A crossinggram for random fields on lattices

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Abstract

The modeling of risk situations that occur in a space framework can be done using max-stable random fields on lattices. Although the summary coefficients for the spatial behaviour do not characterize the finite-dimensional distributions of the random field, they have the advantage of being immediate to interpret and easier to estimate. The coefficients that we propose give us information about the tendency of a random field for local oscillations of its values in relation to real valued high levels. It is not the magnitude of the oscillations that is being evaluated, but rather the greater or lesser number of oscillations, that is, the tendency of the trajectories to oscillate. We can observe surface trajectories more smooth over a region according to higher crossinggram value. It takes value in $[0,1]$ and increases with the concordance of the variables of the random field.

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1 Introduction

Consider that $Y_i(x)$ represents the daily maximum precipitation in year i at a location x belonging to some locations family $A \subset \mathbb{Z}^2$. The stochastic behavior of $\{Y_i(x),\, x \in A,\, i \geq 1\}$ can not be studied using the classical theory of stable distributions because the variables of interest are not sums, thus excluding any modeling with normal multivariate distributions (Embrechts *et al.* [5] 1997). Assessing probabilities of risk events, such as "the maximum, in a region A, of the maximum daily rainfall over n years exceeds u", $P(\bigvee_{x\in A}\bigvee_{i=1}^{n} Y_i(x) > u)$, with notation $a \vee b = \max(a,b)$, requires a theory that provides information about the distributions of variables $\bigvee_{i=1}^{n} Y_i(x)$, $x \in A$, and the dependency structure between them, i.e., the theory of multivariate extreme value distributions (Ribatet *et al.* [21] 2016). In the context of this theory, it is considered that, as $n \to \infty$, the set of approximate distributions for $\bigvee_{i=1}^{n} Y_i(x)$ admits only Fréchet, Weibull and Gumbel laws. Moreover, at any location $(x_1, ..., x_d)$, the approximate copula function $C_{x_1,...,x_d}$ for vector $(\bigvee_{i=1}^{n} Y_i(x_1), ..., \bigvee_{i=1}^{n} Y_i(x_d))$ is max-stable, i.e., satisfies the condition

$$C_{x_1,...,x_d}^k(u_1, ..., u_d) = C_{x_1,...,x_d}(u_1^k, ..., u_d^k),\ \forall k > 0,\ u_i \in [0,1],$$

(de Haan and Ferreira [15] 2006).
The main context of this work considers random fields $\{X(x),\ x \in \mathbb{Z}^2\}$, for which $(X(x_1), ..., X(x_d))$ has multivariate extreme value distribution, regardless the choice at any site $(x_1, ..., x_d)$. Its distribution function is completely characterized by the marginal laws and by its stable tail dependence function. A widely used choice for marginal distributions is the unit Fréchet, for sake of simplicity and without loss of generality. The stable tail dependence function

$$\ell_{x_1,...,x_d}(z_1, ..., z_d) = -\log C_{x_1,...,x_d}(P(X(x_1) \le z_1^{-1}), ..., P(X(x_d) \le z_d^{-1})),\ \forall z_i \ge 0,$$

verifies

$$\ell_{x_1,...,x_d}(z_1, ..., z_d) = \frac{E\left(\bigvee_{i=1}^{d} U(x_i)^{1/z_i}\right)}{1 - E\left(\bigvee_{i=1}^{d} U(x_i)^{1/z_i}\right)},\ \forall z_i \ge 0, \tag{1}$$

where $(U(x_1), ..., U(x_d))$ is a vector of standard uniform distributed marginals having the same copula function $C_{x_1,...,x_d}$ (Ferreira and Ferreira [8], 2012a).
The estimation of the stable tail dependence function presents challenges (see, e.g., Beirlant *et al.* [1] 2004, Ferreira and Ferreira [8] 2012a, Beirlant *et al.* [2] 2016, Escobar *et al.* [6] 2018, Kiriliouk *et al.* [17] 2018 and references therein) and several summary measures of the dependence between the variables of a max-stable random field can be used: extremal coefficients (Tiago de Oliveira [25] 1962/1963, Smith [24] 1990), coefficients of tail dependence (Sibuya [23] 1960, Joe [16] 1997, Li [19] 2009), coefficients of pre-asymptotic tail dependence (Ledford and Tawn [18] 1997, Wadsworth and Tawn [26]-[27] 2012-2013), fragility coefficients (Falk and Tichy [7] 2012, Ferreira and Ferreira [9] 2012b), madogram (Naveau *et al.* [20] 2009, Ferreira and Ferreira [10] 2018), extremogram (Davis and Mikosch [4] 2009), among others.
Although the summary coefficients of the spatial dependence structure do not characterize the finite-dimensional distributions of $\{X(x),\ x \in \mathbb{Z}^2\}$, they are easier to estimate and furnish an immediate reading. Usually, the coefficients of spatial dependence capture an overall dependence of the marginals but do not account for different local dependencies nor a propensity to upcross high risky levels u. The coefficients that we propose, study and apply here give us information about the tendency of $\{X(x),\ x \in A\}$, $A \subset \mathbb{Z}^2$, for local oscillations of their values in relation to real high levels u. It can be observed trajectories $\{x, X(x)\}_{x \in A}$ more or less smooth (or more or less rough) according to the coefficients values.
The tendency for the variables, in close locations, to jointly present extreme values will determine the proportion of exceedances of high levels that are upcrossings of the level. As the joint

