Dynamics of the coquaternionic maps $x^2 + \mathrm{b}\, x$

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Abstract

This paper deals with the dynamics of the one-parameter family of coquaternionic quadratic maps $x^2 + \mathrm{b}x$. By making use of recent results for the zeros of one-sided coquaternionic polynomials, the fixed points are analytically determined. The stability of these fixed points is also addressed, where, in some cases, due to the appearance of sets of non-isolated points, a suitably adapted notion of stability is used. The results obtained show clearly that this family is not dynamically equivalent to the simpler family $x^2 + \mathrm{c}$ previously studied by the authors. Some numerical examples of other dynamics beyond fixed points are also presented.

1 Introduction

Discrete dynamical systems have played an important role in science, by providing very simple models of time evolution for phenomena appearing in many different fields, such as biology, demography, ecology, economics, engineering, finance, and physics.

The dynamics of the quadratic maps in the complex plane, one of the most recognized complex dynamics families, has been object of intense study in the last decades and is now considered a well-established theory [5, 26, 28].

The first and natural attempts to extend this theory to higher dimensions were done in a quaternion framework [6, 7, 10, 18, 20, 21, 22, 25, 29, 35, 36]. The fact that the new results obtained in the quaternionic context appear to be very closely related to the corresponding ones in the complex case does not mean that the use of another four-dimensional hypercomplex real algebra may not lead to interesting and surprising results.

One such algebra is the algebra of coquaternions, also known in the literature as split quaternions, introduced in 1849 by the English mathematician James Cockle [11]. In recent years one can observe an emerging interest among mathematicians and physicists on the study of these hypercomplex numbers [1, 2, 8, 9, 17, 19, 23, 24, 27, 30, 31, 32, 34, 37, 33, 39, 40].

According to Avner Friedman [16], "*While we can expect that established methods in mathematical sciences will be of immediate use, the quantitative analysis of fundamental problems in bioscience will undoubtedly require new ideas and new techniques.*". It is our conviction that the dynamics of coquaternionic maps may be one such technique. From the richness of results already obtained with some preliminary results, we believe

that this area of research will be useful for scientists in general, allowing them to compare and recognize the properties of the time evolution phenomena they study with the characteristics found for the coquaternionic dynamics. It should be noticed that, since the algebra of coquaternions is isomorphic to the algebra of two-by-two real matrices, the actual context of this kind of discrete dynamical systems is not as strange as one may think.

The authors considered in a previous study [13] the family of coquaternionic quadratic maps of the simple form $x^2 + \mathsf{c}$. The fixed points and periodic points of period two of this family of maps were determined and an interesting feature of the coquaternionic dynamics was observed: the appearance of sets of non-isolated such points. For this type of sets, the usual concept of stability has to be appropriately adapted. It turns out that none of the sets of non-isolated fixed points of the map $x^2 + \mathsf{c}$ are attractive[1] (even in this new sense).

As it is well-known, to study the dynamics of complex quadratic maps we only have to consider the particular family of maps of the form $x^2 + c$, since any quadratic map is dynamically equivalent to a member of this family. In the coquaternionic case, the situation is totally different.

We first observe that, due to the non-commutativity of the product of coquaternions, the sum of two mth degree monomials $\mathsf{a}_0 x \mathsf{a}_1 x \cdots \mathsf{a}_{m-1} x \mathsf{a}_m$ and $\mathsf{a}'_0 x \mathsf{a}'_1 x \cdots \mathsf{a}'_{m-1} x \mathsf{a}'_m$ can not be written simply in the form $\mathsf{A}_0 x \mathsf{A}_1 x \cdots \mathsf{A}_{m-1} x \mathsf{A}_m$ and hence, the general expression of a quadratic coquaternionic polynomial is

$$\sum_{j=1}^{n} \mathsf{a}_0^j \, x \, \mathsf{a}_1^j \, x \, \mathsf{a}_2^j + \sum_{j=1}^{k} \mathsf{b}_0^j \, x \, \mathsf{b}_1^j + \mathsf{c}, \quad n, k \in \mathbb{N},$$

with $\mathsf{a}_i^j, \mathsf{b}_i^j$ and c coquaternions. Not surprisingly, contrary to what happens in the commutative case, no conjugacy equivalence of a quadratic coquaternionic polynomial to a simple form is available. In this paper we consider the family of coquaternionic maps of the form $x^2 + \mathsf{b}x$. It is important to observe that, unless $\mathsf{b} \in \mathbb{R}$, this type of map is not conjugate to any map of the form $x^2 + \mathsf{c}$, i.e. to the type of map previously studied [13]. By making use of recent results on the zeros of one-sided coquaternionic polynomials [15], we are able to fully characterize the sets of fixed points of the map $x^2 + \mathsf{b}x$. In particular, we find sets of non-isolated points with a different nature from the ones obtained for the map $x^2 + \mathsf{c}$; in the case of the map $x^2 + \mathsf{c}$, the sets of non-isolated fixed points appear only when $\mathsf{c} \in \mathbb{R}$ and always form hyperboloids (in a certain hyperplane), while in the present case of the map $x^2 + \mathsf{b}x$, for some non-real choices of the parameter b, we obtain lines of fixed points; and, more interestingly, some of these lines turn out to be attractive sets of points.

The rest of the paper is organized as follows: Section 2 contains a revision of the main definitions and results on the algebra of coquaternions and on the zeros of unilateral coquaternionic polynomials. Section 3 is dedicated to the determination and stability analysis of the fixed points of the coquaternionic maps $x^2 + \mathsf{b}x$ and contains the main results of the paper; Section 4 presents some numerical examples of dynamics beyond fixed points and Section 5 concludes.

2 Preliminary results

In this section, we briefly revise the main concepts and results, concerning the algebra of coquaternions, needed for the rest of the paper. More details can be found in previous studies [12, 13, 15].

2.1 The algebra $\mathbb{H}_{\mathrm{coq}}$

Let $\{1, \mathsf{i}, \mathsf{j}, \mathsf{k}\}$ be an orthonormal basis of the Euclidean vector space $\mathbb{R}^4$ with a product given according to the multiplication rules

$$\mathsf{i}^2 = -1, \ \mathsf{j}^2 = \mathsf{k}^2 = 1, \ \ \mathsf{ij} = -\mathsf{ji} = \mathsf{k}.$$

This non-commutative product generates the algebra of real coquaternions, which we will denote by $\mathbb{H}_{\mathrm{coq}}$. We will identify the space $\mathbb{R}^4$ with $\mathbb{H}_{\mathrm{coq}}$ by associating the element $(q_0, q_1, q_2, q_3) \in \mathbb{R}^4$ with the coquaternion $\mathsf{q} = q_0 + q_1\mathsf{i} + q_2\mathsf{j} + q_3\mathsf{k}$. Given $\mathsf{q} = q_0 + q_1\mathsf{i} + q_2\mathsf{j} + q_3\mathsf{k} \in \mathbb{H}_{\mathrm{coq}}$, its *conjugate* $\overline{\mathsf{q}}$ is defined as $\overline{\mathsf{q}} = q_0 - q_1\mathsf{i} - q_2\mathsf{j} - q_3\mathsf{k}$; the number q_0 is called the *real part* of q and is denoted by $\mathrm{re}\,\mathsf{q}$ and the *vector part* of q, denoted by $\mathrm{vec}\,\mathsf{q}$, is $\mathrm{vec}\,\mathsf{q} = q_1\mathsf{i} + q_2\mathsf{j} + q_3\mathsf{k}$. We will identify the set of coquaternions whose vector part is zero with the set $\mathbb{R}$ of real

[1] This is no longer true, however, for the case of periodic points of period two.

numbers. We call *determinant* of q and denote by $\det \mathsf{q}$ the quantity given by $\det \mathsf{q} = \mathsf{q}\,\overline{\mathsf{q}} = q_0^2 + q_1^2 - q_2^2 - q_3^2$. Not all non-zero coquaternions are invertible. It can be shown that a coquaternion q is invertible (also referred to as nonsingular) if and only if $\det \mathsf{q} \neq 0$. In that case, we have, $\mathsf{q}^{-1} = \frac{\overline{\mathsf{q}}}{\det \mathsf{q}}$.

