Global exponential stability of discrete-time Hopfield neural network models with unbounded delays

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ABSTRACT
In this paper, a general setting is presented to study the exponential stability of discrete-time systems with bounded or unbounded delays. Based on the M-matrix theory, we establish sufficient conditions to ensure the global exponential stability of the zero equilibrium of low-order, and high-order, discrete-time Hopfield neural network models with unbounded delays and delay in the leakage terms. A comparison of the literature shows that our results generalize and improve some in recent publications.

KEYWORDS
Neural networks; delay difference equations; unbounded delays; global stability

1. Introduction

In recent decades, neural network models have attracted the attention of a high number of scientists due to their many applications in various engineering and scientific areas such as content-addressable memory, pattern recognition, signal processing, image processing and optimization (see [8, 9, 29]).

In 1984, Hopfield [17] studied the artificial neural network described by the following system of ordinary differential equations

$$x_i'(t) = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t)), \qquad i = 1, \dots, n. \tag{1}$$

In order to reproduce the effect of finite transmission speed of signals among neurons, communication time, or process of moving images, Marcus and Westervelt [21] introduced a discrete delay in (1), making it more realistic. In the same publication, they observed that the delay can destabilize the system. In fact, the delay can affect the dynamic behaviour of neural systems [1], therefore the stability of delay neural network models have been the subject of an intense research activity (see [2–6, 10, 12, 15, 16, 19, 20, 22, 23, 25–28, 31–37] and the references therein).

Roughly, mathematical neural network models can be classified into two types: continuous-time models and discrete-time models. In spite of the former being the

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main focus of mathematicians, it is essential to formulate, and study, discrete-time versions because of computational implementations [23, 24].

In the present work, we consider discrete-time low-order and high-order neural network models with unbounded delay and delay in leakage terms. Using systems with unbounded delay, it is possible to modulate phenomena where the entire history affects the present. At this time, the continuous-time neural network model with unbounded delay is widely studied (see [5, 12, 19, 25, 36] and references therein), while few research works are focus on the discrete-time case [6]. To the best of our knowledge, the global stability of a discrete-time high-order neural network model with unbounded delay has not been studied yet. We should say that many authors pay attention to low-order neural network models, but it is worth studying high-order neural network models because, compared with low-order systems, they have stronger approximation properties, fast convergence speed and higher fault tolerance [14, 30, 31, 36].

Since the work of Golpasamy [15], the continuous neural network models with delay in the leakage terms have been studied by several authors [18, 26, 33] but, as far as we know, there are few results concerning the stability of discrete-time neural network models with delay in the leakage terms [6, 27, 28].

The problem of stability of equilibrium of discrete-time Hopfield neural network models with delays has been studied [6, 10, 16, 22, 23, 27, 28, 31, 35]. However, the models considered have finite delays [10, 16, 22, 23, 27, 28, 31, 35], or with infinite delays but just for low-order models with discrete delays independent of the neurons [6].

In this paper we establish a global exponential stability criterion of zero equilibrium for a discrete-time general system with unbounded delays and we apply it to low-order and high-order Hopfield models to get new stability criteria. The classical method of proof used in the literature [23, 27, 34] consists in constructing a suitable Lyapunov function that assures the global stability of the equilibrium. Here, as in [2, 10, 16], the method of proof goes through applying properties of non-singular M-matrix, which are easier to deal with than Lyapunov functions and, at times, the hypothesis are easy to verify.

Finally, we describe the contents of the paper. In Section 2, we introduce some essential notations, define the phase spaces for discrete-time systems with bounded or unbounded delays in general settings, and establish general global exponential stability criteria. Section 3 is the core of the paper, where the stability results are established for discrete-time, low-order and high-order, Hopfield neural network models with bounded and unbounded delays and delay in the leakage terms. A relevant comparison with results in the literature are presented. In Section 4, numerical examples are presented to illustrate the effectiveness of the main results. The paper ends with a short section of conclusions.

2. Notations and basic stability results

In this paper, we denote by $\mathbb{R}$ the set of real numbers, by $\mathbb{R}^+$ the set of positive real numbers, by $\mathbb{R}^-$ the set of negative real numbers, by $\mathbb{Z}$ the set of integer numbers, and by $\mathbb{N}$ the set of positive integer numbers. For a set $I \subseteq \mathbb{R}$, we define $I_{\mathbb{Z}} = I \cap \mathbb{Z}$ and we denote by $\mathbb{Z}_0^-$ the set of non-positive integer number, i.e. $\mathbb{Z}_0^- = (-\infty, 0]_{\mathbb{Z}}$.

Given $n \in \mathbb{N}$, we are going to consider the cartesian product $\mathbb{R}^n$ equipped with the maximal norm, i.e. $|\overline{d}| = \max\limits_{i \in [1,n]_{\mathbb{Z}}} |d_i|$, for $\overline{d} = (d_1, \ldots, d_n)^T \in \mathbb{R}^n$.

For a positive real number α, we consider the space X_α of the functions

$$\varphi : \mathbb{Z}_0^- \to \mathbb{R}$$

such that $\sup_{j\in\mathbb{Z}_0^-} |\varphi(j)| \mathrm{e}^{\alpha j} < \infty$, equipped with the norm

$$\|\varphi\|_\alpha = \sup_{j\in\mathbb{Z}_0^-} |\varphi(j)| \mathrm{e}^{\alpha j} .$$

For $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}^+$, we denote by X_α^n the space of the functions

$$\begin{array}{rccl} \overline{\varphi}: & \mathbb{Z}_0^- & \to & \mathbb{R}^n \\ & j & \mapsto & (\varphi_1(j), \dots, \varphi_n(j))^T \end{array}$$

such that $\varphi_i \in X_\alpha$ for all $i \in [1,n]_\mathbb{Z}$, i.e.

$$X_\alpha^n = \left\{ \begin{array}{rccl} \overline{\varphi}: & \mathbb{Z}_0^- & \to & \mathbb{R}^n \\ & j & \mapsto & (\varphi_1(j), \dots, \varphi_n(j))^T \end{array} \middle| \max_{i\in[1,n]_\mathbb{Z}} \left(\sup_{j\in\mathbb{Z}_0^-} |\varphi_i(j)| \mathrm{e}^{\alpha j} \right) < \infty \right\},$$

equipped with the norm

$$\|\overline{\varphi}\|_\alpha = \max_{i\in[1,n]_\mathbb{Z}} \|\varphi_i\|_\alpha = \max_{i\in[1,n]_\mathbb{Z}} \left(\sup_{j\in\mathbb{Z}_0^-} |\varphi_i(j)| \mathrm{e}^{\alpha j} \right), \qquad \forall \overline{\varphi} = (\varphi_1, \dots, \varphi_n)^T \in X_\alpha^n.$$

For $\overline{d} \in \mathbb{R}^n$, we also use $\overline{d}$ to denote the constant function $\overline{\varphi}(j) = \overline{d}$ in X_α^n. A vector $\overline{d} = (d_1, \dots, d_n)^T \in \mathbb{R}^n$ is said to be positive if $d_i > 0$ for all $i \in [1,n]_\mathbb{Z}$, and in this case we write $\overline{d} > 0$. We also denote $\overline{d}^{-1} = \left(d_1^{-1}, \dots, d_n^{-1}\right)^T$ in case of $d_i \neq 0$ for all $i \in [1,n]_\mathbb{Z}$. For $\overline{d} = (d_1, \dots, d_n)^T \in \mathbb{R}^n$ and $\overline{q} = (q_1, \dots, q_n)^T \in \mathbb{R}^n$, we write $\overline{d}\overline{q} = (d_1 q_1, \dots, d_n q_n)^T \in \mathbb{R}^n$, which is not the inner product.

Given a function $\overline{x} : \mathbb{Z} \to \mathbb{R}^n$ such that $\sup_{j\in\mathbb{Z}_0^-} |\overline{x}(j)| \mathrm{e}^{\alpha j} < \infty$, we denote the ith component by x_i, i.e. $\overline{x} = (x_1, \dots, x_n)^T$, and, for each $m \in \mathbb{Z}$, we define $\overline{x}_m \in X_\alpha^n$ by

$$\overline{x}_m(j) = \overline{x}(m+j),\ j \in \mathbb{Z}_0^- .$$

The normed space X_α^n is introduced as a possible phase space of difference equations with unbounded delays, where the longer the delay is, the lesser its influence. A general system of delay difference equations is defined by

$$x_i(m+1) = \mathcal{F}_i(m, \overline{x}_m), \qquad \forall m \in [\sigma, \infty)_\mathbb{Z}, \qquad i \in [1,n]_\mathbb{Z}, \tag{2}$$

where $\sigma \in \mathbb{Z}$ and $\overline{\mathcal{F}} : \mathbb{Z} \times X_\alpha^n \to \mathbb{R}^n$ is a function with $\overline{\mathcal{F}}(m, \overline{\varphi}) = (\mathcal{F}_1(m, \overline{\varphi}), \dots, \mathcal{F}_n(m, \overline{\varphi}))^T$.

For each $\sigma \in \mathbb{Z}$ and $\overline{\varphi} \in X_\alpha^n$, we denote by $\overline{x}(\cdot, \sigma, \overline{\varphi})$ the unique solution

$$\overline{x} : \mathbb{Z} \to \mathbb{R}^n$$

of (2) with initial conditions $\overline{x}_\sigma = \overline{\varphi}$.

For difference equations with finite delays, we consider the usual phase space Y_τ^n, where $\tau \in \mathbb{N}_0$ is the delay (there is no delay if $\tau = 0$) and Y_τ^n is the cartesian product with Y_τ the normed space of the functions $\phi : [-\tau, 0]_\mathbb{Z} \to \mathbb{R}$ equipped with the norm

$$\|\phi\| = \max_{j \in [-\tau,0]_\mathbb{Z}} |\phi(j)|.$$

The norm considered in Y_τ^n is the supremum norm, i.e. for $\overline{\phi} = (\phi_1, \ldots, \phi_n)^T \in Y_\tau^n$ we have

$$\|\overline{\phi}\| = \max_{i \in [1,n]_\mathbb{Z}} \|\phi_i\|.$$

For each $\alpha > 0$ and $\tau \in \mathbb{N}_0$, the operator $\overline{\Phi}_{\tau,\alpha} = \overline{\Phi} : Y_\tau^n \to X_\alpha^n$, defined by

$$\begin{array}{rccl} \overline{\Phi}(\overline{\phi}) : & \mathbb{Z}_0^- & \to & \mathbb{R}^n \\ & j & \mapsto & \left\{ \begin{array}{ll} \overline{\phi}(j), & j \in [-\tau, 0]_\mathbb{Z} \\ \overline{0}, & j \in (-\infty, -\tau)_\mathbb{Z} \end{array} \right. , \end{array} \tag{3}$$

is one-one, thus we can regard Y_τ^n as a subset of X_α^n, i.e.

$$Y_\tau^n \equiv \overline{\Phi}(Y_\tau^n) \subseteq X_\alpha^n.$$

We note that

$$\|\overline{\phi}\| \geq \|\overline{\Phi}(\overline{\phi})\|_\alpha, \quad \forall \overline{\phi} \in Y_\tau^n.$$

We now state a global stability result for the general system of difference equations (2).

Theorem 2.1. *Let* $\alpha > 0$ *and* $\overline{\mathcal{F}} : \mathbb{Z} \times X_\alpha^n \to \mathbb{R}^n$*, with* $\overline{\mathcal{F}}(m, \overline{\varphi}) = (\mathcal{F}_1(m, \overline{\varphi}), \ldots, \mathcal{F}_n(m, \overline{\varphi}))^T$*, the function in system* (2).