tendency for extreme values is usually summarized in the literature by upper-tail dependence coefficients, the question arises: how to use these coefficients to summarize the degree of this kind of smoothness for a random field on a lattice? We invite the reader to follow us in a motivated and justified construction of a response to this question.
The objective of this work is to quantify the propensity of a max-stable random field for oscillations over regions $A \subset \mathbb{Z}^2$, through coefficients that are easy to estimate and use in applications. Thus, in the next section it will be introduced the crossinggram $\zeta(A)$, $A \subset \mathbb{Z}^2$. In Section 3 we deduce some of its properties and in Section 3.1 we propose a method for its estimation. Section 3.2 is concerned with oscillations of general random fields, for which we propose a smaller coefficient easier to deal with, but less interesting for the max-stable context. The calculation of $\zeta(A)$ will be illustrated in a model for max-stable random fields in Section 4. We conclude in Section 5.

2 Notations and construction of the crossinggram

Let $\{X(x),\, x \in \mathbb{Z}^2\}$ be a max-stable random field, i.e., the variables $X(x)$ have extreme-type distribution and, for any choice of locations x_1, ..., x_d, the vector $(X(x_1), ..., X(x_d))$ has multivariate extreme value distribution. Without loss of generality for applications, suppose that $X(x)$ has common distribution function unit Fréchet, i.e., $F(x) = \exp(-x^{-1})$, $x > 0$.
For each location $x = \left(x^{(1)}, x^{(2)}\right) \in \mathbb{Z}^2$, let $V(x) := \left\{y \in \mathbb{Z}^2 : \|y - x\| \leq 1\right\}$.
We say that $\{X(x),\, x \in \mathbb{Z}^2\}$ has an oscillation with respect to u, $u \in (0, 1)$, at location x, when the following event occurs

$$\left\{F(X(x)) \leq u < \bigvee_{y \in V(x)} F(X(y))\right\},$$

and it has an exceedance of u, at location x, when $\{F(X(x)) > u\}$ occurs.
Several tail dependence coefficients for bivariate and multivariate distributions have been constructed in the literature and, in our view, the work of Li ([19], 2009) is an important landmark. For our purpose, we take as a good starting point the upper-tail dependence of $\bigvee_{y \in V(x)} F(X(y))$ and $\bigvee_{y \in A} F(X(y))$, for each location $x \in A \subset \mathbb{Z}^2$, where the region A is bounded and its finite cardinal will be denoted $|A|$.
Consider, for some location $x \in A$ such that $V(x) \subset A$,

$$\lambda(V(x)|A) = \lim_{u \uparrow 1} P\left(\bigvee_{y \in V(x)} F(X(y)) > u \middle| \bigvee_{y \in A} F(X(y)) > u\right)$$

and

$$\lambda(x|A) = \lim_{u \uparrow 1} P\left(F(X(x)) > u \middle| \bigvee_{y \in A} F(X(y)) > u\right).$$

We intuitively expect smaller values for the difference

$$\sum_{x\in A}\lambda(V(x)|A)-\sum_{x\in A}\lambda(x|A)$$

in regions where, for each x, the variables $F(X(y))$, $y\in V(x)$, have lower tendency for oscillations relative to high levels.

The following proposition justifies this interpretation for the values of these differences, presenting them as coefficients that summarize the expected number of local oscillations in the spatial context.