We also recall the concepts of similarity and quasi-similarity for coquaternions. We say that a coquaternion q is *similar* to a coquaternion p if there exists an invertible $\mathsf{h} \in \mathbb{H}_{\mathrm{coq}}$ such that $\mathsf{q} = \mathsf{h}^{-1}\mathsf{p}\mathsf{h}$. This is an equivalence relation in $\mathbb{H}_{\mathrm{coq}}$, partitioning $\mathbb{H}_{\mathrm{coq}}$ in the so-called *similarity classes*.

We say that two elements $\mathsf{p}, \mathsf{q} \in \mathbb{H}_{\mathrm{coq}}$ are *quasi-similar* if and only if $\mathrm{re}\,\mathsf{p} = \mathrm{re}\,\mathsf{q}$ and $\det \mathsf{p} = \det \mathsf{q}$ (or, equivalently, if $\mathrm{re}\,\mathsf{p} = \mathrm{re}\,\mathsf{q}$ and $\det(\mathrm{vec}\,\mathsf{p}) = \det(\mathrm{vec}\,\mathsf{q})$). This is also an equivalence relation in $\mathbb{H}_{\mathrm{coq}}$; the class of an element $\mathsf{q} \in \mathbb{H}_{\mathrm{coq}}$ with respect to this relation is denoted by $[\![\mathsf{q}]\!]$ and referred to as the *quasi-similarity class* of q. It can be shown that for two non-real coquaternions, the concepts of quasi-similarity and similarity coincide, i.e. two non-real coquaternions are similar if and only if they have the same real part and the same determinant. However, if $\mathsf{q} = q_0 \in \mathbb{R}$, then q is only similar to itself but quasi-similar to all the coquaternions p of the form $\mathsf{p} = q_0 + \mathrm{vec}\,\mathsf{p}$ with $\det(\mathrm{vec}\,\mathsf{p}) = 0$. Observe that the quasi-similarity class of a coquaternion q is given by

$$[\![\mathsf{q}]\!] = \left\{x_0 + x_1\mathsf{i} + x_2\mathsf{j} + x_3\mathsf{k} : x_0 = q_0 \text{ and } x_1^2 - x_2^2 - x_3^2 = \det(\mathrm{vec}\,\mathsf{q})\right\}$$

and can, therefore, be identified with a hyperboloid in the hyperplane $\{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4 : x_0 = q_0\}$: a hyperboloid of two sheets if $\det(\mathrm{vec}\,\mathsf{q}) > 0$, a hyperboloid of one sheet if $\det(\mathrm{vec}\,\mathsf{q}) < 0$ and a degenerate hyperboloid, i.e. a cone, if $\det(\mathrm{vec}\,\mathsf{q}) = 0$.

2.2 Unilateral coquaternionic polynomials

We now present very briefly some results on the zeros of coquaternionic polynomials [15]. We consider only monic unilateral left polynomials, i.e. polynomials of the form

$$P(x) = x^n + \mathsf{a}_{n-1}x^{n-1} + \cdots + \mathsf{a}_1 x + \mathsf{a}_0,\ \mathsf{a}_i \in \mathbb{H}_{\mathrm{coq}}, \tag{2.1}$$

with addition and multiplication of such polynomials defined as in the commutative case where the variable is allowed to commute with the coefficients.

Given a quasi-similarity class $[\![\mathsf{q}]\!] = [\![q_0 + \mathrm{vec}\,\mathsf{q}]\!]$, the *characteristic polynomial* of $[\![\mathsf{q}]\!]$, denoted by $\Psi_{[\![\mathsf{q}]\!]}$, is the polynomial given by[2]

$$\Psi_{[\![\mathsf{q}]\!]}(x) = x^2 - 2q_0\,x + \det \mathsf{q}.$$

This is a second degree monic polynomial with real coefficients with discriminant $\Delta = -4\det(\mathrm{vec}\,\mathsf{q})$. Hence, $\Psi_{[\![\mathsf{q}]\!]}$ has two complex conjugate roots, if $\det(\mathrm{vec}\,\mathsf{q}) > 0$, and is a polynomial of the form $(x-r_1)(x-r_2)$, with $r_1, r_2 \in \mathbb{R}$, if $\det(\mathrm{vec}\,\mathsf{q}) \leq 0$. Reciprocally, any second degree monic polynomial $S(x)$ with real coefficients is the characteristic polynomial of a uniquely defined quasi-similarity class; if $S(x)$ is irreducible with two complex conjugate roots α and $\overline{\alpha}$, then $S = \Psi_{[\![\alpha]\!]}$; if S has real roots r_1 and r_2 (with, eventually, $r_1 = r_2$), then $S = \Psi_{[\![\mathsf{q}]\!]}$ with $\mathsf{q} = \frac{r_1+r_2}{2} + \frac{r_1-r_2}{2}\mathsf{j}$.

We say that $\mathsf{z} \in \mathbb{H}_{\mathrm{coq}}$ is a *zero* (or a root) of the polynomial P if $P(\mathsf{z}) = 0$ and we denote by $Z(P)$ the *zero set* of P, i.e. the set of all the zeros of P.

Given a polynomial P of the form (2.1), its *conjugate polynomial* is the polynomial defined by $\overline{P}(x) = x^n + \overline{\mathsf{a}}_{n-1}x^{n-1} + \cdots + \overline{\mathsf{a}}_1 x + \overline{\mathsf{a}}_0$ and its *companion polynomial* is the polynomial given by $\mathcal{C}_P(x) = P(x)\overline{P}(x)$.

The following theorem contains an important result relating the characteristic polynomials of the quasi-similarity classes of zeros of a given polynomial P and the companion polynomial of P[15].

