s \int_{\mathbb{R}^d} D^s u^s \cdot \boldsymbol{\varphi} = -\lim_s \int_{\mathbb{R}^d} \widetilde{u}^s D^s \cdot \boldsymbol{\varphi} = -\int_{\mathbb{R}^d} \widetilde{u} D \cdot \boldsymbol{\varphi},
$$

which means that $\boldsymbol{\chi} = D\widetilde{u} \in \boldsymbol{L}^\infty(\mathbb{R}^d)$ and $u \in W^{1,\infty}_0(\Omega)$, and therefore $D\widetilde{u} = \widetilde{Du}$.

Arguing as in (4.10) with an arbitrarily measurable subset $w \subset \mathbb{R}^d$ with finite measure and with $\boldsymbol{\xi} = \frac{Du}{|Du|}\chi_V$, $V = w \cap \{|Du| \neq 0\}$, we obtain

$$
\int_w |D\widetilde{u}| = \int_{\mathbb{R}^d} D\widetilde{u} \cdot \boldsymbol{\xi} = \lim_s \int_{\mathbb{R}^d} D^s \widetilde{u}_s \cdot \boldsymbol{\xi} \le \overline{\lim_s} \int_w |D^s u_s| \le \lim_s \int_w g_s = \int_w g_1
$$

and so $|Du| \le g_1$ a.e. in Ω, i.e. $u \in \mathbb{K}_{g_1}$.

Observe that we still have the lower semicontinuity property

$$
\underline{\lim_s}\, \mathscr{L}^s(u_s, u_s) \ge \mathscr{L}^1(u,u) \tag{5.7}
$$

as we easily see by using (3.5) and taking $\underline{\lim}_s$ in

$$
\int_{\mathbb{R}^d} AD^s u_s \cdot D^s u_s \ge \int_{\mathbb{R}^d} AD^s u_s \cdot D\widetilde{u} + \int_{\mathbb{R}^d} AD\widetilde{u} \cdot D^s u_s - \int_{\mathbb{R}^d} AD\widetilde{u} \cdot D\widetilde{u}.
$$

On the other hand, recalling (2.14), we have $C^\infty_c(\Omega) \subset W^{1,\infty}_0(\Omega) \subset \Lambda^{s,\infty}_0(\Omega)$. Taking the limit $s \to 1$ in $(5.1)_s$ with $v \in C^\infty_c(\Omega)$ we get

$$
\mathscr{L}^1(u,v) + \langle \boldsymbol{\Psi}_\Omega, Dv \rangle = [F, v]_1. \tag{5.8}
$$

Since for each $v \in W^{1,\infty}_0(\Omega)$ we may take a sequence $v_n \in C^\infty_0(\Omega)$ such that $v_n \underset{n}{\longrightarrow} v$ in $H^1_0(\Omega)$ with $Dv_n \underset{n}{\longrightarrow} Dv$ in $\boldsymbol{L}^\infty(\Omega)$-weak*, the equation (5.8) also holds for any $v \in W^{1,\infty}_0(\Omega)$, as $\boldsymbol{\Psi}_\Omega \in \boldsymbol{L}^\infty(\Omega)'$. So $(5.1)_1$ will follows if we show $\boldsymbol{\Psi}_\Omega = \lambda Du$, with $\lambda = \widetilde{\lambda}_\Omega$, which can be done exactly as in the proof of (4.11), by replacing the subscript ν by s in (4.13) and in (4.16).

Similarly, the corresponding limit (4.17) as $s \to 1$ implies $\langle \lambda(g^2 - |Du|^2), 1\rangle = 0$ and the same argument of (4.18) yields $\lambda(g - |Du|) = 0$ in $L^\infty(\Omega)'$, showing that (u, λ) also satisfies $(5.2)_1$ in Ω. □

Remark 5.2. In the coercive case, i.e., if (3.16) and (3.17) hold, it is clear that u_s and u_1 are also the unique solutions of the respective variational inequalities (3.11), with $s \leq 1$. In this case, in particular, with $c = f_\# = 0$ and $\boldsymbol{b} = \boldsymbol{d} = \boldsymbol{0}$, the result was given in [3] only with the convergence $u_s \underset{s}{\longrightarrow} u$ in $H^\sigma_0(\Omega)$, $0 < \sigma < 1$.

Remark 5.3. Under the assumptions of Theorem 3.5 it is easy to obtain the estimates, similarly to (3.18),

$$\|u_s\|_{H^s_0(\Omega)} \leq \tfrac{C_*}{\delta}\|f_*\|_{L^{2^\#}(\Omega)} + \tfrac{1}{\delta}\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\mathbb{R}^d)}$$

and, denoting $\Gamma^s \in \partial I_{\mathbb{K}^s_{g_s}(u_s)}$, as in Remark 3.4, it is easy to conclude that Γ^s is also uniformly bounded in $H^{-s}(\Omega)$, independently of $\sigma < s < 1$. This allows us to take subsequences $u_s \underset{s}{\longrightarrow} u$ in $H^\sigma_0(\Omega)$ and $\Gamma^s \underset{s}{\longrightarrow} \Gamma$ in $H^{-\sigma}(\Omega)$, $\forall \sigma < 1$, with $u \in H^1_0(\Omega)$, $\Gamma \in H^{-1}(\Omega)$ satisfying the local problem $s = 1$.

Remark 5.4. As in Remark 4.4, in Theorem 5.1 we may replace the assumptions on g_s by the weaker assumption $g_s \in L^\infty_{loc}(\mathbb{R}^d)$, with positive lower bound in any compact set, and $\lim\limits_{|x|\to\infty} g_s(x)|x|^{d+s} = \infty$ uniformly in s.

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