Proposition 2.1. *For $x\in A\subset\mathbb{Z}^2$, the tail dependence coefficients $\lambda(V(x)|A)$ and $\lambda(x|A)$ satisfy*

$$\sum_{x\in A}\lambda(V(x)|A)-\sum_{x\in A}\lambda(x|A)=\lim_{u\uparrow 1}E\left(\sum_{x\in A}\mathbf{1}_{\{F(X(x))\leq u<\bigvee_{y\in V(x)}F(X(y))\}}\middle|\sum_{x\in A}\mathbf{1}_{\{F(X(x))>u\}}>0\right).$$

Proof. Observe that

$$\begin{aligned}&\sum_{x\in A}\lambda(V(x)|A)-\sum_{x\in A}\lambda(x|A)\\ &=\lim_{u\uparrow 1}\frac{\sum_{x\in A}P\left(\bigvee_{y\in V(x)}F(X(y))>u\right)-\sum_{x\in A}P(F(X(x))>u)}{P\left(\bigvee_{x\in A}F(X(x))>u\right)}\\ &=\lim_{u\uparrow 1}\frac{\sum_{x\in A}P\left(F(X(x))\leq u<\bigvee_{y\in V(x)}F(X(y))\right)}{P\left(\bigvee_{x\in A}F(X(x))>u\right)}.\end{aligned}$$

□

We depart from the above representation and, with a convenient normalization in order to eliminate the effect of the cardinality $|A|$ of A, we propose coefficients $\zeta(A)$, $A\subset\mathbb{Z}^2$, with values in $[0,1]$. For the normalization here we take into account that $P\left(F(X(x))\leq u<\bigvee_{y\in V(x)}F(X(y))\right)\leq\sum_{y\in V(x)}P(F(X(y))>u)$, which will lead to a usefull representation of the coefficient in Proposition 3.1.

The following crossinggram increases when the local oscillations decrease and, if we consider "roughness" the proximity between the number of upcrossings and exceedances of high levels we can say that crossinggram increases with the local "smoothness" of the random field. For instance, in Figure 1, on the left plot, one can see that by the center of the grounded plane, large peaks occur very closed to each other, but when approaching the edges, the peaks become lower and sparser. Thus, one would expect a small crossinggram value in regions close to the center and large in areas away from the center part, as will be corroborated in Section 4. This assessment clearly indicates where a phenomenon under study presents greater risk.

Definition 2.1. *The crossinggram $\zeta(A)$, $A \subset \mathbb{Z}^2$, for the max-stable random field $\{X(x),\ x \in \mathbb{Z}^2\}$ is defined by*

$$\zeta(A) = 1 - \lim_{u \uparrow 1} \frac{E\left(\sum_{x \in A} \mathbf{1}_{\{F(X(x)) \le u < \bigvee_{y \in V(x)} F(X(y))\}} \middle| \sum_{x \in A} \mathbf{1}_{\{F(X(x)) > u\}} > 0 \right)}{E\left(\sum_{x \in A} \sum_{y \in V(x) - \{x\}} \mathbf{1}_{\{F(X(y)) > u\}} \middle| \sum_{x \in A} \mathbf{1}_{\{F(X(x)) > u\}} > 0 \right)}.$$

As will be pointed in the next section, the limits $\zeta(A)$, $A \subset \mathbb{Z}^2$, exist and have enlightening properties.

3 Properties of the crossinggram

The coefficients of tail dependence can be related to the extremal coefficients (see, e.g., Beirlant *et al.* [1] 2014). We remind that, for any $x_1, ..., x_d \in \mathbb{Z}^2$, we have

$$P\left(\bigcap_{i=1}^{d} \{F(X(x_i)) \le u\} \right) = u^{\theta(x_1,...,x_d)}, \tag{2}$$

with $\theta(x_1, ..., x_d)$ constant in $[1, d]$ and $\theta(x_1, ..., x_d) = \ell_{x_1,...,x_d}(1, ..., 1)$. In the case of $\theta(y,\ y \in V(x))$ we simply write $\theta(V(x))$.
In the particular case of $d = 2$ and isotropic stationary max-stable random fields, we can consider the extremal function

$$\underline{\theta}(h) \equiv \theta(x, x + h) = E(X(x) \vee X(x + h)),$$

since the dependence between $X(x)$ and $X(y)$ will only depend on the distance between x and y. For some models of continuous max-stable random fields found in literature (Smith, Schlather, Brown-Resnick, Extremal-t), an expression for $\underline{\theta}(h)$ is available.
The coefficients of tail dependence and the extremal coefficients can be considered dual when we study their variation with the concordance of the variables: when concordance increases, the bivariate upper-tail dependence rises and the extremal coefficients fall. The proposed $\zeta(A)$ coefficients increase with increasing local concordance of the random field variables, as can easily be seen if we express them from the extremal coefficients. Before we establish the properties that justify the utility and interpretation of the proposed crossinggram, we first present a representation for it through the extremal coefficients, which will also motivate their estimation.