Theorem 2.1. *Let P be a polynomial of the form (2.1). If $\mathsf{z} \in \mathbb{H}_{\mathrm{coq}}$ is a zero of P, then $\Psi_{[\![\mathsf{z}]\!]}$ is a divisor of $\mathcal{C}_P$.*

It can be shown easily that $\mathcal{C}_P$ is a polynomial of degree $2n$ with real coefficients and, as such, considered as a polynomial in $\mathbb{C}$, has $2n$ roots. If these roots are $\alpha_1, \overline{\alpha}_1, \ldots, \alpha_m, \overline{\alpha}_m \in \mathbb{C} \setminus \mathbb{R}$ and $r_1, r_2, \ldots, r_\ell \in \mathbb{R}$, where $\ell = 2(n-m)$, $(0 \leq m \leq n)$, then it is easy to conclude that the characteristic polynomials which divide $\mathcal{C}_P$ are the ones associated with the following quasi-similarity classes:

$$[\![\alpha_k]\!];\ k = 1, \ldots, m, \tag{2.2a}$$

[2] This polynomial is more commonly referred to as the characteristic polynomial of the coquaternion q. Since this polynomial is an invariant of the class, we find it more convenient to adopt our denomination.

$$[\![\mathsf{r}_{ij}]\!];\ i = 1, \ldots, \ell - 1,\ j = i + 1, \ldots, \ell, \tag{2.2b}$$

with

$$\mathsf{r}_{ij} = \frac{r_i + r_j}{2} + \frac{r_i - r_j}{2}\mathsf{j}. \tag{2.2c}$$

We thus have the following result concerning the zero set of P:

Theorem 2.2. *Let P be a polynomial of the form (2.1). Then:*

$$Z(P) \subseteq \bigcup_k [\![\alpha_k]\!] \bigcup_{i,j} [\![\mathsf{r}_{ij}]\!],$$

where $[\![\alpha_k]\!]$ and $[\![\mathsf{r}_{ij}]\!]$ are the quasi-similarity classes defined by (2.2).

We call the classes given by (2.2) the *admissible classes* (with respect to the zeros) of the polynomial P.

The results given in the following theorem show how to find the set of zeros of P belonging to a given admissible class [15].

Theorem 2.3. *Let $P(x)$ be a polynomial of the form (2.1) and let $[\![\mathsf{q}]\!] = [\![q_0 + \operatorname{vec}\mathsf{q}]\!]$ be a given admissible class of $P(x)$. Also, let $\mathsf{A} + \mathsf{B}x$, with $\mathsf{B} = B_0 + B_1\mathsf{i} + B_2\mathsf{j} + B_3\mathsf{k}$, be the remainder of the right division of $P(x)$ by the characteristic polynomial of $[\![\mathsf{q}]\!]$.*[3]

1. *If $\det \mathsf{B} \neq 0$, then $[\![\mathsf{q}]\!]$ contains only one zero of P, given by $\mathsf{z} = -\mathsf{B}^{-1}\mathsf{A}$.*
2. *If $\mathsf{A} = \mathsf{B} = 0$, then $[\![\mathsf{q}]\!] \subseteq Z(P)$.*
3. *If $\mathsf{B} \neq 0, \det \mathsf{B} = 0$ and the equation $\mathsf{A} + \mathsf{B}x = 0$ has a real solution γ_0 satisfying*

$$(q_0 - \gamma_0)^2 = -\det(\operatorname{vec}\mathsf{q}), \tag{2.3}$$

then the zeros of P in $[\![\mathsf{q}]\!]$ form the following line in the hyperplane $x_0 = q_0$,

$$\mathscr{L} = \{q_0 + \alpha\mathsf{i} + (k_2\alpha + k_1(q_0 - \gamma_0))\,\mathsf{j} + (-k_1\alpha + k_2(q_0 - \gamma_0))\,\mathsf{k} : \alpha \in \mathbb{R}\}, \tag{2.4a}$$

with k_1 and k_2 given by

$$k_1 = -\frac{B_0B_2 + B_1B_3}{B_0^2 + B_1^2} \quad \text{and} \quad k_2 = \frac{B_1B_2 - B_0B_3}{B_0^2 + B_1^2}. \tag{2.4b}$$

4. *If $\mathsf{B} \neq 0, \det \mathsf{B} = 0$ and the equation $\mathsf{A} + \mathsf{B}x = 0$ has a nonreal solution $\gamma = \gamma_0 + \gamma_1\mathsf{i}$, then the class $[\![\mathsf{q}]\!]$ contains only one zero of P, given by*

$$\mathsf{z} = q_0 + (\beta + \gamma_1)\mathsf{i} + (k_2\beta + k_1(q_0 - \gamma_0))\,\mathsf{j} + (-k_1\beta + k_2(q_0 - \gamma_0))\,\mathsf{k},$$

where

$$\beta = \frac{\det(\operatorname{vec}\mathsf{q}) + (q_0 - \gamma_0)^2 - \gamma_1^2}{2\gamma_1}$$

and k_1 and k_2 are given by (2.4b).

5. *If none of the above conditions holds, then there are no zeros of P in $[\![\mathsf{q}]\!]$.*

In cases (1) and (4), we say that the zero z is an *isolated zero* of P; in case (2), we say that the class $[\![\mathsf{q}]\!]$ (or any of its elements) is a *hyperboloidal zero* of P and in case (3) we call the line $\mathscr{L}$ (or any of its elements) a *linear zero* of P.

[3] Since the product of two polynomials in $\mathbb{H}_{\mathrm{coq}}$ is defined in the usual manner, we can use the "Euclidean Division Algorithm" to perform the division of two polynomials, provided that the leading coefficient of the divisor is non-singular, which is obviously the case here.

3 The map $\mathrm{f}_{\mathrm{b}}(x) = x^2 + \mathrm{b}x$

In a previous paper, the authors studied the dynamics of the family of quadratic coquatenionic maps of the form $x^2 + \mathrm{c}$, with c a coquaternionic parameter.

We are now concerned with the dynamics of a different family of quadratic coquaternionic maps, namely, the maps $x^2 + \mathrm{b}x$, with b a coquaternionic parameter.

We first recall several basic dynamical systems concepts and results which will play an important role in the remaining part of the paper. Finally, it is worthwhile to mention that, since the algebra of coquaternions is isomorphic to the algebra $M_2(\mathbb{R})$ of 2×2 real matrices, the research on the iteration of functions defined on matrix algebras developed by Baptista *et al* [4] and Serenevy [38] can also be seen — although with a different approach — as a first and important contribution to the study of the dynamics of coquaternionic maps.

3.1 Preliminaries

Let X be a metric space and f a map from X to itself. We say that (X, f) is a discrete dynamical system. For $k \in \mathbb{N}_0$, we denote by f^k the k-th iterate of the map f, inductively defined by $f^0 = \mathrm{id}_X$ and $f^k = f \circ f^{k-1}$, $k \in \mathbb{N}$. For a given initial point $x_0 \in X$, the *orbit* of x_0 under f is the sequence $\left(f^k(x_0)\right)_{k\in\mathbb{N}_0}$. A point $x \in X$ is said to be a *periodic point of f with period* $n \in \mathbb{N}$, if $f^n(x) = x$, with $f^k(x) \neq x$ for $0 < k < n$; in this case, we say that the set $\{x, f(x), \ldots, f^{n-1}(x)\}$ is an *n-cycle* for f. Periodic points of period one are called *fixed points*.

Two maps $f : X \to X$ and $g : X \to X$ are said to be *conjugate* if there exists an invertible map ϕ such that $f \circ \phi = \phi \circ g$. In this case, we say that the corresponding dynamical systems (X, f) and (X, g) are *dynamically equivalent*, since they share the same dynamical characteristics.