If

$$|\mathcal{F}_i(m, \overline{\varphi})| \leq \mathrm{e}^{-\alpha} \|\overline{\varphi}\|_\alpha, \quad \forall \overline{\varphi} \in X_\alpha^n, \forall m \in \mathbb{Z}, \forall i \in [1, n]_\mathbb{Z}, \tag{4}$$

then the zero solution of (2) *is globally exponentially stable, i.e.*

$$\|\overline{x}_m(\cdot, \sigma, \overline{\varphi})\|_\alpha \leq \mathrm{e}^{-\alpha(m-\sigma)} \|\overline{\varphi}\|_\alpha, \quad \forall \sigma \in \mathbb{Z}, \forall \overline{\varphi} \in X_\alpha^n, \forall m \in [\sigma, \infty)_\mathbb{Z}.$$

Proof. Considering $\sigma \in \mathbb{Z}$ and $\overline{\varphi} = (\varphi_1, \ldots, \varphi_n)^T \in X_\alpha^n$, we define

$$\begin{array}{rccl} V : & [\sigma, \infty)_\mathbb{Z} & \to & \mathbb{R} \\ & m & \mapsto & \mathrm{e}^{-\alpha(m-\sigma)} \|\overline{\varphi}\|_\alpha \end{array}$$

and we denote $\overline{x}(m) = \overline{x}(m, \sigma, \overline{\varphi})$ the solution of (2) with initial condition $\overline{x}_\sigma = \overline{\varphi}$.

By induction on $m \in [\sigma, \infty)_\mathbb{Z}$, we prove that

$$\|\overline{x}_m\|_\alpha \leq V(m). \tag{5}$$

For $m = \sigma$, trivially we have

$$\|\overline{x}_\sigma\|_\alpha = \|\overline{x}_\sigma(\cdot, \sigma, \overline{\varphi})\|_\alpha = \|\overline{\varphi}\|_\alpha = V(\sigma).$$

Now, we assume that, for some $m \in [\sigma, \infty)_{\mathbb{Z}}$, we have

$$\|\overline{x}_r\|_\alpha \leq V(r), \qquad \forall r \in [\sigma, m]_{\mathbb{Z}}. \tag{6}$$

From equation (2), induction hypothesis (6), and (4), for each $i \in [1, n]_{\mathbb{Z}}$, we have

$$|x_i(m+1)| = |\mathcal{F}_i(m, \overline{x}_m)| \leq \mathrm{e}^{-\alpha} \|\overline{x}_m\|_\alpha \leq \mathrm{e}^{-\alpha} V(m) = \mathrm{e}^{-\alpha(m+1-\sigma)} \|\overline{\varphi}\|_\alpha = V(m+1).$$

From (6), we also have $|x_i(r)| \leq V(r)$ for all $r \in [\sigma, m]_{\mathbb{Z}}$ and $i \in [1, n]_{\mathbb{Z}}$, thus

$$|x_i(r)| \leq V(r), \qquad \forall r \in [\sigma, m+1]_{\mathbb{Z}}, \, \forall i \in [1, n]_{\mathbb{Z}}.$$

Consequently,

$$\begin{aligned}
\|\overline{x}_{m+1}\|_\alpha &= \max_{i \in [1,n]_{\mathbb{Z}}} \left(\sup_{j \in \mathbb{Z}_0^-} |x_i(m+1+j)| \, \mathrm{e}^{\alpha j} \right) \\
&= \max_{i \in [1,n]_{\mathbb{Z}}} \left\{ \sup_{j \in (-\infty, \sigma-m-1]_{\mathbb{Z}}} |x_i(m+1+j)| \, \mathrm{e}^{\alpha j}, \max_{j \in (\sigma-m-1, 0]_{\mathbb{Z}}} |x_i(m+1+j)| \, \mathrm{e}^{\alpha j} \right\} \\
&\leq \max_{i \in [1,n]_{\mathbb{Z}}} \left\{ \sup_{j \in (-\infty, \sigma-m-1]_{\mathbb{Z}}} |\varphi_i(j+m+1-\sigma)| \, \mathrm{e}^{\alpha j}, \max_{j \in (\sigma-m-1, 0]_{\mathbb{Z}}} V(m+1+j) \, \mathrm{e}^{\alpha j} \right\} \\
&= \max_{i \in [1,n]_{\mathbb{Z}}} \left\{ \sup_{j \in (-\infty, 0]_{\mathbb{Z}}} |\varphi_i(j)| \, \mathrm{e}^{\alpha(j+\sigma-m-1)}, \max_{j \in (\sigma-m-1, 0]_{\mathbb{Z}}} \mathrm{e}^{-\alpha(m+1+j-\sigma)} \|\overline{\varphi}\|_\alpha \, \mathrm{e}^{\alpha j} \right\} \\
&= \max_{i \in [1,n]_{\mathbb{Z}}} \left\{ \sup_{j \in (-\infty, 0]_{\mathbb{Z}}} |\varphi_i(j)| \, \mathrm{e}^{\alpha j} \, \mathrm{e}^{-\alpha(m+1-\sigma)}, \mathrm{e}^{-\alpha(m+1-\sigma)} \|\overline{\varphi}\|_\alpha \right\} \\
&= \mathrm{e}^{-\alpha(m+1-\sigma)} \|\overline{\varphi}\|_\alpha = V(m+1).
\end{aligned}$$

Thus (5) holds and, by definition of V, we obtain

$$\|\overline{x}_m(\cdot, \sigma, \overline{\varphi})\|_\alpha \leq \mathrm{e}^{-\alpha(m-\sigma)} \|\overline{\varphi}\|_\alpha,$$

and the proof is concluded.

□

The exponential stability result given in Theorem 2.1, can also be applied to difference equations with finite delays, i.e. to models which can be written in the form

$$y_i(m+1) = \mathcal{G}_i(m, \overline{y}_m), \qquad \forall m \in [\sigma, \infty)_{\mathbb{Z}}, \qquad i \in [1, n]_{\mathbb{Z}}, \tag{7}$$

where $\sigma \in \mathbb{Z}$, $\overline{\mathcal{G}} : \mathbb{Z} \times Y_\tau^n \to \mathbb{R}^n$ a function with $\overline{\mathcal{G}}(m, \overline{\phi}) = (\mathcal{G}_1(m, \overline{\phi}), \ldots, \mathcal{G}_n(m, \overline{\phi}))^T$, and $\overline{y}_m \in Y_\tau^n$ defined by $\overline{y}_m(j) = \overline{y}(m+j)$ for $j \in [-\tau, 0]_{\mathbb{Z}}$.

Corollary 2.2. *Let $\tau \in \mathbb{N}_0$, $\xi > 0$, and $\overline{\mathcal{G}} : \mathbb{Z} \times Y_\tau^n \to \mathbb{R}^n$, with $\overline{\mathcal{G}}(m, \overline{\phi}) = (\mathcal{G}_1(m, \overline{\phi}), \ldots, \mathcal{G}_n(m, \overline{\phi}))^T$, the function in system (7).*

If

$$|\mathcal{G}_i(m,\overline{\phi})| \le \mathrm{e}^{-\xi}\,\|\overline{\phi}\|, \qquad \forall \overline{\phi} \in Y_\tau^n, \forall m \in \mathbb{Z}, \forall i \in [1,n]_{\mathbb{Z}}, \tag{8}$$

then the zero solution of (7) *is globally exponentially stable, i.e. there are* $C \ge 1$ *and* $\alpha > 0$ *such that*

$$\|\overline{y}_m(\cdot,\sigma,\overline{\phi})\| \le C\,\mathrm{e}^{-\alpha(m-\sigma)}\,\|\overline{\phi}\|, \qquad \forall \sigma \in \mathbb{Z}, \forall \overline{\phi} \in Y_\tau^n, \forall m \in [\sigma,\infty)_{\mathbb{Z}}.$$

Proof. Let $\sigma \in \mathbb{Z}$, $\overline{\phi} \in Y_\tau^n$, and denote $\overline{y}(m,\sigma,\overline{\phi}) = \overline{y}(m) = (y_1(m),\ldots,y_n(m))^T$ the solution of (7) such that $\overline{y}_\sigma = \overline{\phi}$.

Fix $\alpha > 0$ such that $\xi > \alpha(1+\tau)$. For $m \in [\sigma,\infty)_{\mathbb{Z}}$ and $i \in [1,n]_{\mathbb{Z}}$, we have

$$y_i(m+1) = \mathcal{G}_i(m,\overline{y}_m) = \mathcal{F}_i(m,\overline{\Phi}(\overline{y}_m)),$$

where $\overline{\Phi}$ is defined by (3) and

$$\begin{array}{rccl} \mathcal{F}_i: & \mathbb{Z}\times X_\alpha^n & \to & \mathbb{R} \\ & (m,\overline{\varphi}) & \mapsto & \mathcal{G}_i\left(m,\overline{\varphi}_{|_{[-\tau,0]_{\mathbb{Z}}}}\right) \end{array}.$$

This means that $\overline{y}$ is the solution of

$$x_i(m+1) = \mathcal{F}_i(m,\overline{x}_m), \qquad i \in [1,n]_{\mathbb{Z}},$$

with initial conditions $\overline{x}_\sigma = \overline{\Phi}(\overline{\phi})$, i.e. $\overline{y}(m) = \overline{x}(m,\sigma,\overline{\Phi}(\overline{\phi}))$.

By (8), for each $\overline{\varphi} = (\varphi_1,\ldots,\varphi_n)^T \in X_\alpha^n$ and $i \in [1,n]_{\mathbb{Z}}$, we have

$$\begin{aligned} |\mathcal{F}_i(m,\overline{\varphi})| &= \left|\mathcal{G}_i\left(m,\overline{\varphi}_{|_{[-\tau,0]_{\mathbb{Z}}}}\right)\right| \le \mathrm{e}^{-\xi}\left\|\overline{\varphi}_{|_{[-\tau,0]_{\mathbb{Z}}}}\right\| \\ &\le \mathrm{e}^{-\xi}\max_{i\in[1,n]_{\mathbb{Z}}}\left(\max_{j\in[-\tau,0]_{\mathbb{Z}}}|\varphi_i(j)|\,\mathrm{e}^{\alpha(j+\tau)}\right) \\ &\le \mathrm{e}^{\alpha\tau-\xi}\max_{i\in[1,n]_{\mathbb{Z}}}\left(\sup_{j\in\mathbb{Z}_0^-}|\varphi_i(j)|\,\mathrm{e}^{\alpha j}\right) = \mathrm{e}^{-(\xi-\alpha\tau)}\,\|\overline{\varphi}\|_\alpha. \end{aligned}$$

As $\xi - \alpha\tau > \alpha$, we conclude that

$$|\mathcal{F}_i(m,\overline{\varphi})| \le \mathrm{e}^{-\alpha}\,\|\overline{\varphi}\|_\alpha, \qquad \forall \overline{\varphi} \in X_\alpha^n,\ \forall i \in [1,n]_{\mathbb{Z}}.$$

From Theorem 2.1, we obtain

$$\|\overline{x}_m(\cdot,\sigma,\overline{\Phi}(\overline{\phi}))\|_\alpha \le \mathrm{e}^{-\alpha(m-\sigma)}\,\|\overline{\Phi}(\overline{\phi})\|_\alpha, \qquad \forall m \in [\sigma,\infty)_{\mathbb{Z}}.$$

Consequently,

$$\mathrm{e}^{\tau\alpha}\,\|\overline{x}_m(\cdot,\sigma,\overline{\Phi}(\overline{\phi}))\|_\alpha \le \mathrm{e}^{\tau\alpha}\,\mathrm{e}^{-\alpha(m-\sigma)}\,\|\overline{\Phi}(\overline{\phi})\|_\alpha \le \mathrm{e}^{\tau\alpha}\,\mathrm{e}^{-\alpha(m-\sigma)}\,\|\overline{\phi}\|,$$

and as

$$\begin{aligned}\|\overline{y}_m\| &= \sup_{j\in[-\tau,0]_{\mathbb{Z}}} \left|\overline{x}(m+j,\sigma,\overline{\Phi}(\overline{\phi}))\right| \leq \sup_{j\in[-\tau,0]_{\mathbb{Z}}} \left|\overline{x}(m+j,\sigma,\overline{\Phi}(\overline{\phi}))\right| \mathrm{e}^{\alpha(\tau+j)} \\ &\leq \mathrm{e}^{\tau\alpha}\, \|\overline{x}_m(\cdot,\sigma,\overline{\Phi}(\overline{\phi}))\|_{\alpha},\end{aligned}$$

finally we have

$$\|\overline{y}_m(\cdot,\varphi,\overline{\phi})\| \leq C\,\mathrm{e}^{-\alpha(m-\sigma)}\,\|\overline{\phi}\|,$$

with $C = \mathrm{e}^{\tau\alpha}$. □

The stability results in the next section involve the concept of non-singular M-matrix. Thus we recall the definition here.