Observe that

$$\lambda(V(x)|A) = \lim_{u\uparrow 1} \frac{P\left(\bigvee_{y\in V(x)} F(X(y)) > u\right)}{P\left(\bigvee_{y\in A} F(X(y)) > u\right)} = \lim_{u\uparrow 1} \frac{1-u^{\theta(V(x))}}{1-u^{\theta(A)}} = \frac{\theta(V(x))}{\theta(A)}.$$

Simple calculations allow then to obtain the following representation for the crossinggram, from the proposition 2.1.

Proposition 3.1. *The crossinggram $\zeta(A)$, $A \subset \mathbb{Z}^2$, for the max-stable random field $\{X(x),\ x \in \mathbb{Z}^2\}$ satisfies*

$$\zeta(A) = \frac{\mathcal{V}(A) - \sum_{x\in A} \theta(V(x))}{\mathcal{V}(A) - |A|}, \tag{3}$$

where $\mathcal{V}(A) = \sum_{x\in A} |V(x)|$.

If $\{X(x),\ x \in \mathbb{Z}^2\}$ is isotropic, stationary and all $V(x)$, $x \in A$, have the same cardinality, then

$$\zeta(A) = \frac{|V(x_0)| - \theta(V(x_0))}{|V(x_0)| - 1}$$

for some $x_0 \in A$.
Several extremal coefficients type-functionals can be defined as indicators of dependence of the variables over A but this is not de purpose of the proposition 3.1. For instance, $\gamma(A) = \frac{\theta(A)-1}{|A|-1}$ captures the overall dependence of the variables X_i, $i \in A$, but does not incorporate the discrepancy between local dependences neither the propensity for local upcrossings. The expression (3) is a usefull representation for the crossinggram. The insight to $\zeta(A)$ is not available from this representation rather from its definition.
We will apply the above proposition to derive the next properties and propose an estimator, beyond the moment estimator suggested from the definition.

Proposition 3.2. *The crossinggram $\zeta(A)$, $A \subset \mathbb{Z}^2$, for the max-stable random field $\{X(x),\ x \in \mathbb{Z}^2\}$ satisfies*

(i) $\zeta(A) \in [0,1]$.

(ii) $\zeta(A) = 0$ *if and only if the variables of* $\{X(x),\ x \in A\}$ *are independent;*

(iii) $\zeta(A) = 1$ *if and only if the variables of* $\{X(x),\ x \in A\}$ *are totally dependent;*

(iv) $\zeta(A)$ *increases with the concordance between the variables of* $\{X(x),\ x \in A\}$.

Proof. (i) The statement results from $1 \le \theta(V(x)) \le |V(x)|$, $\forall x \in A$.
(ii) $\zeta(A) = 0 \Leftrightarrow \theta(V(x)) = |V(x)|$, $\forall x \in A$, which occurs if and only if, variables $X(x)$, $x \in A$, are independent.
(iii) $\zeta(A) = 1 \Leftrightarrow \theta(V(x)) = 1$, $\forall x \in A$, which occurs if and only if, variables $X(x)$, $x \in A$, are

totally dependent.
(iv) Suppose that the variables of $\{Y(x),\ x\in A\}$ are more concordant than those of $\{X(x),\ x\in A\}$. This means that, for any $z(x)\in[0,1]$, with $x\in A$,

$$P\left(\bigcap_{x\in A}\{F(Y(x))>z(x)\}\right)\geq P\left(\bigcap_{x\in A}\{F(X(x))>z(x)\}\right)$$

and

$$P\left(\bigcap_{x\in A}\{F(Y(x))\leq z(x)\}\right)\geq P\left(\bigcap_{x\in A}\{F(X(x))\leq z(x)\}\right).$$

Then (Shaked and Shanthikumar [22] 2007) we have

$$E\left(\bigvee_{y\in V(x)}F(Y(y))\right)\leq E\left(\bigvee_{y\in V(x)}F(X(y))\right)$$

and

$$\frac{E\left(\bigvee_{y\in V(x)}F(Y(y))\right)}{1-E\left(\bigvee_{y\in V(x)}F(Y(y))\right)}\leq\frac{E\left(\bigvee_{y\in V(x)}F(X(y))\right)}{1-E\left(\bigvee_{y\in V(x)}F(X(y))\right)},$$

that is, by (1) and (2), $\theta^{(Y)}(V(x))\leq\theta^{(X)}(V(x))$, $\forall x\in A$. Thus, from the previous proposition it results $\zeta^{(Y)}(A)\geq\zeta^{(X)}(A)$, where the upper indexes distinguish the fields to which the coefficients refer. □