As already referred, in the complex case, any quadratic map, in particular a map of the form $x^2 + \mathrm{b}x$, is conjugate to a map $x^2 + \mathrm{c}$, for a suitable c. Since a real number commutes with any coquaternion, it is easy to recognize that this result still holds for maps of a coquaternionic variable in the particular case where the parameter b is real. For other types of parameters, however, as the results of the next sections show, the maps $x^2 + \mathrm{b}x$ admit behaviors never occurring in the case $x^2 + \mathrm{c}$, allowing us to conclude that these two families of maps cannot be conjugate.

When studying a dynamical system, besides the existence of periodic points of the map, it is also important to discuss their stability, i.e. to determine if these points are attractive, in the following sense.

Definition 3.1. A periodic point x of period $n \in \mathbb{N}$ of a map $f : X \to X$ is said to be *attractive* if, given any x' sufficiently close to x, the sequence of iterates $\left((f^n)^k(x')\right)_k$ converges to x.

A cycle is said to be attractive if all its points are attractive.

3.2 Existence of fixed points

In this section we study the existence of fixed points for the family of coquaternionic quadratic maps of the form

$$f_{\mathrm{b}}(x) = x^2 + \mathrm{b}x, \tag{3.1}$$

with b a non-real coquaternionic parameter.

We start by stating a lemma whose proof is a simple adaptation of the proof of Theorem 3.2 given in a previous study by the authors [13].

Lemma 3.2. *To study the dynamics of the quadratic map $f_{\mathrm{b}}(x) = x^2 + \mathrm{b}x$, $\mathrm{b} \in \mathbb{H}_{\mathrm{coq}} \setminus \mathbb{R}$, there is no loss of generality in assuming that* b *has one of the following forms:*

$\mathrm{b} = b_0 + b_1\mathrm{i},\ b_1 > 0$,

$\mathrm{b} = b_0 + b_2\mathrm{j},\ b_2 > 0$, *or*

$\mathrm{b} = b_0 + \mathrm{i} + \mathrm{j}$.

The results of following lemma show that the choice of the parameter b can be restricted even further.

Lemma 3.3. *Let* b *be a non-real coquaternion with any of the forms referred to in Lemma 3.2. Then, the dynamical system $(\mathbb{H}_{\mathrm{coq}}, f_{\mathrm{b}})$ is dynamically equivalent to the dynamical system $(\mathbb{H}_{\mathrm{coq}}, f_{2-\mathrm{b}})$.*

Proof. Let $x = x_0 + x_1\mathsf{i} + x_2\mathsf{j} + x_3\mathsf{k} \in \mathbb{H}_{\mathrm{coq}}$ and consider the involutions [8]

$$\begin{aligned}\kappa_{\mathsf{i}}(x) &= -\mathsf{i}\,\bar{x}\,\mathsf{i} = x_0 - x_1\mathsf{i} + x_2\mathsf{j} + x_3\mathsf{k},\\ \kappa_{\mathsf{j}}(x) &= \mathsf{j}\,\bar{x}\,\mathsf{j} \quad = x_0 + x_1\mathsf{i} - x_2\mathsf{j} + x_3\mathsf{k},\\ \kappa_{\mathsf{k}}(x) &= \ \mathsf{k}\,\bar{x}\,\mathsf{k} = x_0 + x_1\mathsf{i} + x_2\mathsf{j} - x_3\mathsf{k}.\end{aligned}$$

Introducing the function $\phi_{\mathsf{b}} : \mathbb{H}_{\mathrm{coq}} \to \mathbb{H}_{\mathrm{coq}}$, such that $\phi_{\mathsf{b}}(x) = \kappa(x) + \mathsf{b} - 1$, where

$$\kappa(x) = \begin{cases} \kappa_{\mathsf{j}}(x), & \text{if } \mathsf{b} = b_0 + b_1\mathsf{i},\\ \kappa_{\mathsf{i}}(x), & \text{if } \mathsf{b} = b_0 + b_2\mathsf{j},\\ \kappa_{\mathsf{k}}(x), & \text{if } \mathsf{b} = b_0 + \mathsf{i} + \mathsf{j},\end{cases}$$

one can easily verify that, for any $x \in \mathbb{H}_{\mathrm{coq}}$, one has $(\phi_{\mathsf{b}} \circ f_{\mathsf{b}})(x) = (f_{2-\mathsf{b}} \circ \phi_{\mathsf{b}})(x)$, which establishes the result. □

As an immediate consequence of the two previous lemmas, we immediately obtain the following important result.

Lemma 3.4. *To study the dynamics of the quadratic map* $f_{\mathsf{b}}(x) = x^2 + \mathsf{b}\,x$, $\mathsf{b} \in \mathbb{H}_{\mathrm{coq}} \setminus \mathbb{R}$, *there is no loss of generality in assuming that* b *has one of the following forms:*

$\mathcal{C}_1$ $\quad \mathsf{b} = b_0 + b_1\mathsf{i}, \quad b_0 \geq 1, b_1 > 0$

$\mathcal{C}_2$ $\quad \mathsf{b} = b_0 + b_2\mathsf{j}, \quad b_0 \geq 1, b_2 > 0$

$\mathcal{C}_3$ $\quad \mathsf{b} = b_0 + \mathsf{i} + \mathsf{j}, \ \ b_0 \geq 1.$

We will refer to the three different forms of the parameter b given above as the *canonical forms*.

We now observe that the fixed points of the map (3.1) are, naturally, the zeros of the polynomial

$$P_{\mathsf{b}}(x) = x^2 + (\mathsf{b} - 1)x. \tag{3.2}$$

We can make use of Theorem 2.3 to fully discuss the number and nature of zeros of P_{b}, or, in other words, to characterize the fixed points of the map f_{b}, for each of the three different canonical forms of the parameter b referred to in Lemma 3.4. This leads us to the results contained in the following theorem.

Theorem 3.5. *Let* $f_{\mathsf{b}}(x) = x^2 + \mathsf{b}\,x$, *with* b *a coquaternion with one of the forms* $\mathcal{C}_1$–$\mathcal{C}_3$. *The fixed points of the map* f_{b} *can be characterized as follows.*

1. *For* b *of the form* $\mathcal{C}_1$, *there are two fixed points,* 0 *and* $1 - \mathsf{b}$.

2. *For* b *of the form* $\mathcal{C}_2$, *there are two lines of fixed points*

$$\mathscr{L}_1 = \left\{ \tfrac{1-b_0-b_2}{2} + \alpha\mathsf{i} + \tfrac{1-b_0-b_2}{2}\mathsf{j} - \alpha\mathsf{k} : \alpha \in \mathbb{R} \right\} \tag{3.3}$$

and

$$\mathscr{L}_2 = \left\{ \tfrac{1-b_0+b_2}{2} + \alpha\mathsf{i} - \tfrac{1-b_0+b_2}{2}\mathsf{j} + \alpha\mathsf{k} : \alpha \in \mathbb{R} \right\}, \tag{3.4}$$

and, if $b_0 - b_2 \neq 1$, *two additional fixed points,* 0 *and* $1 - \mathsf{b}$.

3. *For* b *of the form* $\mathcal{C}_3$, *there is a line of fixed points*

$$\mathscr{L} = \left\{ \tfrac{1-b_0}{2} + \alpha\mathsf{i} + \alpha\mathsf{j} + \tfrac{1-b_0}{2}\mathsf{k} : \alpha \in \mathbb{R} \right\} \tag{3.5}$$

and, if $b_0 \neq 1$, *two additional fixed points,* 0 *and* $1 - \mathsf{b}$.