Definition 2.3. Let $\mathcal{M} = [m_{ij}]$ a square real matrix with non-positive off-diagonal entries, i.e. $m_{ij} \leq 0$ for all $i \neq j$.

The matrix $\mathcal{M}$ is called non-singular M-matrix if all the eigenvalues have positive real part.

The matrix $\mathcal{M}$ is called M-matrix if all the eigenvalues have non-negative real part.

There is a large number of equivalent properties to identify a non-singular M-matrix and we refer to Chapter 5 of [13] to see them and to study further properties.

3. Main results

In this section, by a non-singular M-matrix method, we establish global exponential stability criteria of zero equilibrium of discrete-time Hopfield neural network models, of low-order and high-order, with unbounded delays and delay in the leakage terms.

3.1. Low-order Hopfield model

First, we consider the following discrete-time low-order Hopfield neural network model

$$\begin{aligned}x_i(m+1) = a_i x_i(m-\delta_i(m)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(m)) + \sum_{j=1}^{n} c_{ij} f_j(x_j(m-\tau_{ij}(m))) \\ + \sum_{j=1}^{n} d_{ij} \sum_{l=1}^{\infty} \rho_{ijl} f_j(x_j(m-l)),\; i\in[1,n]_{\mathbb{Z}}, \quad (9)\end{aligned}$$

where $n \in \mathbb{N}$ is the number of neurons, $x_i(m)$ is the state of i-th neuron at moment $m \in \mathbb{Z}$, $A = diag(a_1,\ldots,a_n)$ is the self-feedback connection weight matrix with $a_i \in (-1,1)$, $B = [b_{ij}]$, $C = [c_{ij}]$, and $D = [d_{ij}] \in \mathbb{R}^{n\times n}$ are, respectively, the connection weight matrix, the discrete delay connection weight matrix and distributive delay connection weight matrix, $f_j : \mathbb{R} \to \mathbb{R}$ are the neuron activation functions, $\delta_i : \mathbb{Z} \to \mathbb{N}_0$ are the delays in leakage terms, $\tau_{ij} : \mathbb{Z} \to \mathbb{N}_0$ are the discrete time delays, and $(\rho_{ijl})_{l\in\mathbb{N}}$ are non-negative sequences in the infinitely distributed delay terms.

Under different setting, the stability of (9) was studied by X. Chen et. al. [6] in the complex field.

Remark 1. As is refereed in [6], the so-called infinitely distributed delay terms in (9),

$$\sum_{l=1}^{\infty} \rho_{ijl} f_j(x_j(m-l)),$$

can be regarded as the discretization of infinite integral form

$$\int_0^{\infty} k_{ij}(s) f_j(x_j(t-s)) ds$$

for the continuous-time Hopfield neural network models (see for example [7, 12, 20]).

To deal with the model (9), we assume the following hypotheses set:

(H1) for each $j \in [1, n]_{\mathbb{Z}}$, there exists $F_j > 0$ such that

$$|f_j(u)| \leq F_j |u|, \quad \forall u \in \mathbb{R};$$

(H2) for each $i, j \in [1, n]_{\mathbb{Z}}$, there exist $\delta, \tau \geq 0$ such that

$$\delta_i(m) \leq \delta \text{ and } \tau_{ij}(m) \leq \tau, \quad \forall m \in \mathbb{Z};$$

(H3) for each $i, j \in [1, n]_{\mathbb{Z}}$, the sequence $(\rho_{ijl})_{l \in \mathbb{N}}$, with $\rho_{ijl} \geq 0$, satisfies the convergence conditions

$$\sum_{l=1}^{\infty} \rho_{ijl} = 1 \quad \text{and} \quad \sum_{l=1}^{\infty} \mathrm{e}^{\xi l} \rho_{ijl} < \infty,$$

for some $\xi > 0$;

Remark 2. The hypothesis (H1) implies that $\overline{x}(t) = \overline{0}$ is an equilibrium of (9).

Remark 3. In the studies about global stability of neural networks models, discrete and continuous, it is usually assumed that the activation functions, f_j, are Lipschitz [2, 6, 7, 10, 12, 25, 37]. Here we do not assume that f_j are Lipschitz and hypothesis (H1) only implies the continuity of f_j at $u = 0$. Condition (H1) is assumed in [10] for discrete-time models and in [32] for continuous-time models.

The most famous activation functions used in neural networks, such as linear ReLu (rectified linear unit), leaky ReLu, sigmoid, and tanh (hyperbolic tangent), verify hypothesis (H1).

Remark 4. The hypothesis (H3) can be regard as the discretization of the integral conditions

$$\int_0^{\infty} k_{ij}(s) ds = 1 \quad \text{and} \quad \int_0^{\infty} k_{ij}(s) \, \mathrm{e}^{\xi s} \, ds < \infty,$$

usual in several studies about global exponential stability of continuous neural network models with unbounded distributed delays (see for example [19, 25]).

To define a convenient phase space for system (9), we need to prove the following lemma.

Lemma 3.1. *Assume hypothesis (H3).*
If $\gamma > 0$*, then there is* $\eta > 0$ *such that*

$$\sum_{l=1}^{\infty} \mathrm{e}^{tl}\, \rho_{ijl} < 1 + \gamma, \qquad \forall t \in [0, \eta],\, \forall i, j \in [1, n]_{\mathbb{Z}}. \tag{10}$$

Proof. Let $\gamma > 0$.

Fix $i, j \in [1, n]_{\mathbb{Z}}$ and consider the function $G(t) := G_{ij}(t) = \sum_{l=1}^{\infty} \mathrm{e}^{tl}\, \rho_{ijl}$, for $t \in [0, \xi]$. As $\rho_{ijl} \geq 0$, for all $l \in \mathbb{N}$, $G(t)$ is a non-decreasing function and, from (H3), we have

$$G(0) = \sum_{l=1}^{\infty} \rho_{ijl} = 1 \quad \text{and} \quad G(\xi) = \sum_{l=1}^{\infty} \mathrm{e}^{\xi l}\, \rho_{ijl} < \infty.$$

We now claim that G is continuous on $[0, \xi]$.

Fix $\varepsilon > 0$. From (H3), there is $N \in \mathbb{N}$ such that $\sum_{l=N}^{\infty} \mathrm{e}^{\xi l}\, \rho_{ijl} < \frac{\varepsilon}{3}$ and consequently

$$\sum_{l=N}^{\infty} \mathrm{e}^{tl}\, \rho_{ijl} < \frac{\varepsilon}{3}, \qquad \forall t \in [0, \xi].$$

Since $g(t, l) = \mathrm{e}^{tl}$ is uniformly continuous on $[0, \xi] \times ([1, N]_{\mathbb{Z}})$, there is $\beta > 0$ such that

$$\forall t, s \in [0, \xi], \forall l \in [1, N]_{\mathbb{Z}} : |t - s| < \beta \Rightarrow \left|\mathrm{e}^{tl} - \mathrm{e}^{sl}\right| < \frac{\varepsilon}{3}.$$

Thus, for $t, s \in [0, \xi]$ with $|t - s| < \beta$, we have

$$\begin{aligned}
|G(t) - G(s)| &= \left|\sum_{l=1}^{\infty} \mathrm{e}^{tl}\, \rho_{ijl} - \sum_{l=1}^{\infty} \mathrm{e}^{sl}\, \rho_{ijl}\right| \\
&\leq \left|\sum_{l=1}^{N-1} \left(\mathrm{e}^{tl} - \mathrm{e}^{sl}\right) \rho_{ijl}\right| + \left|\sum_{l=N}^{\infty} \mathrm{e}^{tl}\, \rho_{ijl} - \sum_{l=N}^{\infty} \mathrm{e}^{sl}\, \rho_{ijl}\right| \\
&\leq \left(\sum_{l=1}^{N-1} \left|\mathrm{e}^{tl} - \mathrm{e}^{sl}\right| \rho_{ijl}\right) + \sum_{l=N}^{\infty} \mathrm{e}^{tl}\, \rho_{ijl} + \sum_{l=N}^{\infty} \mathrm{e}^{sl}\, \rho_{ijl} \\
&< \frac{\varepsilon}{3} \left(\sum_{l=1}^{\infty} \rho_{ijl}\right) + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Consequently G is continuous on $[0, \xi]$. From Intermediate Value Theorem, we conclude that there is $\eta_{ij} \in (0, \xi)$ such that $G(\eta_{ij}) = G_{ij}(\eta_{ij}) = \sum_{l=1}^{\infty} \mathrm{e}^{\eta_{ij} l}\, \rho_{ijl} < 1 + \gamma$. As G_{ij} are non-decreasing, condition (10) holds taking

$$\eta = \min_{ij} \eta_{ij}.$$

$\square$

Now, we state our result on the global exponential stability of zero equilibrium of (9).

Theorem 3.2. *Assume (H1)-(H3).*
If

$$\mathcal{M} = diag(1 - |a_1|, \ldots, 1 - |a_n|) - \big[F_j(|b_{ij}| + |c_{ij}| + |d_{ij}|)\big]$$

is a non-singular M-matrix, then the zero equilibrium of (9) *is globally exponentially stable, i.e. there are* $C \geq 1$ *and* $\alpha > 0$ *such that*

$$\|\overline{x}_m(\cdot, \sigma, \overline{\varphi})\|_\alpha \leq C\, \mathrm{e}^{-\alpha(m-\sigma)}\, \|\overline{\varphi}\|_\alpha, \quad \forall (\sigma, \overline{\varphi}) \in \mathbb{Z} \times X_\alpha^n,\ \forall m \in [\sigma, \infty)_\mathbb{Z}.$$

Proof. If $\mathcal{M}$ is a non-singular M-matrix, then (see Fiedler [13, Theorem 5.1]) there is $\overline{p} = (p_1, \ldots, p_n)^T > 0$ such that $\mathcal{M}\overline{p} > 0$, i.e.

$$p_i - p_i|a_i| - \sum_{j=1}^{n} p_j F_j(|b_{ij}| + |c_{ij}| + |d_{ij}|) > 0, \quad \forall i \in [1, n]_\mathbb{Z}.$$

Consequently, there is $\gamma > 0$ such that

$$p_i\, \mathrm{e}^{-\gamma} - p_i|a_i|\, \mathrm{e}^{\gamma\delta} - \sum_{j=1}^{n} p_j F_j\big(|b_{ij}| + |c_{ij}|\, \mathrm{e}^{\gamma\tau} + |d_{ij}|(1+\gamma)\big) > 0, \quad \forall i \in [1, n]_\mathbb{Z}. \quad (11)$$