3.1 Estimation of $\zeta(A)$

We recall that the extremal coefficient corresponds to the stable tail dependence function at the unit vector and thus, considering (1), we have

$$\theta(x_1,...,x_d)=\ell_{x_1,...,x_d}(1,...,1)=\frac{E\left(\bigvee_{i=1}^{d}U(x_i)\right)}{1-E\left(\bigvee_{i=1}^{d}U(x_i)\right)}=\frac{1}{1-E\left(\bigvee_{i=1}^{d}U(x_i)\right)}-1,\qquad(4)$$

where $U(x_i)=F(X(x_i))$, $i=1,\ldots,d$. Ferreira and Ferreira ([8] 2012a) presented an estimator for $\theta(x_1,...,x_d)$ by taking the sample mean in place of the expected value. More precisely,

$$\hat{\theta}(x_1,...,x_d)=\frac{1}{1-\frac{1}{n}\sum_{j=1}^{n}\bigvee_{i=1}^{d}\hat{U}_j(x_i)}-1,$$

with

$$\hat{U}_j(x_i)=\hat{F}(X_j(x_i))=\frac{1}{n+1}\sum_{l=1}^{n}\mathbf{1}_{\{X_l(x_i)\leq X_j(x_i)\}},$$

where $\hat{F}$ is a modified empirical distribution function, more convenient for high order statistics (Beirlant *et al.* [1], 2004). The performance of $\hat{\theta}(x_1,...,x_d)$ was analyzed in Ferreira ([12] 2018).

Strong consistency was also addressed in Ferreira and Ferreira ([8] 2012a). See also Foseca *et al.* ([13] 2014). Here we consider an estimator of the crossigram $\zeta(A)$ based on $\hat{\theta}(x_1, ..., x_d)$. More precisely, consider $\{X_j(x),\, x \in A\}$, $j = 1, ..., n$, a random sample coming from $\{X(x),\, x \in A\}$. Given (3) and (4), we propose

$$\hat{\zeta}(A) = \frac{\mathcal{V}(A) - \sum_{x \in A} \hat{\theta}(V(x))}{\mathcal{V}(A) - |A|}, \tag{5}$$

where

$$\hat{\theta}(V(x)) = \frac{1}{1 - \frac{1}{n}\sum_{j=1}^{n} \bigvee_{y \in V(x)} \hat{U}_j(y)} - 1,$$

and

$$\hat{U}_j(y) = \hat{F}(X_j(y)) = \frac{1}{n+1}\sum_{l=1}^{n} \mathbf{1}_{\{X_l(y) \leq X_j(y)\}}\,.$$

Observe that $\hat{\theta}(V(x))$ is a rank based estimator since it involves an empirical estimator $\hat{F}$ of the unknown marginal distribution function. The procedure to deduce the asymptotic behavior is usually based on the convergence of the empirical copula process to a limit approaching a centered Gaussian process, under some conditions mainly on the continuity and partial derivatives of the copula process. See, e.g., Ferreira and Ferreira ([8], 2012), Gudendorf and Segers ([14], 2012), Bücher *et al.* ([3], 2017) and references therein. The histogram of sample estimates obtained in Figure 2 corroborates the Gaussian behavior.

3.2 The crossinggram outside the max-stable context

For a random field $\{X(x),\, x \in \mathbb{Z}^2\}$, not necessarily max-stable, we can define $\zeta(A)$ as previously

$$\zeta(A) = 1 - \lim_{u \uparrow 1} \frac{E\left(\sum_{x \in A} \mathbf{1}_{\{U(x) \leq u < \bigvee_{y \in V(x)} U(y)\}} \Big| \sum_{x \in A} \mathbf{1}_{\{U(x) > u\}} > 0\right)}{E\left(\sum_{x \in A} \sum_{y \in V(x) - \{x\}} \mathbf{1}_{\{U(y) > u\}} \Big| \sum_{x \in A} \mathbf{1}_{\{U(x) > u\}} > 0\right)},$$

with $U(x) = F_{X(x)}(X(x))$, $x \in \mathbb{Z}^2$, provided the limit exists. We can consider different marginals and the relationship with the tail dependence coefficients remains valid. However, we don't have the relation between the tail dependence coefficients λ and the extremal coefficients θ, and therefore Proposition 3.1 is not valid. Consequently, the estimation method proposed for $\zeta(A)$ can not be used. The estimation of the coefficients could be done through the moment estimation for the expectations in its definition, or estimation methods for tail dependence coefficients, already mentioned.