Proof. The companion polynomial of the polynomial P_{b} given by (3.2) is

$$\mathcal{C}_{P_{\mathsf{b}}}(x) = x^2 \Psi_{[\![1-\mathsf{b}]\!]}, \tag{3.6}$$

whose roots, in $\mathbb{C}$, are the double real root 0 together with the roots of $\Psi_{[\![1-\mathsf{b}]\!]}$. First, observe that 0 and $1-\mathsf{b}$ are always zeros of P_{b}, and hence $[\![0]\!]$ and $[\![1-\mathsf{b}]\!]$ are always admissible classes of P_{b}. The characteristic polynomial of $[\![0]\!]$ is simply x^2 and so the remainder of the division of $P_{\mathsf{b}}(x)$ by $\Psi_{[\![0]\!]}$ is the polynomial $(\mathsf{b}-1)x$ i.e. the values of A and B referred to in Theorem 2.3 are given by:

$$\mathsf{A} = 0 \quad \text{and} \quad \mathsf{B} = b_0 - 1 + \operatorname{vec}\mathsf{b}. \tag{3.7}$$

As for the characteristic polynomial of $[\![1-\mathsf{b}]\!]$, we have $\Psi_{[\![1-\mathsf{b}]\!]}(x) = x^2 - 2(1-b_0)x + \det(1-\mathsf{b})$ and so, in this case, we obtain the following values for A and B:

$$\mathsf{A} = -(1-b_0)^2 - \det(\operatorname{vec}\mathsf{b}) \quad \text{and} \quad \mathsf{B} = 1 - b_0 + \operatorname{vec}\mathsf{b}. \tag{3.8}$$

1. Case b with form $\mathcal{C}_1$

   In this case, the polynomial $\Psi_{[\![1-\mathsf{b}]\!]}$ has two complex conjugate roots and so there are only two admissible classes, $[\![0]\!]$ and $[\![1-\mathsf{b}]\!]$. Let us now determine the zeros in each of these classes.

   (a) *Zeros in* $[\![0]\!]$

   In this case, we have from (3.7), $\mathsf{B} = b_0 - 1 + b_1\mathsf{i}$. Since $\det\mathsf{B} = (b_0-1)^2 + b_1^2 > 0$, we conclude that there is only one zero of $P_mathsfb$ in $[\![0]\!]$, which is, naturally, $x = 0$.

   (b) *Zeros in* $[\![1-\mathsf{b}]\!]$

   We have, from (3.8), $\mathsf{B} = 1 - b_0 + b_1\mathsf{i}$ and so, we have again that $\det\mathsf{B} > 0$, showing that there is only one zero of P_{b} in $[\![1-\mathsf{b}]\!]$, which is $x = 1-\mathsf{b}$.

2. Case b with form $\mathcal{C}_2$

   In this case, the characteristic polynomial $\Psi_{[\![1-\mathsf{b}]\!]}$ has two distinct real roots $\alpha_{\mp} = 1 - b_0 \mp b_2$ and the admissible classes of the polynomial P_{b} are

   $$[\![0]\!],\ [\![1-\mathsf{b}]\!],\ \left[\!\!\left[\tfrac{1-b_0-b_2}{2} + \tfrac{1-b_0-b_2}{2}\mathsf{j}\right]\!\!\right] \quad \text{and} \quad \left[\!\!\left[\tfrac{1-b_0+b_2}{2} + \tfrac{1-b_0+b_2}{2}\mathsf{j}\right]\!\!\right].$$

   Note that, if $b_0 - b_2 \neq 1$, we have four distinct admissible classes, whilst if $b_0 - b_2 = 1$, the number of admissible classes reduces to two: $[\![0]\!]$ and $[\![1-\mathsf{b}]\!]$. Also, note that, since $\mathsf{b} \in \mathcal{C}_2$,

   $$(b_0-1)^2 - b_2^2 = 0 \iff b_0 - b_2 = 1. \tag{3.9}$$

   (a) Case $b_0 - b_2 \neq 1$

   i. *Zeros in* $[\![0]\!]$

   In this case, from (3.7), we have $\mathsf{B} = b_0 - 1 + b_2\mathsf{j}$ and so $\det\mathsf{B} = (b_0-1)^2 - b_2^2 \neq 0$; see (3.9). Hence, there is only one root of P_{b} in $[\![0]\!]$, which is $x = 0$.

   ii. *Zeros in* $[\![1-\mathsf{b}]\!]$

   We have, from (3.8), $\mathsf{B} = 1 - b_0 + b_2\mathsf{j}$ and, again, the use of (3.9) guarantees that $\det\mathsf{B} \neq 0$. Thus, there is only zero of P_{b} in $[\![1-\mathsf{b}]\!]$, which is is $x = 1-\mathsf{b}$.

   iii. *Zeros in* $\left[\!\!\left[\tfrac{1-b_0-b_2}{2} + \tfrac{1-b_0-b_2}{2}\mathsf{j}\right]\!\!\right]$

   The characteristic polynomial of this class is $x^2 - (1-b_0-b_2)x$ and the remainder of the division of P_{b} by this polynomial is $((\mathsf{b}-1) + (1-b_0-b_2))\,x = (-b_2 + b_2\mathsf{j})x$, i.e. we have, in this case

   $$\mathsf{A} = 0 \quad \text{and} \quad \mathsf{B} = -b_2 + b_2\mathsf{j}.$$

   So, $\mathsf{B} \neq 0$ and $\det\mathsf{B} = 0$; also, there exists $\gamma_0 = 0 \in \mathbb{R}$ such that $\mathsf{A} + \mathsf{B}\gamma_0 = 0$ and condition (2.3) holds; so we conclude that there is a linear zero of P_{b} in this class, given by (cf. (2.4)): $\left\{\frac{1-b_0-b_2}{2} + \alpha\mathsf{i} + \frac{1-b_0-b_2}{2}\mathsf{j} - \alpha\mathsf{k} : \alpha \in \mathbb{R}\right\}$.

iv. *Zeros in* $\left[\!\left[\frac{1-b_0+b_2}{2} + \frac{1-b_0+b_2}{2}\mathsf{j}\right]\!\right]$
An analysis similar to the previous case leads us to conclude that there is a linear zero of P_{b} in this class, given by: $\left\{\frac{1-b_0+b_2}{2} + \alpha\mathsf{i} - \frac{1-b_0+b_2}{2}\mathsf{j} + \alpha\mathsf{k} : \alpha \in \mathbb{R}\right\}$.

(b) Case $b_0 - b_2 = 1$
In this case, there are only two classes: $[\![0]\!]$ and $[\![1-\mathsf{b}]\!]$.

i. *Zeros in* $[\![0]\!]$
The values of A and B for this class are given by (3.7), i.e. are $\mathsf{A} = 0$ and $\mathsf{B} = (b_0 - 1) + b_2\mathsf{j} = b_2 + b_2\mathsf{j}$, but contrary to what happened before, we now have (cf. (3.9)) $\det \mathsf{B} = 0$. In this case, there is $\gamma_0 = 0$ such that $\mathsf{A} + \mathsf{B}\gamma_0 = 0$ and the condition (2.3) is satisfied. Taking into account the expression of B, we conclude that there is the following linear zero in this class: $\{\alpha\mathsf{i} + \alpha\mathsf{k} : \alpha \in \mathbb{R}\}$.

ii. *Zeros in* $[\![1-\mathsf{b}]\!]$
Following a procedure similar to the one used in the previous case, we conclude that $[\![1-\mathsf{b}]\!]$ contains the following line of zeros: $\{1 - b_0 + \alpha\mathsf{i} + (1-b_0)\mathsf{j} - \alpha\mathsf{k} : \alpha \in \mathbb{R}\}$.