From Lemma 3.1, we conclude that there is $\alpha \in (0, \gamma)$ such that

$$\sum_{l=1}^{\infty} \mathrm{e}^{\alpha l}\, \rho_{ijl} < 1 + \gamma, \quad \forall i, j \in [1, n]_\mathbb{Z}, \quad (12)$$

and, as $0 < \alpha < \gamma$, from (11) we obtain

$$p_i\, \mathrm{e}^{-\alpha} - p_i|a_i|\, \mathrm{e}^{\alpha\delta} - \sum_{j=1}^{n} p_j F_j\big(|b_{ij}| + |c_{ij}|\, \mathrm{e}^{\alpha\tau} + |d_{ij}|(1+\gamma)\big) > 0, \quad \forall i \in [1, n]_\mathbb{Z}$$

and consequently

$$\mathrm{e}^{-\alpha} > |a_i|\, \mathrm{e}^{\alpha\delta} + \sum_{j=1}^{n} \frac{p_j}{p_i} F_j\big(|b_{ij}| + |c_{ij}|\, \mathrm{e}^{\alpha\tau} + |d_{ij}|(1+\gamma)\big), \quad \forall i \in [1, n]_\mathbb{Z}. \quad (13)$$

The change of variables $y_i(m) = p_i^{-1} x_i(m)$ transforms the model (9) into

$$y_i(m+1) = a_i y_i(m - \delta_i(m)) + \sum_{j=1}^{n} \frac{b_{ij}}{p_i} f_j(p_j y_j(m)) + \sum_{j=1}^{n} \frac{c_{ij}}{p_i} f_j(p_j y_j(m - \tau_{ij}(m)))$$
$$+ \sum_{j=1}^{n} \frac{d_{ij}}{p_i} \sum_{l=1}^{\infty} \rho_{ijl} f_j(p_j y_j(m - l)), \quad i \in [1, n]_\mathbb{Z}. \quad (14)$$

Considering X_α^n as the phase space of model (14), then it assumes the form

$$y_i(m+1) = \mathcal{F}_i(m, \overline{y}_m), \quad i \in [1,n]_{\mathbb{Z}},$$

where

$$\mathcal{F}_i(m,\overline{\varphi}) = a_i\varphi_i(-\delta_i(m)) + \sum_{j=1}^{n} \frac{b_{ij}}{p_i} f_j(p_j\varphi_j(0)) + \sum_{j=1}^{n} \frac{c_{ij}}{p_i} f_j(p_j\varphi_j(-\tau_{ij}(m))) + \sum_{j=1}^{n} \frac{d_{ij}}{p_i} \sum_{l=1}^{\infty} \rho_{ijl} f_j(p_j\varphi_j(-l)), \quad i \in [1,n]_{\mathbb{Z}},$$

for all $\overline{\varphi} = (\varphi_1, \ldots, \varphi_n)^T \in X_\alpha^n$ and $m \in \mathbb{Z}$.

Now, for $i \in [1,n]_{\mathbb{Z}}$, $\overline{\varphi} = (\varphi_1, \ldots, \varphi_n)^T \in X_\alpha^n$, and $m \in \mathbb{Z}$, from (H1) and (H2) we have

$$\begin{aligned}
|\mathcal{F}_i(m,\overline{\varphi})| &\leq |a_i\varphi_i(-\delta_i(m))| + \sum_{j=1}^{n} \frac{|b_{ij}|}{p_i} |f_j(p_j\varphi_j(0))| + \sum_{j=1}^{n} \frac{|c_{ij}|}{p_i} |f_j(p_j\varphi_j(-\tau_{ij}(m)))| \\
&\quad + \sum_{j=1}^{n} \frac{|d_{ij}|}{p_i} \sum_{l=1}^{\infty} \rho_{ijl} |f_j(p_j\varphi_j(-l))| \\
&\leq |a_i| \frac{|\varphi_i(-\delta_i(m))|\, \mathrm{e}^{-\alpha\delta_i(m)}}{\mathrm{e}^{-\alpha\delta_i(m)}} + \sum_{j=1}^{n} \frac{|b_{ij}|}{p_i} F_j p_j |\varphi_j(0)| \\
&\quad + \sum_{j=1}^{n} \frac{|c_{ij}|}{p_i} F_j p_j \frac{|\varphi_j(-\tau_{ij}(m))|\, \mathrm{e}^{-\alpha\tau_{ij}(m)}}{\mathrm{e}^{-\alpha\tau_{ij}(m)}} \\
&\quad + \sum_{j=1}^{n} \frac{|d_{ij}|}{p_i} \sum_{l=1}^{\infty} \rho_{ijl} F_j p_j \frac{|\varphi_j(-l)|\, \mathrm{e}^{-\alpha l}}{\mathrm{e}^{-\alpha l}} \\
&\leq |a_i| \frac{\|\overline{\varphi}\|_\alpha}{\mathrm{e}^{-\alpha\delta_i(m)}} + \sum_{j=1}^{n} \frac{|b_{ij}|}{p_i} F_j p_j \|\overline{\varphi}\|_\alpha + \sum_{j=1}^{n} \frac{|c_{ij}|}{p_i} F_j p_j \frac{\|\overline{\varphi}\|_\alpha}{\mathrm{e}^{-\alpha\tau_{ij}(m)}} \\
&\quad + \sum_{j=1}^{n} \frac{|d_{ij}|}{p_i} \sum_{l=1}^{\infty} \rho_{ijl} F_j p_j \frac{\|\overline{\varphi}\|_\alpha}{\mathrm{e}^{-\alpha l}} \\
&\leq |a_i|\, \mathrm{e}^{\alpha\delta} \|\overline{\varphi}\|_\alpha + \sum_{j=1}^{n} \frac{p_j}{p_i} F_j |b_{ij}| \|\overline{\varphi}\|_\alpha + \sum_{j=1}^{n} \frac{p_j}{p_i} F_j |c_{ij}|\, \mathrm{e}^{\alpha\tau} \|\overline{\varphi}\|_\alpha \\
&\quad + \sum_{j=1}^{n} \frac{p_j}{p_i} F_j |d_{ij}| \sum_{l=1}^{\infty} \mathrm{e}^{\alpha l} \rho_{ijl} \|\overline{\varphi}\|_\alpha \\
&= \left[|a_i|\, \mathrm{e}^{\alpha\delta} + \sum_{j=1}^{n} \frac{p_j}{p_i} F_j \left(|b_{ij}| + |c_{ij}|\, \mathrm{e}^{\alpha\tau} + |d_{ij}| \sum_{l=1}^{\infty} \mathrm{e}^{\alpha l} \rho_{ijl} \right) \right] \|\overline{\varphi}\|_\alpha,
\end{aligned}$$

and from (12) and (13), we obtain

$$|\mathcal{F}_i(m,\overline{\varphi})| \leq \left[|a_i|\,\mathrm{e}^{\alpha\delta} + \sum_{j=1}^{n}\frac{p_j}{p_i}F_j\left(|b_{ij}| + |c_{ij}|\,\mathrm{e}^{\alpha\tau} + |d_{ij}|(1+\gamma)\right)\right]\|\overline{\varphi}\|_\alpha \leq \mathrm{e}^{-\alpha}\,\|\overline{\varphi}\|_\alpha.$$

From Theorem 2.1, we conclude that

$$\|\overline{y}_m(\cdot,\sigma,\overline{p}^{-1}\overline{\varphi})\|_\alpha \leq \mathrm{e}^{-\alpha(m-\sigma)}\,\|\overline{p}^{-1}\overline{\varphi}\|_\alpha, \quad \forall\sigma\in\mathbb{Z}, \forall\overline{\varphi}\in X_\alpha^n, \forall m\in[\sigma,\infty)_\mathbb{Z},$$

which implies that

$$\min_i\left\{p_i^{-1}\right\}\|\overline{x}_m(\cdot,\sigma,\overline{\varphi})\|_\alpha \leq \mathrm{e}^{-\alpha(m-\sigma)}\,\|\overline{\varphi}\|_\alpha \max_i\{p_i^{-1}\}, \quad \forall\sigma\in\mathbb{Z}, \forall\overline{\varphi}\in X_\alpha^n, \forall m\in[\sigma,\infty)_\mathbb{Z},$$

where $\overline{x}(\cdot,\sigma,\overline{\varphi})$ is the solution of (9) with $\overline{x}_\sigma = \overline{\varphi}$. Finally we obtain

$$\|\overline{x}_m(\cdot,\sigma,\overline{\varphi})\|_\alpha \leq \frac{\max_i\{p_i\}}{\min_i\{p_i\}}\,\mathrm{e}^{-\alpha(m-\sigma)}\,\|\overline{\varphi}\|_\alpha, \quad \forall\sigma\in\mathbb{Z}, \forall\overline{\varphi}\in X_\alpha^n, \forall m\in[\sigma,\infty)_\mathbb{Z}.$$

□

As a particular situation of model (9), we have the discrete-time Hopfield neural network model with finite delays and delay in the leakage terms

$$\begin{aligned} x_i(m+1) &= a_i x_i(m-\delta_i(m)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(m)) \\ &\quad + \sum_{j=1}^{n} c_{ij} f_j(x_j(m-\tau_{ij}(m))), \quad i\in[1,n]_\mathbb{Z}. \end{aligned} \tag{15}$$

From Corollary 2.2, we obtain the following criterion for the global exponential stability of the zero equilibrium of (15).

Proposition 3.3. *Assume (H1) and (H2).*
If

$$\mathcal{N} = diag(1-|a_1|,\ldots,1-|a_n|) - \left[F_j(|b_{ij}|+|c_{ij}|)\right] \tag{16}$$

is a non-singular M-matrix, then the zero equilibrium of (15) *is globally exponentially stable, i.e. there are* $C\geq 1$ *and* $\alpha>0$ *such that*

$$\|\overline{x}_m(\cdot,\sigma,\overline{\phi})\| \leq C\,\mathrm{e}^{-\alpha(m-\sigma)}\,\|\overline{\phi}\|, \quad \forall(\sigma,\overline{\phi})\in\mathbb{Z}\times Y_\omega^n,\ \forall m\in[\sigma,\infty)_\mathbb{Z},$$

where $\omega = \max\{\delta,\tau\}$.

Proof. Consider $\omega=\max\{\delta,\tau\}$, where δ and τ are defined in (H2).
As $\mathcal{N}$ is a non-singular M-matrix, then (by Fiedler [13, Theorem 5.1] again) there

is $\overline{p} = (p_1, \dots, p_n)^T > 0$ such that

$$p_i - p_i|a_i| - \sum_{j=1}^{n} p_j F_j(|b_{ij}| + |c_{ij}|) > 0, \qquad \forall i \in [1,n]_{\mathbb{Z}}.$$

Consequently, there is $\xi > 0$ such that

$$p_i \, \mathrm{e}^{-\xi} > p_i|a_i| + \sum_{j=1}^{n} p_j F_j\big(|b_{ij}| + |c_{ij}|\big), \qquad \forall i \in [1,n]_{\mathbb{Z}}. \tag{17}$$

The change of variables $y_i(m) = p_i^{-1} x_i(m)$ transforms the model (15) into

$$\begin{aligned} y_i(m+1) &= a_i y_i(m - \delta_i(m)) + \sum_{j=1}^{n} \frac{b_{ij}}{p_i} f_j(p_j y_j(m)) \\ &\quad + \sum_{j=1}^{n} \frac{c_{ij}}{p_i} f_j(p_j y_j(m - \tau_{ij}(m))), \; i \in [1,n]_{\mathbb{Z}}. \end{aligned} \tag{18}$$

Considering Y_ω^n as the phase space of model (18), then it assumes the form

$$y_i(m+1) = \mathcal{G}_i(m, \overline{y}_m), \qquad i \in [1,n]_{\mathbb{Z}},$$

where

$$\mathcal{G}_i(m, \overline{\phi}) = a_i \phi_i(-\delta_i(m)) + \sum_{j=1}^{n} \frac{b_{ij}}{p_i} f_j(p_j \phi_j(0)) + \sum_{j=1}^{n} \frac{c_{ij}}{p_i} f_j(p_j \phi_j(-\tau_{ij}(m))), \, i \in [1,n]_{\mathbb{Z}},$$

for all $\overline{\phi} = (\phi_1, \dots, \phi_n)^T \in Y_\omega^n$ and $m \in \mathbb{Z}$.