Since bivariate tail dependence coefficients can be more easily computed and estimated than multivariate tail dependence coefficients, we propose now a smaller measure $\zeta^*(A) \leq \zeta(A)$ dependent only on bivariate marginal distributions. Its main drawback is to not take into account joint exceedances of u in $V(x)$ and, for isotropic and stationary random fields, it reduces to bivariate λ. Its advantages over $\zeta(A)$ are the availability of several models for bivariate tail dependence in the literature and a simpler estimation.

Definition 3.1. *The crossinggram $\zeta^*(A)$, $A \subset \mathbb{Z}^2$, for the random field $\{X(x), x \in \mathbb{Z}^2\}$ is defined by*

$$\zeta^*(A) = 1 - \lim_{u \uparrow 1} \frac{E\left(\sum_{x\in A}\sum_{y\in V(x)-\{x\}} \mathbf{1}_{\{U(x)\leq u<U(y)\}} \middle| \sum_{x\in A} \mathbf{1}_{\{U(x)>u\}} > 0\right)}{E\left(\sum_{x\in A}\sum_{y\in V(x)-\{x\}} \mathbf{1}_{\{U(y)>u\}} \middle| \sum_{x\in A} \mathbf{1}_{\{U(x)>u\}} > 0\right)},$$

where $U(x) = F_{X(x)}(X(x))$, $x \in \mathbb{Z}^2$, provided the limit exists.

4 Example

Let $\{Y(x), x \in \mathbb{Z}^2\}$ be a random field with independent variables and independent of the random variable R. Suppose that $F_{Y(x)}(z) = F_R(z) = \exp(-z^{-1})$, $z > 0$, and that $\{\beta_1, \ldots, \beta_k\}$ is a family of constants in $(0, 1]$.
For a fixed partition $\mathcal{P} = \{A_1, \ldots, A_k\}$ of $\mathbb{Z}^2$, we define

$$X(x) = Y(x)\beta(x) \vee R(1 - \beta(x)),\ x \in \mathbb{Z}^2,\ n \geq 1,$$

with $\beta(x) = \sum_{i=1}^k \beta_i \mathbb{1}_{A_i}(x)$, $x \in \mathbb{Z}^2$.
We have $F_{X(x)}(z) = P(X(x) \leq z) = e^{-z^{-1}}$ and, for any choice of locations $x_1, \ldots, x_d$,

$$\begin{aligned} F_{(X(x_1),\ldots,X(x_d))}(z_1, \ldots, z_d) &= P\left(\bigcap_{j=1}^d \{X(x_j) \leq z_j\}\right) \\ &= \exp\left(-\sum_{j=1}^d z_j^{-1}\beta(x_j) - \bigvee_{j=1}^d z_j^{-1}(1 - \beta(x_j))\right). \end{aligned}$$

The copula function of $(X(x_1), \ldots, X(x_d))$ is given by

$$\begin{aligned} C_{x_1,\ldots,x_d}(u_1, \ldots, u_d) &= \sum_{(i_1,\ldots,i_d)\in\{1,\ldots,k\}^d} \mathbb{1}_{A_{i_1}\times\cdots\times A_{i_d}}(x_1, \ldots, x_d) \cdot \prod_{j=1}^d u_j^{\beta_{i_j}} \bigwedge_{j=1}^d u_j^{(1-\beta_{i_j})} \\ &= \prod_{j=1}^d u_j^{\beta(x_j)} \bigwedge_{j=1}^d u_j^{(1-\beta(x_j))}, \end{aligned} \tag{6}$$

which is max-stable. We remark that, if locations $x_1, \ldots, x_d$ belong to the same region A_s, we have

$$C_{x_1,\ldots,x_d}(u_1,\ldots,u_d) = \prod_{j=1}^{d} u_j^{\beta_s} \bigwedge_{j=1}^{d} u_j^{(1-\beta_s)} = \left(\prod_{j=1}^{d} u_j\right)^{\beta_s} \left(\bigwedge_{j=1}^{d} u_j\right)^{(1-\beta_s)},$$

which is a geometric mean of the product copula and the minimum copula.
In general, if $x_1 \in A_{i_1}, \ldots, x_d \in A_{i_d}$, we have

$$C_{x_1,\ldots,x_d}(u_1,\ldots,u_d) = C_{\prod}\left(u_1^{\beta_{i_1}},\ldots,u_d^{\beta_{i_d}}\right) C_{\bigwedge}\left(u_1^{1-\beta_{i_1}},\ldots,u_d^{1-\beta_{i_d}}\right),$$