3. Case $\mathsf{b} = b_0 + \mathsf{i} + \mathsf{j}$

In this case, the characteristic polynomial of $[\![1-\mathsf{b}]\!]$ is $\Psi_{[\![1-\mathsf{b}]\!]}(x) = (x-(1-b_0))^2$. Hence, the admissible classes of P_{b} are $[\![0]\!]$, $[\![1-b_0]\!] = [\![1-\mathsf{b}]\!]$ and $[\![\frac{1-b_0}{2} + \frac{1-b_0}{2}\mathsf{j}]\!]$. Note that, there are three distinct classes if $b_0 \neq 1$ and a unique class, $[\![0]\!]$, when $b_0 = 1$.

(a) Case $b_0 \neq 1$

i. *Zeros in* $[\![0]\!]$
In this case $\mathsf{A} = 0$ and $\mathsf{B} = \mathsf{b} - 1 = (b_0 - 1) + \mathsf{i} + \mathsf{j}$; since $\det \mathsf{B} = (b_0 - 1)^2 \neq 0$, B is non-singular and so the only root of P_{b} in $[\![0]\!]$ is $x = 0$.

ii. *Zeros in* $[\![1-\mathsf{b}]\!]$
In this case, $\mathsf{A} = -\det(1-\mathsf{b}) = -(1-b_0)^2$ and $\mathsf{B} = \mathsf{b} + 1 - 2b_0 = -b_0 + 1 + \mathsf{i} + \mathsf{j}$; we have $\det \mathsf{B} = (1-b_0)^2 \neq 0$ which shows that the only root of the polynomial in $[\![1-\mathsf{b}]\!]$ is $x = 1 - \mathsf{b}$.

iii. *Zeros in* $[\![\frac{1-b_0}{2} + \frac{1-b_0}{2}\mathsf{j}]\!]$
The characteristic polynomial of this class is $x^2 - (1+b_0)x$, leading to

$$\mathsf{A} = 0, \quad \mathsf{B} = \mathsf{b} - b_0 = \mathsf{i} + \mathsf{j}. \tag{3.10}$$

We have $\mathsf{B} \neq 0$, B singular, there exists $\gamma_0 = 0$ such that $\mathsf{A} + \mathsf{B}\gamma_0 = 0$ and condition (3.9) is satisfied. So we have the following line of zeros: $\left\{\frac{1-b_0}{2} + \alpha\mathsf{i} + \alpha\mathsf{j} + \frac{1-b_0}{2}\mathsf{k} : \alpha \in \mathbb{R}\right\}$.

(b) Case $b_0 = 1$
In this case, it can easily be shown that there is a line of zeros in $[\![0]\!]$ given by $\{\alpha\mathsf{i} + \alpha\mathsf{j} : \alpha \in \mathbb{R}\}$, which completes the proof.

□

Remark 3.6.

1. In case (2), if $b_0 - b_2 = 1$, then $\mathscr{L}_1 = \{-b_2 + \alpha\mathsf{i} - b_2\mathsf{j} - \alpha\mathsf{k} : \alpha \in \mathbb{R}\} \subset [\![1-\mathsf{b}]\!]$ and $\mathscr{L}_2 = \{\alpha\mathsf{i} + \alpha\mathsf{k} : \alpha \in \mathbb{R}\} \subset [\![0]\!]$.
2. In case (3), if $b_0 = 1$, then $\mathscr{L} = \{\alpha\mathsf{i} + \alpha\mathsf{j} : \alpha \in \mathbb{R}\} \subset [\![0]\!]$.

We end this section by observing that the determination of periodic points with period $n > 1$ for the map $f_{\mathsf{b}}(x) = x^2 + \mathsf{b}x$ is an extremely difficult problem. Note that, even for the case of periodic points of period two, this corresponds to solving the equation

$$x^4 + x^2\mathsf{b}x + \mathsf{b}x^3 + \mathsf{b}x\mathsf{b}x + \mathsf{b}x^2 + (\mathsf{b}^2 - 1)x = 0,$$

and that the main tool that we have used when determining fixed points — Theorem 2.3 — can no longer be applied, since the polynomial whose zeros we seek to determine are not one-sided polynomials.

3.3 Stability of the fixed points

We now want to study the stability of the fixed points determined in the previous section. As we have seen, and in analogy to what happened in the case of the map $x^2 + \mathsf{c}$, we now have a situation which does not occur in the classical case of the complex quadratic map: the existence of sets of *non-isolated* fixed points. It is simple to recognize that points in this type of sets will never be attractive in the usual sense; in fact, since, given any point in the set, all its neighborhoods always contain other fixed points, it will be impossible to find a neighborhood of a given point totally formed by points whose dynamics evolve to it. Hence, for sets of non-isolated fixed points, it becomes necessary to work with a different notion of attractivity [3]. We propose to adopt the following definition, already used in our previous studies [13].

Definition 3.7. A set $\mathscr{F}$ of non-isolated fixed points of the coquaternionic map f_{b} is said to be *attractive* if, given any coquaternion x sufficiently close to $\mathscr{F}$, the sequence of iterates $\left({f_{\mathsf{b}}}^k(x)\right)_k$ converges to a point belonging to $\mathscr{F}$.

Since there is no appropriate notion of derivative for coquaternionic maps, the natural approach to study the stability of a given fixed point is to consider the function f_{b} as a function from $\mathbb{R}^4$ into $\mathbb{R}^4$ and discuss the magnitude of the modulus of the eigenvalues of the respective Jacobian matrix. As it is well known, if all the eigenvalues of this matrix have modulus less than one, then the fixed point is attractive and, if there exists one eigenvalue with modulus greater than one, the point is not attractive. In the case of sets of non-isolated fixed points, the situation where all the eigenvalues of the Jacobian matrix have absolute value less than one will never occur, since, as observed before, it is impossible to have attractivity in the usual sense.

In what follows, we will denote by $\mathbf{J}(x)$ the Jacobian matrix of the map f_{b} computed at a given point $x = x_0 + x_1\mathsf{i} + x_2\mathsf{j} + x_3\mathsf{k} \in \mathbb{H}_{\text{coq}}$, i.e.

$$\mathbf{J}(x) = \begin{pmatrix} 2x_0 + b_0 & -2x_1 - b_1 & 2x_2 + b_2 & 2x_3 + b_3 \\ 2x_1 + b_1 & 2x_0 + b_0 & b_3 & -b_2 \\ 2x_2 + b_2 & b_3 & 2x_0 + b_0 & -b_1 \\ 2x_3 + b_3 & -b_2 & b_1 & 2x_0 + b_0 \end{pmatrix}.$$

We now consider different cases corresponding to the distinct forms of the parameter b.