Now, for $i \in [1,n]_{\mathbb{Z}}$, $\overline{\phi} = (\phi_1, \dots, \phi_n)^T \in Y_\omega^n$, and $m \in \mathbb{Z}$, from (H1) and (H2) we have

$$\begin{aligned} |\mathcal{G}_i(m, \overline{\phi})| &\leq |a_i \phi_i(-\delta_i(m))| + \sum_{j=1}^{n} \frac{|b_{ij}|}{p_i} |f_j(p_j \phi_j(0))| + \sum_{j=1}^{n} \frac{|c_{ij}|}{p_i} |f_j(p_j \phi_j(-\tau_{ij}(m)))| \\ &\leq |a_i| \|\overline{\phi}\| + \sum_{j=1}^{n} \frac{|b_{ij}|}{p_i} F_j p_j \|\overline{\phi}\| + \sum_{j=1}^{n} \frac{|c_{ij}|}{p_i} F_j p_j \|\overline{\phi}\| \end{aligned}$$

and from (17) we obtain

$$|\mathcal{G}_i(m, \overline{\phi})| \leq \left[|a_i| + \sum_{j=1}^{n} \frac{p_j}{p_i} F_j \left(|b_{ij}| + |c_{ij}| \right) \right] \|\overline{\phi}\| \leq \mathrm{e}^{-\xi} \, \|\overline{\phi}\|.$$

From Corollary 2.2, there are $C^* \geq 1$ and $\alpha > 0$ such that

$$\|\overline{y}_m(\cdot, \sigma, \overline{p}^{-1}\overline{\phi})\| \leq C^* \, \mathrm{e}^{-\alpha(m-\sigma)} \, \|\overline{p}^{-1}\overline{\phi}\|, \qquad \forall \sigma \in \mathbb{Z}, \forall \overline{\phi} \in Y_\omega^n, \forall m \in [\sigma, \infty)_{\mathbb{Z}},$$

and finally we conclude that

$$\|\overline{x}_m(\cdot,\sigma,\overline{\phi})\| \leq C\,\mathrm{e}^{-\alpha(m-\sigma)}\,\|\overline{\phi}\|, \qquad \forall \sigma\in\mathbb{Z}, \forall\overline{\phi}\in Y_\omega^n, \forall m\in[\sigma,\infty)_\mathbb{Z},$$

where $C = C^* \frac{\max_i\{p_i\}}{\min_i\{p_i\}}$. □

Remark 5. For model (15) with $b_{ij}=0$, with constant delays, and without delay in the leakage terms, i.e. $\tau_{ij}(m)=\tau_{ij}$ and $\delta_i(m)=0$ for all $m\in\mathbb{Z}$, $i,j\in[1,n]_\mathbb{Z}$, Y. Hong and W. Ma [16, Theorem 3.1] proved the global attractivity of zero equilibrium assuming that: the matrix $\mathcal{N}$, defined in (16), is an M-matrix and the activation functions f_j, $j\in[1,n]_\mathbb{Z}$, are differentiable and satisfy

i) $f_j(0)=0$, $|f_j(u)|\leq 1$, for all $u\in\mathbb{R}$, $\lim\limits_{u\to\infty} f_j(u)=1$, and $\lim\limits_{u\to-\infty} f_j(u)=-1$;

ii) $f_j'(u)>0$ for all $u\in\mathbb{R}$ and $f_j'(0)=\sup\limits_{u\in\mathbb{R}} f_j'(u)=1$.

In this work, we do not assume differentiable activation functions and conditions i) and ii) imply (H1) with $F_j=1$. However, Proposition 3.3 does not improve [16, Theorem 3.1] because we assume the restrictive condition of $\mathcal{N}$ being non-singular M-matrix. In fact, this restrictive condition is needed since we get the global exponential stability of the equilibrium while in [16, Theorem 3.1] the authors obtained the global attractivity of the equilibrium.

3.2. High-order Hopfield model

Now, we consider the following discrete-time high-order Hopfield neural network model with unbounded delays and delay in the leakage terms,

$$\begin{aligned} x_i(m+1) \;=\;& a_i x_i(m-\delta_i(m)) + \sum_{j=1}^n b_{ij} f_j(x_j(m)) \\ &+ \sum_{j=1}^n\sum_{k=1}^n c_{ijk}\, g_j(x_j(m-\tau_{ijk}(m)))\, g_k(x_k(m-\tau_{ijk}(m))) \\ &+ \sum_{j=1}^n\sum_{k=1}^n d_{ijk}\left(\sum_{l=1}^\infty \rho_{ijl}\, g_j(x_j(m-l))\right)\left(\sum_{l=1}^\infty \rho_{ikl}\, g_k(x_k(m-l))\right) \end{aligned} \tag{19}$$

with $i\in[1,n]_\mathbb{Z}$, where $n\in\mathbb{N}$ is the number of neurons, $x_i(m)$ is the state of i-th neuron at moment $m\in\mathbb{Z}$, $A=diag(a_1,\dots,a_n)$ is the self-feedback connection weight matrix with $a_i\in(-1,1)$, $B=[b_{ij}]$ is the low-order connection weight matrix, $c_{ijk}\in\mathbb{R}$ are the high-order connection weights to discrete delay terms, and $d_{ijk}\in\mathbb{R}$ are the high-order connection weights to distributed delay terms, $f_j, g_j:\mathbb{R}\to\mathbb{R}$ are the neuron activation functions, $\delta_i:\mathbb{Z}\to\mathbb{N}_0$ are the delays in leakage terms, $\tau_{ij_k}:\mathbb{Z}\to\mathbb{N}_0$ are the discrete time delays, and $(\rho_{ijl})_{l\in\mathbb{N}}$ are non-negative sequences in the infinite distributed delay high-order terms.

To deal with the model (19), we assume the hypotheses (H1) and (H3) from the model (9) joint with:

(HO1) for each $j\in[1,n]_\mathbb{Z}$, there exist $G_j, M_j>0$ such that

$$|g_j(u)|\leq\min\{M_j, G_j|u|\}, \qquad \forall u\in\mathbb{R};$$

(HO2) for each $i, j, k \in [1, n]_{\mathbb{Z}}$, there exist $\delta, \tau \geq 0$ such that

$$\delta_i(m) \leq \delta \text{ and } \tau_{ijk}(m) \leq \tau, \quad \forall m \in \mathbb{Z}.$$

Remark 6. We should remark again that hypotheses (H1) and (HO1) imply that $\overline{x}(t) = \overline{0}$ is an equilibrium of (19).

Using similar arguments to those present in the proof of Theorem 3.2, we obtain the following exponential stability criterion for the zero solution of (19). For convenience of the reader, we put the proof here.

Theorem 3.4. *Assume (H1), (H3), (HO1), and (HO2).*
If

$$\mathcal{Q} = diag(1 - |a_1|, \dots, 1 - |a_n|) - \left[F_j|b_{ij}|\right] - \left[G_j \sum_{k=1}^{n} \left(M_k(|c_{ijk}| + |d_{ijk}|)\right)\right]$$

is a non-singular M-matrix, then the zero equilibrium of (19) *is globally exponentially stable, i.e. there are $C \geq 1$ and $\alpha > 0$ such that*

$$\|\overline{x}_m(\cdot, \sigma, \overline{\varphi})\|_\alpha \leq C\, \mathrm{e}^{-\alpha(m-\sigma)}\, \|\overline{\varphi}\|_\alpha, \quad \forall (\sigma, \overline{\varphi}) \in \mathbb{Z} \times X^n_\alpha, \forall m \in [\sigma, \infty)_{\mathbb{Z}}.$$

Proof. As $\mathcal{Q}$ is a non-singular M-matrix, then (see Fiedler [13, Theorem 5.1]) there is $\overline{p} = (p_1, \dots, p_n)^T > 0$ such that

$$p_i - p_i|a_i| - \sum_{j=1}^{n} p_j F_j|b_{ij}| - \sum_{j=1}^{n} p_j G_j \sum_{k=1}^{n} \left(M_k(|c_{ijk}| + |d_{ijk}|)\right) > 0, \quad \forall i \in [1, n]_{\mathbb{Z}}.$$

Consequently, there is $\gamma > 0$ such that

$$p_i\, \mathrm{e}^{-\gamma} - p_i|a_i|\, \mathrm{e}^{\gamma\delta} - \sum_{j=1}^{n} p_j F_j|b_{ij}| - \sum_{j=1}^{n} p_j G_j \sum_{k=1}^{n} \left(M_k(|c_{ijk}|\, \mathrm{e}^{\gamma\tau} + |d_{ijk}|(1+\gamma))\right) > 0, \tag{20}$$

for all $i \in [1, n]_{\mathbb{Z}}$. As in the proof of Theorem 3.2, from Lemma 3.1 we conclude that there is $\alpha \in (0, \gamma)$ such that (12) holds and

$$\mathrm{e}^{-\alpha} > |a_i|\, \mathrm{e}^{\alpha\delta} + \sum_{j=1}^{n} \frac{p_j}{p_i} F_j|b_{ij}| + \sum_{j=1}^{n} \frac{p_j}{p_i} G_j \sum_{k=1}^{n} \left(M_k(|c_{ijk}|\, \mathrm{e}^{\alpha\tau} + |d_{ijk}|(1+\gamma))\right), \tag{21}$$

for all $i \in [1, n]_{\mathbb{Z}}$.

Now, consider X^n_α the phase space of (19). Using again the change of variables $y_i(m) = p_i^{-1} x_i(m)$, the model (19) assumes the form

$$y_i(m+1) = \mathcal{F}_i(m, \overline{y}_m), \quad i \in [1, n]_{\mathbb{Z}},$$

where

$$\begin{aligned}
\mathcal{F}_i(m,\overline{\varphi}) &= a_i\varphi_i(-\delta_i(m)) + \sum_{j=1}^{n}\frac{b_{ij}}{p_i}f_j(p_j\varphi_j(0)) \\
&\quad + \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{c_{ijk}}{p_i}g_j(p_j\varphi_j(-\tau_{ijk}(m)))g_k(p_k\varphi_k(-\tau_{ijk}(m))) \\
&\quad + \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{d_{ijk}}{p_i}\left(\sum_{l=1}^{\infty}\rho_{ijl}g_j(p_j\varphi_j(-l))\right)\left(\sum_{l=1}^{\infty}\rho_{ikl}g_k(p_k\varphi_k(-l))\right),
\end{aligned}$$

for all $\overline{\varphi} = (\varphi_1, \ldots, \varphi_n)^T \in X_\alpha^n$ and $m \in \mathbb{Z}$.