where $C_{\prod}$ and $C_{\bigwedge}$ respectively denote the copulas of vectors with independent and totally dependent marginals.
From (6) we obtain, for $V(x) = \{x_1, \ldots, x_d\}$,

$$\theta(V(x)) = \sum_{y\in V(x)} \beta(y) + \bigvee_{y\in V(x)} (1-\beta(y)).$$

In particular,

$$\theta(x_1,x_2) = \begin{cases} 1+\beta_s & \text{, if } x_1,x_2 \in A_s \\ 1+\beta_s \vee \beta_{s'} & \text{, if } x_1 \in A_s, x_2 \in A_{s'}. \end{cases}$$

The expression of $\theta(x_1, x_2)$ suggests an estimation method for the model constants β_i, $i = 1, \ldots, k$.
For each i, if we choose two locations x_1 and x_2 in A_i, we have

$$\widehat{\beta}_i = \widehat{\theta}(x_1,x_2) - 1,$$

where $\widehat{\theta}(x_1, x_2)$ can be obtained as we proposed in Section 3.1.
From the expression

$$\theta(V(x)) = 1 + \sum_{y\in V(x)} \beta(y) - \bigwedge_{y\in V(x)} \beta(y),$$

we can also conclude that, in this model, $\theta(V(x)) \in]1, |V(x)|]$, thus excluding total dependence.
For $A \subset \mathbb{Z}^2$, by applying the proposition 3.1, we obtain

$$\begin{aligned} \zeta(A) = & \frac{\mathcal{V}(A) - \sum_{x\in A}\left(\sum_{y\in V(x)}\beta(y) + \bigvee_{y\in V(x)}(1-\beta(y))\right)}{\mathcal{V}(A) - |A|} \\ = & \ 1 - \frac{\sum_{x\in A}\sum_{y\in V(x)}\beta(y) - \sum_{x\in A}\bigwedge_{y\in V(x)}\beta(y)}{\mathcal{V}(A) - |A|}. \end{aligned}$$

Consider, to easily illustrate this crossinggram, regions $A \subset A_i$ and such that $V(x) \subset A_i$ for each $x \in A$. Then, for these regions, we get $\theta(V(x)) = 1 + (|V(x)| - 1)\beta_i$ and $\zeta(A) = 1 - \beta_i$. Therefore, the lower the beta the greater the dependence between variables on A (smaller θ) and

the greater the crossinggram value , which indicates a smaller proportion of upcrossings among exceedances of high levels. This result agrees with what we would expect from the definition of the random field since lower β value potentiates the leveling effect of the factor R.
We simulate this random field, for $\mathcal{P} = \cup_{i=1}^{3} A_i$, with $A_i = \{(x, y) \in \mathbb{Z}^2 : d_{i-1}^2 \leq x^2 + y^2 < d_i^2\}$, $i = 1, 2, 3$, $d_0 = 0$, $d_1 = 12$, $d_2 = 34$, $d_3 = +\infty$, $\beta_1 = 8/10$, $\beta_2 = 6/10$, $\beta_3 = 1/10$, over $C = \{(x, y) \in \mathbb{Z}^2 : x^2 + y^2 < 50^2\}$. The generated trajectory can be seen in Figure 1 (left) and the respective projection on the plane of the x and y axes (right). Despite the plots can't provide a quantitative information for the intensity of local upcrossings we can see over $A_1 \cap C$ a more rugged trajectory, corresponding to a $\zeta(A_1 \cap C) = \frac{2}{10}$, and a smoother trajectory over $A_3 \cap C$ corresponding to $\zeta(A_3 \cap C) = \frac{9}{10}$.

To briefly illustrate the behavior of estimator (5), we have considered 500 replicates of size $n = 1000$ of the random field and computed $\hat{\zeta}(A_1 \cap C)$. The histogram of the obtained estimates is in Figure 2. The bias is -0.0046 and the standard deviation 0.0222.

5 Conclusion

In the trajectories of a random field on a lattice, the propensity for oscillations, meaning the proportion of exceedances of high levels which are upcrossings, is inversely related to the degree of dependence and concordance between the random variables that generate it. We intended to quantify this propensity through coefficients that are easy to estimate and use in applications. We defined a crossinggram for max-stable random fields on lattices, which take values in [0,1] and are larger the more dependent and concordant the variables in the field are. These coefficients are related to the extremal coefficients usually found in the literature of extreme values. They also have a representation from the expected values of local maxima of the random field. This representation motivates the estimation method proposed and applied. The coefficients range from 0 to 1, where 0 represents a very local rough random field and 1 maximum local smoothness. We propose an intuitive and simple estimator that can be used in practical applications. The new coefficients give good insight into the propensity for upcrossings by a max-stable random field from the theoretical point of view and in the example considered. They are easy to estimate and can be widely used.