1. Case b of the form $\mathcal{C}_1$

   (a) *Fixed point* $x = 0$

   The Jacobian of f_{b} at the point $x = 0$ has eigenvalues $\lambda_1 = \lambda_2 = \mathsf{b}$ and $\lambda_3 = \lambda_4 = \bar{\mathsf{b}}$; since the conditions $b_0 \geq 1$ and $b_1 > 0$ imply that $|\lambda_i| > 1$, we conclude that the fixed point $x = 0$ is not attractive.

   (b) *Fixed point* $x = 1 - \mathsf{b}$

   The Jacobian at the point $x = 1 - \mathsf{b}$ has eigenvalues $\lambda_1 = \lambda_2 = 2 - \mathsf{b}, \lambda_3 = \lambda_4 = 2 - \bar{\mathsf{b}}$, so, we conclude that $x = 1 - \mathsf{b}$ is an attractive fixed point for b such that $|2 - \mathsf{b}| < 1$.

2. Case b of the form $\mathcal{C}_2$

   In discussing the case b with the form $\mathsf{b} = b_0 + b_2\mathsf{j}$, $b_0 \geq 1$, $b_2 > 0$, we make the additional assumption that $b_0 - b_2 \neq 1$.

   (a) *Fixed point* $x = 0$

   The Jacobian of f_{b} at the point $x = 0$ has eigenvalues $\lambda_1 = \lambda_2 = b_0 - b_2$ and $\lambda_3 = \lambda_4 = b_0 + b_2$. The conditions $b_0 \geq 1,\ b_2 > 0$ imply that $\lambda_3 = \lambda_4 > 1$. Hence, we conclude that $x = 0$ is not an attractive fixed point.

   (b) *Fixed point* $x = 1 - \mathsf{b}$

   The Jacobian at the point $x = 1 - \mathsf{b}$ has eigenvalues $\lambda_1 = \lambda_2 = 2 - b_0 - b_2, \lambda_3 = \lambda_4 = 2 - b_0 + b_2$ and so, we conclude that $x = 1 - \mathsf{b}$ is an attractive fixed point for b such that $|2 - b_0 - b_2| < 1$ and $|2 - b_0 + b_2| < 1$; these conditions, for $b_0 \geq 1$ and $b_2 > 0$, are equivalent to $b_2 + 1 < b_0 < 3 - b_2$; see Figure 1.

(c) *Fixed points in the line* $\mathscr{L}_1$

Let x be any point in the line given by (3.3), i.e. let x be of the form $x = \frac{1-b_0-b_2}{2} + \alpha\mathsf{i} + \frac{1-b_0-b_2}{2}\mathsf{j} - \alpha\mathsf{k}$, with $\alpha \in \mathbb{R}$. The eigenvalues of the Jacobian of f_{b} at x are

$$\lambda_1 = 1,\ \lambda_2 = 1 - 2b_2,\ \lambda_3 = 2 - b_0 - b_2,\ \lambda_4 = b_0 - b_2.$$

and the eigenvalue $\lambda_1 = 1$ has $v_1 = (0, 1, 0, -1)$ as an associated eigenvector.

Let $\tilde{x}$ be a perturbation of x given by $\tilde{x} = x + \varepsilon_1\, v_1 + \varepsilon_2\, v_2 + \varepsilon_3\, v_3 + \varepsilon_4\, v_4$, where $\{v_1, v_2, v_3, v_4\}$ is a basis of $\mathbb{R}^4$ formed by eigenvectors[4] of $\mathbf{J}(x)$, with $v_1 = (0, 1, 0, -1)$. We have

$$\tilde{x} = x^* + \varepsilon_2\, v_2 + \varepsilon_3\, v_3 + \varepsilon_4\, v_4,$$

where $x^* = x + \varepsilon_1 v_1 \in \mathcal{L}_1$, due to the expression of v_1. Hence, for $\mathsf{b} = b_0 + b_2\mathsf{j}$ with b_0, b_2 satisfying the conditions $|1 - 2b_2| < 1,\ |2 - b_0 - b_2| < 1,\ |b_0 - b_2| < 1$, we will have $|\lambda_i| < 1, i = 2, 3, 4$ and (provided the ε_i are sufficiently small) the sequence of iterates $(f_{\mathsf{b}}^k(\tilde{x}))$ will converge to the point x^*. We thus conclude that, for values of the parameter $\mathsf{b} = b_0 + b_2\mathsf{j}$ with b_0 and b_2 satisfying the conditions $b_0 \geq 1, b_0 - 1 < b_2 < 1$ – see Figure1 – the line $\mathscr{L}_1$ is an attractive set of fixed points.

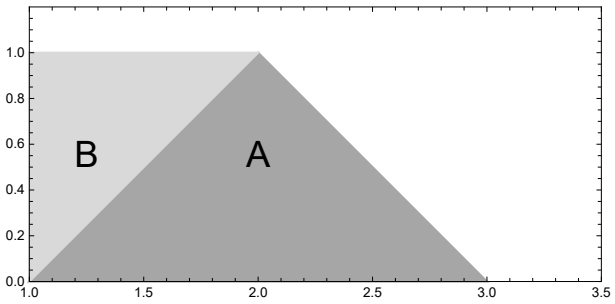

Figure 1: Stability regions for the case b with form $\mathcal{C}_2$: A – stability region for the fixed point $x = 1 - \mathsf{b}$; B – stability region for the line of fixed points $\mathscr{L}_1$.

(d) *Fixed points in the line* $\mathscr{L}_2$

Let now x be any point in the line (3.4), i.e. let x be of the form $x = \frac{1-b_0+b_2}{2} + \alpha\mathsf{i} - \frac{1-b_0+b_2}{2}\mathsf{j} + \alpha\mathsf{k}$, with $\alpha \in \mathbb{R}$. In this case, the Jacobian of f_{b} at x has eigenvalues

$$\lambda_1 = 1,\ \lambda_2 = 2 - b_0 + b_2,\ \lambda_3 = b_0 + b_2,\ \lambda_4 = 1 + 2b_2.$$

Since, in this case, we have $\lambda_3 > 1$ (and also $\lambda_4 > 1$), we can conclude that the line $\mathscr{L}_2$ is not an attractive set of points.

3. Case b of the form $\mathcal{C}_3$

We exclude from our discussion the case where $b_0 = 1$, i.e. only discuss the case of parameters b of the form $\mathsf{b} = b_0 + \mathsf{i} + \mathsf{j}$ with $b_0 > 1$.

(a) *Fixed point* $x = 0$

The Jacobian of f_{b} at the point $x = 0$ has a unique eigenvalue $\lambda = b_0$.[5] We thus conclude that the fixed point is not attractive.

(b) *Fixed point* $x = 1 - \mathsf{b}$

At the fixed point $x = 1 - \mathsf{b}$, the Jacobian has a single eigenvalue $\lambda = 2 - b_0$. Hence, the fixed point is attractive for $1 < b_0 < 3$.

[4] Note that, in the case we are considering, we have four distinct eigenvalues and hence such a basis always exists.

[5] In this case, we do not have four linearly independent eigenvectors, but we can consider a basis of $\mathbb{R}^4$ formed by generalized eigenvectors.