Now, for $i \in [1, n]_{\mathbb{Z}}$, $\overline{\varphi} = (\varphi_1, \ldots, \varphi_n)^T \in X_\alpha^n$, and $m \in \mathbb{Z}$, from (H1) and (HO1) we have

$$\begin{aligned}
|\mathcal{F}_i(m,\overline{\varphi})| &\leq |a_i\varphi_i(-\delta_i(m))| + \sum_{j=1}^{n}\frac{|b_{ij}|}{p_i}|f_j(p_j\varphi_j(0))| \\
&\quad + \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{|c_{ijk}|}{p_i}|g_j(p_j\varphi_j(-\tau_{ijk}(m)))||g_k(p_k\varphi_k(-\tau_{ijk}(m)))| \\
&\quad + \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{|d_{ijk}|}{p_i}\left(\sum_{l=1}^{\infty}\rho_{ijl}|g_j(p_j\varphi_j(-l))|\right)\left(\sum_{l=1}^{\infty}\rho_{ikl}|g_k(p_k\varphi_k(-l))|\right) \\
&\leq |a_i|\frac{|\varphi_i(-\delta_i(m))|\,\mathrm{e}^{-\alpha\delta_i(m)}}{\mathrm{e}^{-\alpha\delta_i(m)}} + \sum_{j=1}^{n}\frac{|b_{ij}|}{p_i}F_jp_j|\varphi_j(0)| \\
&\quad + \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{|c_{ijk}|}{p_i}G_jp_j\frac{|\varphi_j(-\tau_{ijk}(m))|\,\mathrm{e}^{-\alpha\tau_{ijk}(m)}}{\mathrm{e}^{-\alpha\tau_{ijk}(m)}}M_k \\
&\quad + \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{|d_{ijk}|}{p_i}\left(\sum_{l=1}^{\infty}\rho_{ijl}G_jp_j\frac{|\varphi_j(-l)|\,\mathrm{e}^{-\alpha l}}{\mathrm{e}^{-\alpha l}}\right)\left(\sum_{l=1}^{\infty}\rho_{ikl}M_k\right),
\end{aligned}$$

and consequently, from (HO3) and (H3), we obtain

$$\begin{aligned}|\mathcal{F}_i(m,\overline{\varphi})| &\leq |a_i|\frac{\|\overline{\varphi}\|_\alpha}{\mathrm{e}^{-\alpha\delta_i(m)}}+\sum_{j=1}^n\frac{|b_{ij}|}{p_i}F_jp_j\|\overline{\varphi}\|_\alpha+\sum_{j=1}^n\sum_{k=1}^n\frac{|c_{ijk}|}{p_i}G_jp_j\frac{\|\overline{\varphi}\|_\alpha}{\mathrm{e}^{-\alpha\tau_{ijk}(m)}}M_k\\
&\quad+\sum_{j=1}^n\sum_{k=1}^n\frac{|d_{ijk}|}{p_i}\left(\sum_{l=1}^\infty\rho_{ijl}G_jp_j\frac{\|\overline{\varphi}\|_\alpha}{\mathrm{e}^{-\alpha l}}\right)M_k\\
&\leq |a_i|\,\mathrm{e}^{\alpha\delta}\,\|\overline{\varphi}\|_\alpha+\sum_{j=1}^n\frac{p_j}{p_i}F_j|b_{ij}|\|\overline{\varphi}\|_\alpha+\sum_{j=1}^n\sum_{k=1}^n\frac{p_j}{p_i}G_jM_k|c_{ijk}|\,\mathrm{e}^{\alpha\tau}\,\|\overline{\varphi}\|_\alpha\\
&\quad+\sum_{j=1}^n\sum_{k=1}^n\frac{p_j}{p_i}G_jM_k|d_{ijk}|\left(\sum_{l=1}^\infty\mathrm{e}^{\alpha l}\,\rho_{ijl}\right)\|\overline{\varphi}\|_\alpha\\
&= \Bigg[|a_i|\,\mathrm{e}^{\alpha\delta}+\sum_{j=1}^n\frac{p_j}{p_i}F_j|b_{ij}|\\
&\quad+\sum_{j=1}^n\frac{p_j}{p_i}G_j\sum_{k=1}^nM_k\left(|c_{ijk}|\,\mathrm{e}^{\alpha\tau}+|d_{ijk}|\left(\sum_{l=1}^\infty\mathrm{e}^{\alpha l}\,\rho_{ijl}\right)\right)\Bigg]\|\overline{\varphi}\|_\alpha.\end{aligned}$$

Finally, from (12) and (21), we have

$$\begin{aligned}|\mathcal{F}_i(m,\overline{\varphi})| &\leq \Bigg[|a_i|\,\mathrm{e}^{\alpha\delta}+\sum_{j=1}^n\frac{p_j}{p_i}F_j|b_{ij}|\\
&\quad+\sum_{j=1}^n\frac{p_j}{p_i}G_j\sum_{k=1}^n\big(M_k(|c_{ijk}|\,\mathrm{e}^{\alpha\tau}+|d_{ijk}|(1+\gamma))\big)\Bigg]\|\overline{\varphi}\|_\alpha\leq\mathrm{e}^{-\alpha}\,\|\overline{\varphi}\|_\alpha.\end{aligned}$$

From Theorem 2.1, we conclude that

$$\|\overline{y}_m(\cdot,\sigma,\overline{p}^{-1}\overline{\varphi})\|_\alpha\leq\mathrm{e}^{-\alpha(m-\sigma)}\,\|\overline{p}^{-1}\overline{\varphi}\|_\alpha,\quad\forall\sigma\in\mathbb{Z},\forall\overline{\varphi}\in X_\alpha^n,\forall m\in[\sigma,\infty)_\mathbb{Z},$$

and consequently

$$\|\overline{x}_m(\cdot,\sigma,\overline{\varphi})\|_\alpha\leq\frac{\max_i\{p_i\}}{\min_i\{p_i\}}\,\mathrm{e}^{-\alpha(m-\sigma)}\,\|\overline{\varphi}\|_\alpha,\quad\forall\sigma\in\mathbb{Z},\forall\overline{\varphi}\in X_\alpha^n,\forall m\in[\sigma,\infty)_\mathbb{Z}.$$

□

As a particular situation of model (19), we have the discrete-time high-order Hopfield neural network model with finite delays and delay in the leakage terms

$$\begin{aligned}x_i(m+1)&=a_ix_i(m-\delta_i(m))+\sum_{j=1}^nb_{ij}f_j(x_j(m))\\
&\quad+\sum_{j=1}^n\sum_{k=1}^nc_{ijk}g_j(x_j(m-\tau_{ijk}(m)))g_k(x_k(m-\tau_{ijk}(m))),\ i\in[1,n]_\mathbb{Z}.\end{aligned}\tag{22}$$

From Corollary 2.2, and following similar arguments to those present in the proof of Proposition 3.3, we obtain the following criterion for the global exponential stability of the zero equilibrium of (22). We write the proof for the convenience of the reader.

Proposition 3.5. *Assume (H1), (HO1), and (HO2).*
If

$$\mathcal{R} = diag(1-|a_1|,\dots,1-|a_n|) - \left[F_j|b_{ij}|\right] - \left[G_j\sum_{k=1}^{n} M_k|c_{ijk}|\right] \tag{23}$$

is a non-singular M-matrix, then the zero equilibrium of (22) *is globally exponentially stable, i.e. there are* $C \geq 1$ *and* $\alpha > 0$ *such that*

$$\|\overline{x}_m(\cdot,\sigma,\overline{\phi})\| \leq C\,\mathrm{e}^{-\alpha(m-\sigma)}\,\|\overline{\phi}\|, \quad \forall (\sigma,\overline{\phi}) \in \mathbb{Z}\times Y_\omega^n,\ \forall m \in [\sigma,\infty)_{\mathbb{Z}},$$

where $\omega = \max\{\delta,\tau\}$.

Proof. Let $\omega = \max\{\delta,\tau\}$, where δ and τ are in (HO2), and consider Y_ω^n the phase space of (22).

As $\mathcal{R}$ is a non-singular M-matrix, then, by [13, Theorem 5.1], there is $\overline{p} = (p_1,\dots,p_n)^T > 0$ such that

$$p_i - p_i|a_i| - \sum_{j=1}^{n} p_jF_j|b_{ij}| + \sum_{j=1}^{n}\sum_{k=1}^{n} p_jG_jM_k|c_{ijk}| > 0, \quad \forall i \in [1,n]_{\mathbb{Z}}.$$

Consequently, there is $\xi > 0$ such that

$$\mathrm{e}^{-\xi} > |a_i| + \sum_{j=1}^{n}\frac{p_j}{p_i}F_j|b_{ij}| + \sum_{j=1}^{n}\frac{p_j}{p_i}G_j\left(\sum_{k=1}^{n} M_k|c_{ijk}|\right), \quad \forall i \in [1,n]_{\mathbb{Z}}. \tag{24}$$

Using again the change of variables $y_i(m) = p_i^{-1}x_i(m)$, the model (22) assumes the form

$$y_i(m+1) = \mathcal{G}_i(m,\overline{y}_m), \quad i \in [1,n]_{\mathbb{Z}},$$

where

$$\begin{aligned}\mathcal{G}_i(m,\overline{\phi}) \;&=\; a_i\phi_i(-\delta_i(m)) + \sum_{j=1}^{n}\frac{b_{ij}}{p_i}f_j(p_j\phi_j(0)) \\ &\quad+ \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{c_{ijk}}{p_i}g_j(p_j\phi_j(-\tau_{ijk}(m)))g_k(p_k\phi_k(-\tau_{ijk}(m))), \quad i \in [1,n]_{\mathbb{Z}},\end{aligned}$$

for all $\overline{\phi} = (\phi_1,\dots,\phi_n)^T \in Y_\omega^n$ and $m \in \mathbb{Z}$.

Now, for $i \in [1,n]_{\mathbb{Z}}$, $\overline{\phi} = (\phi_1,\dots,\phi_n)^T \in Y_\omega^n$, and $m \in \mathbb{Z}$, from (H1), (HO1), and

(HO2) we have

$$\begin{aligned}
|\mathcal{G}_i(m,\overline{\phi})| &\leq |a_i\phi_i(-\delta_i(m))| + \sum_{j=1}^{n}\frac{|b_{ij}|}{p_i}|f_j(p_j\phi_j(0))| \\
&\quad + \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{|c_{ijk}|}{p_i}|g_j(p_j\phi_j(-\tau_{ij}(m)))||g_k(p_k\phi_k(-\tau_{ijk}(m)))| \\
&\leq |a_i|\|\overline{\phi}\| + \sum_{j=1}^{n}\frac{|b_{ij}|}{p_i}F_jp_j\|\overline{\phi}\| + \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{|c_{ijk}|}{p_i}G_jp_j\|\overline{\phi}\|M_k
\end{aligned}$$

and from (24) we obtain

$$|\mathcal{G}_i(m,\overline{\phi})| \leq \left[|a_i| + \sum_{j=1}^{n}\frac{p_j}{p_i}F_j|b_{ij}| + \sum_{j=1}^{n}\frac{p_j}{p_i}G_j\left(\sum_{k=1}^{n}M_k|c_{ijk}|\right)\right]\|\overline{\phi}\| \leq \mathrm{e}^{-\xi}\,\|\overline{\phi}\|.$$

From Corollary 2.2, there are $C^* \geq 1$ and $\alpha > 0$ such that

$$\|\overline{y}_m(\cdot,\sigma,\overline{p}^{-1}\overline{\phi})\| \leq C^*\,\mathrm{e}^{-\alpha(m-\sigma)}\,\|\overline{p}^{-1}\overline{\phi}\|, \quad \forall\sigma\in\mathbb{Z}, \forall\overline{\phi}\in Y_\omega^n, \forall m\in[\sigma,\infty)_{\mathbb{Z}},$$

and finally we conclude that

$$\|\overline{x}_m(\cdot,\sigma,\overline{\phi})\| \leq C\,\mathrm{e}^{-\alpha(m-\sigma)}\,\|\overline{\phi}\|, \quad \forall\sigma\in\mathbb{Z}, \forall\overline{\phi}\in Y_\omega^n, \forall m\in[\sigma,\infty)_{\mathbb{Z}},$$

where $C = C^*\frac{\max_i\{p_i\}}{\min_i\{p_i\}}$. □

Remark 7. For model (22) without delay in the leakage term, i.e. $\delta_i(m) = 0$ for all $m\in\mathbb{Z}$, $i\in[1,n]_{\mathbb{Z}}$, Z. Dong et al. [10] proved the global exponential stability of zero equilibrium assuming that: the matrix $\mathcal{R}$ defined in (23) is a non-singular M-matrix; there is $\tau > 0$ such that $\tau_{ijk}(m) < \tau$ for all $i,j,k\in[1,n]_{\mathbb{Z}}$, $m\in\mathbb{Z}$; and the assumptions

Assumption 1. [10] "The activation functions f_j ($j\in[1,n]_{\mathbb{Z}}$) satisfy

$$f_j(0) = 0, \quad |f_j(u) - f_j(v)| \leq F_j|u-v|, \quad \forall u,v\in\mathbb{R},$$

where $F_j > 0$ is a known constant",

Assumption 2. [10] "The activation functions g_j ($j\in[1,n]_{\mathbb{Z}}$) satisfy

$$g_j(0) = 0, \quad |g_j(u)| \leq M_j, \quad |g_j(u) - g_j(v)| \leq G_j|u-v|, \quad \forall u,v\in\mathbb{R},$$

where $M_j > 0$ and $G_j > 0$ are known constants".