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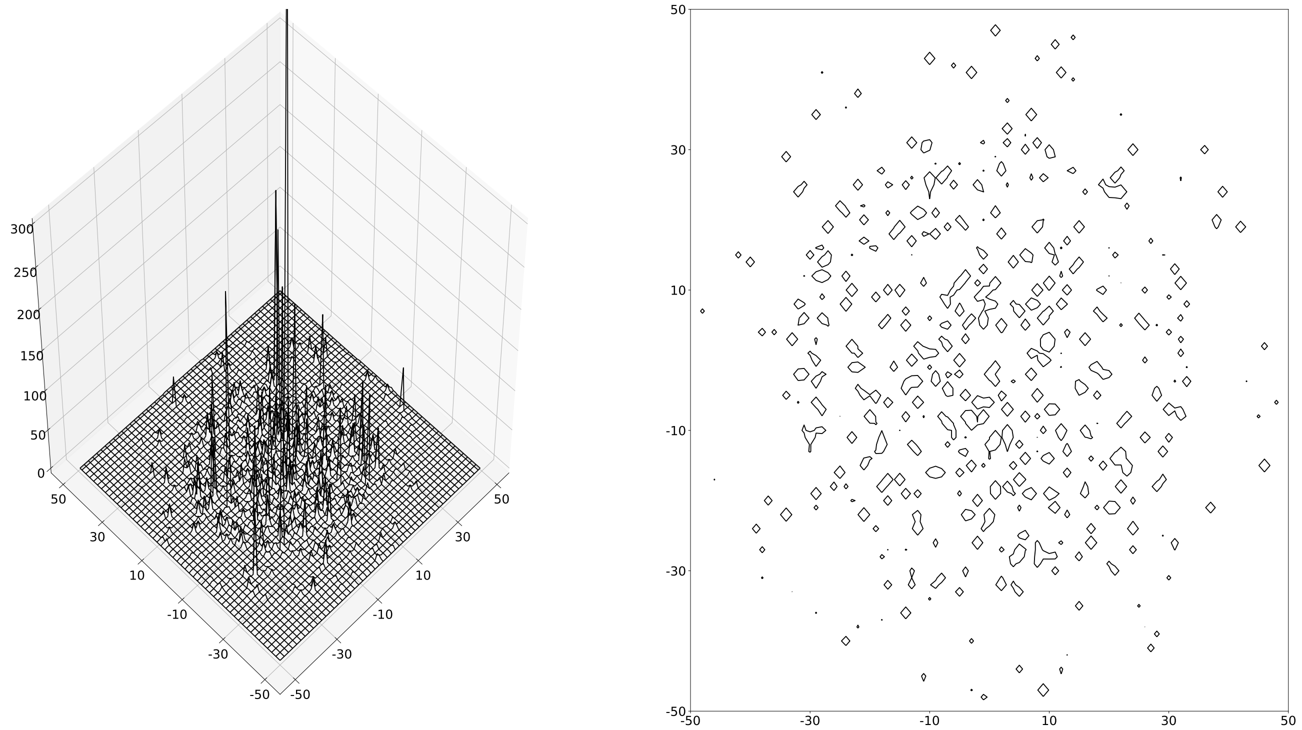

Figure 1: The simulated random field over $C = \{(x, y) \in \mathbb{Z}^2 : x^2 + y^2 \leq 50^2\}$.

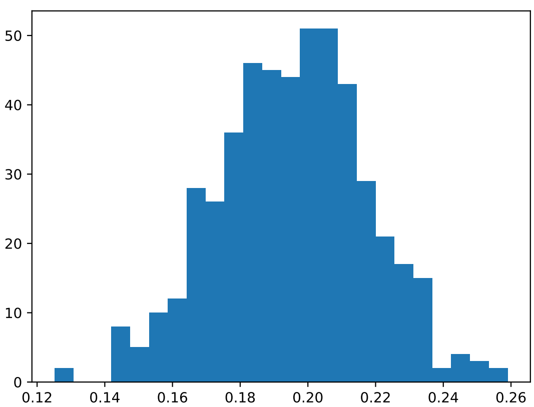

Figure 2: Histogram of 500 simulated estimates of $\zeta(A_1 \cap C) = \frac{2}{10}$, based on samples of size $n = 1000$ of the random field in Example 4.