(c) *Fixed points in the line $\mathscr{L}$*
Let x be any point in the line given by (3.5), i.e. let x be of the form $x = \frac{1-b_0}{2} + \alpha\mathsf{i} + \alpha\mathsf{j} + \frac{1-b_0}{2}\mathsf{k}$. In this case, the eigenvalues of the Jacobian of f_{b} at x are

$$\lambda_1 = \lambda_2 = 1,\ \lambda_3 = 2 - b_0,\ \lambda_4 = b_0.$$

Since $\lambda_4 > 1$, we can conclude that the line $\mathscr{L}$ is not an attractive set of fixed points.

Remark 3.8. In the previous discussion, we left out some special cases, namely, the cases where $\mathsf{b} = b_0 + b_2\mathsf{j}$ with $b_0 - b_2 = 1$ and the case $\mathsf{b} = b_0 + \mathsf{i} + \mathsf{j}$ with $b_0 = 1$. In these cases, when considering the lines of fixed points, we are faced with a difficulty that does not allow us to fully discuss their stability: the fact that the Jacobian of $f_{\mathsf{b}}(x)$ at any point in one of those lines has $\lambda = 1$ as a defective eigenvalue.

4 Numerical experiments

In this section we present the results of some numerical experiments obtained for some specific choices of the parameter b, which already illustrate the richness of admissible dynamics for the coquatenionic map $f_{\mathsf{b}}(x) = x^2 + \mathsf{b}x$.

Example 4.1. Our first example is for the complex parameter value $\mathsf{b} = 2.84 + 0.58\mathsf{i}$. For this specific parameter, the corresponding map f_{b} has two aperiodic coquaternionic attractors. The first is the circle

$$\mathscr{C} = \left\{-1.82731 - 0.584057\mathsf{i} + q_2\mathsf{j} + q_3\mathsf{k} : q_2^2 + q_3^2 = 0.0255611\right\}$$

and the second is made up of the following five circles:

$$\mathscr{C}_1 = \left\{-1.65717 - 0.842274\mathsf{i} + \alpha\mathsf{j} + \beta\mathsf{k} : \alpha^2 + \beta^2 = 0.00520474\right\},$$

$$\mathscr{C}_2 = \left\{-2.17585 - 0.561637\mathsf{i} + \alpha\mathsf{j} + \beta\mathsf{k} : \alpha^2 + \beta^2 = 0.00292192\right\},$$

$$\mathscr{C}_3 = \left\{-1.43185 - 0.412965\mathsf{i} + \alpha\mathsf{j} + \beta\mathsf{k} : \alpha^2 + \beta^2 = 0.00766025\right\},$$

$$\mathscr{C}_4 = \left\{-1.93962 - 0.820685\mathsf{i} + \alpha\mathsf{j} + \beta\mathsf{k} : \alpha^2 + \beta^2 = 0.00258121\right\},$$

$$\mathscr{C}_5 = \left\{-1.94134 - 0.272091\mathsf{i} + \alpha\mathsf{j} + \beta\mathsf{k} : \alpha^2 + \beta^2 = 0.00365608\right\}.$$

This example shows a big difference between the dynamics of coquaternionic maps and the ones for the complex case: the possibility of coexistence of attractors. It also interesting to observe that the complex 5-cycle

$$\begin{aligned} z_1 &= -1.39283 - 0.40758\mathsf{i},\ z_2 = -1.94539 - 0.82999\mathsf{i},\\ z_3 &= -1.94786 - 0.256193\mathsf{i},\ z_4 = -1.65481 - 0.85929\mathsf{i},\\ z_5 &= -2.20126 - 0.556251\mathsf{i}, \end{aligned}$$

which is an attractor for the restriction of the map f_{b} to the complex plane, $f_{\mathsf{b}}|_{\mathbb{C}}$, looses its stability when allowing coquaternionic arguments.

Example 4.2. We now consider the case of a parameter of the form $\mathcal{C}_2$, $\mathsf{b} = \frac{24}{10} + \frac{9}{10}\mathsf{j}$. For this parameter value, the lines

$$\mathscr{L} = \left\{q_0 + \alpha\mathsf{i} + (q_0 + \tfrac{1}{2})\mathsf{j} + \alpha\mathsf{k} : \alpha \in \mathbb{R}\right\}$$

and

$$\mathscr{L}' = \left\{q_0' + \alpha\mathsf{i} + (q_0 + \tfrac{1}{2})\mathsf{j} + \alpha\mathsf{k} : \alpha \in \mathbb{R}\right\},$$

where $q_0 = \frac{1}{40}(-53 - \sqrt{129})$ and $q_0' = \frac{1}{40}(-53 + \sqrt{129})$, are attractive sets of periodic points of period two such that $f_{\mathsf{b}}(\mathscr{L}) = \mathscr{L}'$, i.e. the sets $\mathscr{L}$ and $\mathscr{L}'$ form what we call a 2-set cycle [14].

Example 4.3. Our last example is again for a parameter value of the form $\mathcal{C}_2$, $\mathsf{b} = 1.375 + 1.1\mathsf{j}$. In this case, the map admits the following two hyperbolas

$$\mathscr{H} = \left\{-0.458371 + \alpha\mathsf{i} - 0.642342\mathsf{j} + \beta\mathsf{k} : \alpha^2 - \beta^2 = 0.2025\right\}$$

and

$$\mathscr{H}' = \left\{-0.916629 + \alpha\mathsf{i} - 0.798567\mathsf{j} + \beta\mathsf{k} : \beta^2 - \alpha^2 = 0.2025\right\}$$

as an attracting 2-set cycle, i.e. $\mathscr{H}$ and $\mathscr{H}'$ are attractive sets of periodic points of period two such that $f_{\mathsf{b}}(\mathscr{H}) = \mathscr{H}'$.

5 Conclusions

This paper is a first approach to the study of the dynamics of the one-parameter family of coquaternionic quadratic maps $f_{\mathsf{b}}(x) = x^2 + \mathsf{b}x$. This may be seen as a continuation of our previous work [13], where the simpler one-parameter family of maps, $f_{\mathsf{c}}(x) = x^2 + \mathsf{c}$, was considered.

Recent results for the zeros of one-sided coquaternionic polynomials [15] led us to conclude that this new family of maps should demonstrate even further the richness of dynamics allowed by the use of coquaternions, when compared with the complex case.

By making use of the aforementioned results, the fixed points of $f_{\mathsf{b}}(x) = x^2 + \mathsf{b}x$ are analytically determined, with very interesting results: for complex values of the parameter, the fixed points are exactly the ones already known for the corresponding complex maps; however, for certain choices of non-complex parameter values, the maps f_{b} have lines of fixed points, something that never happens with the fixed points of the quadratic maps $f_{\mathsf{c}}(x) = x^2 + \mathsf{c}$ previously studied.

The study of the stability of the fixed points of f_{b} is also addressed, where, in some cases, due to the appearance of sets of non-isolated points, a suitably adapted notion of stability is used.

Some examples of dynamics beyond fixed points are also presented. These few numerical examples already show a richness of dynamics compared with the complex case — the possibility of coexistence of attractors, for example — which makes us to believe that the dynamics of coquaternionic maps is a subject deserving further investigation.

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Material and Code availability

A *Mathematica* add-on application implementing the algebra of coquaternions, written by the authors, is available at the site `https://w3.math.uminho.pt/Coquaternions`. Some *Matlab* programs used in our computations can be obtained upon request to the corresponding author.

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