Clearly, Assumption 1 implies (H1) and Assumption 2 implies (HO1) but the reverses do not hold. Thus above Proposition 3.5 improves the main results in [10]. We should say that Theorem 3.4 extends the results in [10] to discrete-time high-order Hopfield neural network models with delay in leakage term and unbounded distributed delays.

Now, we consider the following discrete-time high-order Hopfield neural network model with S-type distributed unbounded delays in the low-order and high-order terms,

$$
\begin{aligned}
x_i(m+1) &= a_i x_i(m-\delta_i(m)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(m)) \\
&\quad + \sum_{j=1}^{n}\sum_{r=1}^{R} c_{ij}^{(r)} g_j\left(\sum_{l=1}^{\infty} \rho_{ijl}^{(r)} x_j(m-l)\right) \qquad (25) \\
&\quad + \sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{r=1}^{R} d_{ijk}^{(r)} g_j\left(\sum_{l=1}^{\infty} \rho_{ijl}^{(r)} x_j(m-l)\right) g_k\left(\sum_{l=1}^{\infty} \rho_{ikl}^{(r)} x_k(m-l)\right),
\end{aligned}
$$

with $i \in [1,n]_{\mathbb{Z}}$, where n, $x_i(m)$, $A = diag(a_1,\dots,a_n)$, $B = [b_{ij}]$, $\delta_i : \mathbb{Z} \to \mathbb{N}_0$, and $f_j, g_j : \mathbb{R} \to \mathbb{R}$ have the same meanings as in model (19), $R \in \mathbb{N}$, $c_{ij}^{(r)} \in \mathbb{R}$ are the low-order connection weights to distributed delays terms, $d_{ijk}^{(r)} \in \mathbb{R}$ are the high-order connection weights to distributed delays terms, and $\left(\rho_{ijl}^{(r)}\right)_{l\in\mathbb{N}}$ are non-negative sequences in the infinite distributed delay terms.

The discrete-time model (25) looks like the discrete version of the continuous-time high-order Hopfield neural network model with S-type distributed delays studied in [37].

For model (25), we assume the hypotheses (H1), (HO1), and (HO2) joint with:

(HO3) for each $i,j \in [1,n]_{\mathbb{Z}}$ and $r \in [1,R]_{\mathbb{Z}}$, the sequence $\left(\rho_{ijl}^{(r)}\right)_{l\in\mathbb{N}}$, with $\rho_{ijl}^{(r)} \geq 0$, satisfies the convergence conditions

$$
\sum_{l=1}^{\infty} \rho_{ijl}^{(r)} = 1 \quad \text{and} \quad \sum_{l=1}^{\infty} e^{\xi l}\, \rho_{ijl}^{(r)} < \infty,
$$

for some $\xi > 0$.

Using the same arguments to those present in the proof of Theorem 3.4, we obtain the following exponential stability criterion for the zero solution of (25).

To avoid repetition of arguments, the proof of the next result is omitted.

Theorem 3.6. *Assume (H1), (HO1), (HO2), and (HO3).*
If

$$
\mathcal{S} = diag(1-|a_1|,\dots,1-|a_n|) - \left[F_j|b_{ij}| + G_j \sum_{r=1}^{R}\left(|c_{ij}^{(r)}| + \sum_{k=1}^{n} M_k |d_{ijk}^{(r)}|\right)\right]
$$

is a non-singular M-matrix, then the zero equilibrium of (25) *is globally exponentially stable.*

4. Numerical examples

In this section, we give two numerical examples to illustrate the effectiveness of the results presented in Theorems 3.2 and 3.4.

In the first example, we consider a continuous-time Hopfield neural network with unbounded delays and, following the ideas in [22, 23], we obtain a discrete-time model analogous to the continuous-time model. We should say that the discretization process present in [22, 23] can not be applied to models with delay in the leakage terms.

Example 4.1. Consider the model

$$\begin{cases} x_1'(t) &= -10x_1(t) + 2\tanh(x_2(t-1)) + 15\displaystyle\int_{-\infty}^{0} 4^s \tanh(x_2(t+s))ds \\ x_2'(t) &= -10x_2(t) + \tanh(x_1(t-3)) + 2\displaystyle\int_{-\infty}^{0} 2^s \tanh(x_1(t+s))ds \end{cases}, \quad t \geq 0. \tag{26}$$

Consider also the following approximation of (26)

$$\begin{cases} x_1'(t) &= -10x_1(t) + 2\tanh(x_2([t/h]h-1)) \\ & \quad +15\displaystyle\int_{-\infty}^{0} 4^{[s/h]h} \tanh(x_2([t/h]h+[s/h]/h))ds \\ x_2'(t) &= -10x_2(t) + \tanh(x_1([t/h]h-3)) \\ & \quad +2\displaystyle\int_{-\infty}^{0} 2^{[s/h]h} \tanh(x_1([t/h]h+[s/h]h))ds \end{cases}, \tag{27}$$

for $t \in [mh, (m+1)h]$, where $h > 0$ is the discretization step size and $[u]$ denotes the integer part of $u \in \mathbb{R}$. For $t \in [mh, (m+1)h]$, we have $[t/h] = m$ and model (27) has the form

$$\begin{cases} x_1'(t) &= -10x_1(t) + 2\tanh(x_2(mh-1)) \\ & \quad +15\displaystyle\int_{-\infty}^{0} 4^{[s/h]h} \tanh(x_2(mh+[s/h]/h))ds \\ x_2'(t) &= -10x_2(t) + \tanh(x_1(mh-3)) \\ & \quad +2\displaystyle\int_{-\infty}^{0} 2^{[s/h]h} \tanh(x_1(mh+[s/h]h))ds \end{cases}. \tag{28}$$

For $s \in [-lh, -(l-1)h]$, with $l \in \mathbb{N}$, we have $[s/h] = -l$ and (28) assumes the form

$$\begin{cases} x_1'(t) &= -10x_1(t) + 2\tanh(x_2(mh-1)) + 15\displaystyle\sum_{l=1}^{\infty} 4^{-lh} \tanh(x_2(mh-lh)) \\ x_2'(t) &= -10x_2(t) + \tanh(x_1(mh-3)) + 2\displaystyle\sum_{l=1}^{\infty} 2^{-lh} \tanh(x_1(mh-lh)) \end{cases},$$

and multiplying by e^{10t}, we get

$$\begin{cases} x_1'(t)\,\mathrm{e}^{10t} + 10\,\mathrm{e}^{10t}\,x_1(t) = \mathrm{e}^{10t}\left(2\tanh(x_2(mh-1)) + 15\displaystyle\sum_{l=1}^{\infty} 4^{-lh}\tanh(x_2(mh-lh))\right) \\ x_2'(t)\,\mathrm{e}^{10t} + 10\,\mathrm{e}^{10t}\,x_2(t) = \mathrm{e}^{10t}\left(\tanh(x_1(mh-3)) + 2\displaystyle\sum_{l=1}^{\infty} 2^{-lh}\tanh(x_1(mh-lh))\right) \end{cases}.$$

Integrating over $[mh, t]$, with $t < (m+1)h$, we obtain

$$\begin{cases} \displaystyle\int_{mh}^{t} \left(x_1(s)\, \mathrm{e}^{10s}\right)' ds &= \dfrac{\mathrm{e}^{10t} - \mathrm{e}^{10mh}}{10} \\ & \quad \cdot \left(2\tanh(x_2(mh-1)) + 15 \displaystyle\sum_{l=1}^{\infty} 4^{-lh} \tanh(x_2(mh-lh))\right) \\ \displaystyle\int_{mh}^{t} \left(x_2(s)\, \mathrm{e}^{10s}\right)' ds &= \dfrac{\mathrm{e}^{10t} - \mathrm{e}^{10mh}}{10} \\ & \quad \cdot \left(\tanh(x_1(mh-3)) + 2 \displaystyle\sum_{l=1}^{\infty} 2^{-lh} \tanh(x_1(mh-lh))\right) \end{cases},$$

which is equivalent to

$$\begin{cases} x_1(t) &= \mathrm{e}^{10(mh-t)}\, x_1(mh) + \dfrac{1 - \mathrm{e}^{10(mh-t)}}{10} \\ & \quad \cdot \left(2\tanh(x_2(mh-1)) + 15 \displaystyle\sum_{l=1}^{\infty} 4^{-lh} \tanh(x_2(mh-lh))\right) \\ x_2(t) &= \mathrm{e}^{10(mh-t)}\, x_2(mh) + \dfrac{1 - \mathrm{e}^{10(mh-t)}}{10} \\ & \quad \cdot \left(\tanh(x_1(mh-3)) + 2 \displaystyle\sum_{l=1}^{\infty} 2^{-lh} \tanh(x_1(mh-lh))\right) \end{cases}.$$

Letting $t \to (m+1)h$ and identifying mh with m and lh with l, we obtain

$$\begin{cases} x_1(m+1) &= \mathrm{e}^{-10h}\, x_1(m) + \dfrac{1 - \mathrm{e}^{-10h}}{10} \\ & \quad \cdot \left(2\tanh(x_2(m-1)) + 15 \displaystyle\sum_{l=1}^{\infty} 4^{-l} \tanh(x_2(m-l))\right) \\ x_2(m+1) &= \mathrm{e}^{-10h}\, x_2(m) + \dfrac{1 - \mathrm{e}^{-10h}}{10} \\ & \quad \cdot \left(\tanh(x_1(m-3)) + 2 \displaystyle\sum_{l=1}^{\infty} 2^{-l} \tanh(x_1(m-l))\right) \end{cases}.$$

Choosing the discretization step size $h = 1$, we obtain the discrete-time model

$$\left\{\begin{array}{rcl} x_1(m+1) & = & \mathrm{e}^{-10}\, x_1(m) + \dfrac{1-\mathrm{e}^{-10}}{10} \\ & & \cdot\left(2\tanh(x_2(m-1)) + 5\displaystyle\sum_{l=1}^{\infty}\frac{3}{4^l}\tanh(x_2(m-l))\right) \\ x_2(m+1) & = & \mathrm{e}^{-10}\, x_2(m) + \dfrac{1-\mathrm{e}^{-10}}{10} \\ & & \cdot\left(\tanh(x_1(m-3)) + 2\displaystyle\sum_{l=1}^{\infty}\frac{1}{2^l}\tanh(x_1(m-l))\right) \end{array}\right. . \tag{29}$$

Model (29) is a particular situation of (9) with $n = 2$, $a_1 = a_2 = \mathrm{e}^{-10}$, $\delta_1(m) = \delta_2(m) = 0$, $b_{11} = b_{12} = b_{21} = b_{22} = 0$, $c_{11} = c_{22} = 0$, $c_{12} = \frac{1-\mathrm{e}^{-10}}{5}$, $c_{21} = \frac{1-\mathrm{e}^{-10}}{10}$, $\tau_{12}(m) = 1$, $\tau_{21}(m) = 3$, $d_{11} = d_{22} = 0$, $d_{12} = \frac{1-\mathrm{e}^{-10}}{2}$, $d_{21} = \frac{1-\mathrm{e}^{-10}}{5}$, $\rho_{12l} = \frac{3}{4^l}$, $\rho_{21l} = \frac{1}{2^l}$, and $f_j(u) = \tanh(u)$. Thus hypothesis (H1) holds with $F_1 = F_2 = 1$, (H2) holds with $\delta = 0$, $\tau = 3$, and hypothesis (H3) holds with $\varepsilon \in (0, \ln 2)$. In this case, the matrix $\mathcal{M}$ in Theorem 3.2 reads as

$$\begin{aligned} \mathcal{M} &= \begin{bmatrix} 1-\mathrm{e}^{-10} & 0 \\ 0 & 1-\mathrm{e}^{-10} \end{bmatrix} - \begin{bmatrix} 0 & \frac{1-\mathrm{e}^{-10}}{5} + \frac{1-\mathrm{e}^{-10}}{2} \\ \frac{1-\mathrm{e}^{-10}}{10} + \frac{1-\mathrm{e}^{-10}}{5} & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1-\mathrm{e}^{-10} & -\frac{7(1-\mathrm{e}^{-10})}{10} \\ -\frac{3(1-\mathrm{e}^{-10})}{10} & 1-\mathrm{e}^{-10} \end{bmatrix}. \end{aligned}$$

Since $\mathcal{M}$ is a non-singular M-matrix (the eigenvalues are $\left(1-\mathrm{e}^{-10}\right)\left(1+\frac{\sqrt{21}}{10}\right)$ and $\left(1-\mathrm{e}^{-10}\right)\left(1-\frac{\sqrt{21}}{10}\right)$), by Theorem 3.2, the zero solution of (29) is globally exponentially stable. In Figure 1, see the numerical simulation of the solution of (29) with initial condition $\sigma = 0$ and

$$\overline{x}_0(j) = \left\{\begin{array}{ll} (\cos(j), \sin(j))^T, & j \in [-9, 0]_{\mathbb{Z}} \\ (0,0)^T, & j \in (-\infty, -10]_{\mathbb{Z}} \end{array}\right. .$$

Remark 8. We should say that example model (26) is a particular situation of [25, model (4.7)] and from [25, Corollary 4.2] we know that zero solution of (26) is globally exponentially stable.

Remark 9. The stability criterion in [16, Theorem 3.1] can not be applied to the model (29) because we are dealing with unbounded distributed delays. However, if the model had finite delays i.e., putting $\tanh(x_2(m-\tau_{12}))$ and $\tanh(x_1(m-\tau_{21}))$ with $\tau_{12}, \tau_{21} \in \mathbb{R}^+$, instead of $\displaystyle\sum_{l=1}^{\infty}\frac{3}{4^l}\tanh(x_2(m-l))$ and $\displaystyle\sum_{l=1}^{\infty}\frac{1}{2^l}\tanh(x_1(m-l))$ respectively, then [16, Theorem 3.1] would be applicable. The conclusion would be the global attractivity of the zero solution of the model. This is a weaker conclusion than to conclude the global exponential stability of the zero solution.

Example 4.2. Letting $n = 2$, $a_1 = \frac{1}{3}$, $a_2 = -\frac{2}{9}$, $b_{11} = \frac{4}{9}$, $b_{12} = b_{21} = 0$, $b_{22} = \frac{1}{3}$, $c_{111} = c_{112} = c_{122} = c_{211} = c_{221} = c_{222} = 0$, $c_{121} = c_{212} = \frac{1}{9}$, $d_{111} = d_{112} =$

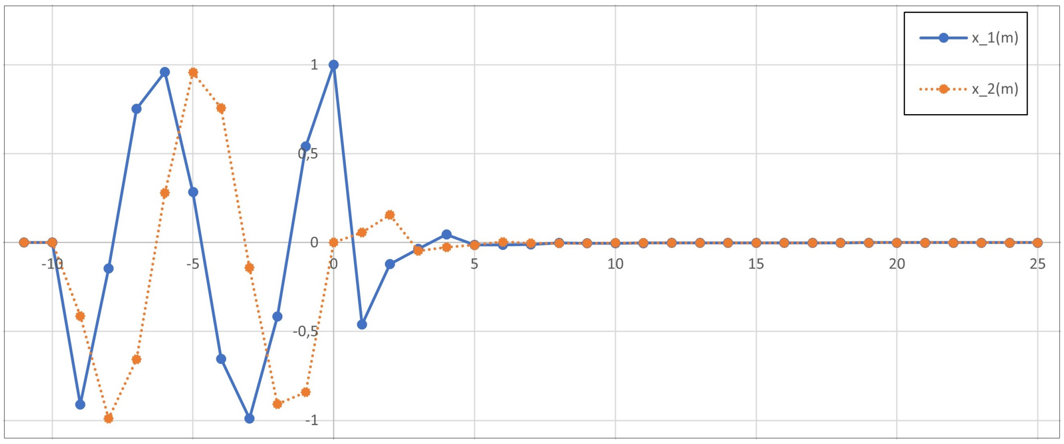

Figure 1. Solution $(x_1(m), x_2(m))^T$ of system (29) with initial condition $\sigma = 0$ and $\overline{x}_0(j) = \begin{cases} (\cos(j), \sin(j))^T, & j \in [-9, 0]_{\mathbb{Z}} \\ (0,0)^T, & j \in (-\infty, -10]_{\mathbb{Z}} \end{cases}$.

$d_{122} = d_{211} = d_{221} = d_{222} = 0$, $d_{121} = \frac{2}{9}$, $d_{212} = \frac{1}{9}$, $\rho_{ijl} = \frac{1}{2^l}$, $\delta_1(m) = \delta_2(m) = 2$, $\tau_{ijk}(m) = 2 + \cos(\pi m)$, $f_1(u) = f_2(u) = \tanh(u)$, and $g_1(u) = g_2(u) = \sin(u^2)$ in the high-order Hopfield neural network model (19), we have the delay difference system.

$$\begin{aligned} x_1(m+1) &= \frac{1}{3}x_1(m-2) + \frac{4}{9}\tanh(x_1(m)) \\ &+\frac{1}{9}\sin\left(x_2(m-2-\cos(\pi m))^2\right)\sin\left(x_1(m-2-\cos(\pi m))^2\right) \\ &+\frac{2}{9}\left(\sum_{l=1}^{\infty}\frac{1}{2^l}\sin\left(x_2(m-l)^2\right)\right)\left(\sum_{l=1}^{\infty}\frac{1}{2^l}\sin\left(x_1(m-l)^2\right)\right) \\ x_2(m+1) &= -\frac{2}{9}x_2(m-2) + \frac{1}{3}\tanh(x_2(m)) \\ &+\frac{1}{9}\sin\left(x_1(m-2-\cos(\pi m))^2\right)\sin\left(x_2(m-2-\cos(\pi m))^2\right) \\ &+\frac{1}{9}\left(\sum_{l=1}^{\infty}\frac{1}{2^l}\sin\left(x_1(m-l)^2\right)\right)\left(\sum_{l=1}^{\infty}\frac{1}{2^l}\sin\left(x_2(m-l)^2\right)\right) \end{aligned} \tag{30}$$

It is easy to conclude that hypothesis (H1) holds with $F_1 = F_2 = 1$, (H3) holds with $\xi \in (0, \ln 2)$, (HO1) holds with $M_1 = M_2 = G_1 = G_2 = 1$, and (HO2) holds with $\delta = 2$ and $\tau = 3$. For system (30), the matrix $\mathcal{Q}$ defined in Theorem 3.4 assumes the form

$$\mathcal{Q} = \begin{bmatrix} \frac{2}{3} & 0 \\ 0 & \frac{7}{9} \end{bmatrix} - \begin{bmatrix} \frac{4}{9} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} - \begin{bmatrix} 0 & \frac{1}{9}+\frac{2}{9} \\ \frac{1}{9}+\frac{1}{9} & 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{9} & -\frac{3}{9} \\ -\frac{2}{9} & \frac{4}{9} \end{bmatrix}$$

Since $\mathcal{Q}$ is a non-singular M-matrix (the eigenvalues are $\frac{3+\sqrt{7}}{9}$ and $\frac{3-\sqrt{7}}{9}$), by Theorem 3.4, the zero solution of (30) is globally exponentially stable. In Figure 2, see the numerical simulation of the solution of (30) with initial condition $\sigma = 0$ and $\overline{x}_0(j) = \begin{cases} (7\cos(j), 7\sin(j))^T, & j \in [-9, 0]_{\mathbb{Z}} \\ (0,0)^T, & j \in (-\infty, -10]_{\mathbb{Z}} \end{cases}$.

Remark 10. In comparison with the results in the recent paper of Z. Dong et al. [10], mainly with [10, Theorem 2], example (30) illustrates the improvements of our

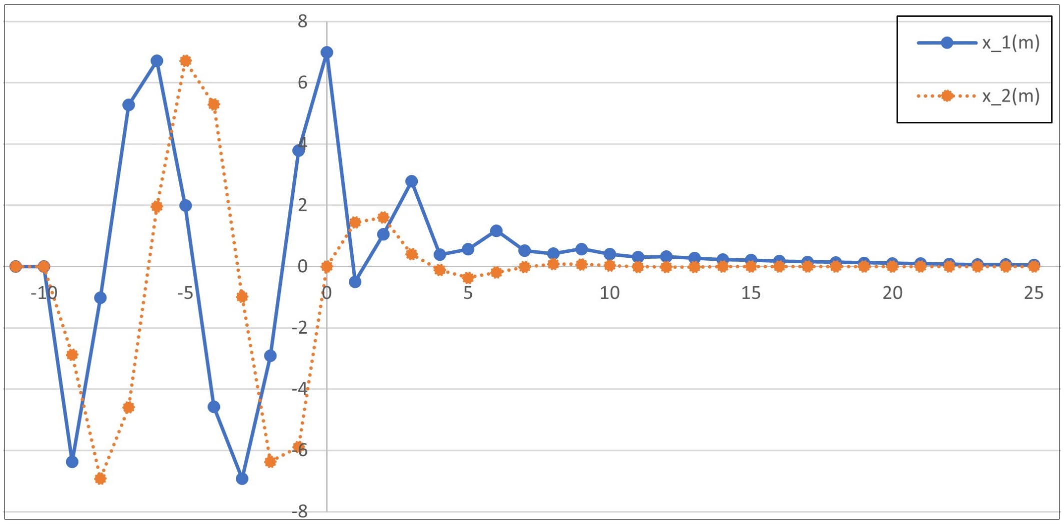

Figure 2. Solution $(x_1(m), x_2(m))^T$ of system (30) with initial condition $\sigma = 0$ and
$\overline{x}_0(j) = \begin{cases} (7\cos(j), 7\sin(j))^T, & j \in [-9, 0]_{\mathbb{Z}} \\ (0,0)^T, & j \in (-\infty, -10]_{\mathbb{Z}} \end{cases}$.

Theorem 3.4. In fact, the main result in [10] can not be applied to the system (30) because it has delay in the leakage terms ($\delta_i(m) = 2$), unbounded distributed delays and the activation function $g_1(u) = g_2(u) = \sin(u^2)$, $u \in \mathbb{R}$, is not a Lipschitz function.

5. Conclusions

In this paper, we present criteria for global exponential stability of zero equilibrium for classes of discrete-time, low-order and high-order, Hopfield neural network models with unbounded delays and delay in the leakage term (Theorems 3.2 and 3.4). A general stability criterion is first presented for a discrete-time delay system in general settings which can be applied to other delay models than Hopfield models (Theorem 2.1).

The proof method based on non-singular M-matrix is easier to apply than the usual Lyapunov method and the hypotheses are normally easy to verify. In comparison with the literature, the obtained stability results for low-order models have less computational complexity and the obtained stability results for high-order models generalize the previous results in [10] for the situation with unbounded delays and delay in leakage terms.

In the next work, we expect to extend the results here established for impulsive Hopfield, or Cohen-Grossberg, neural network models [11, 31].

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