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RESEARCH ARTICLE

Type O pure radiation metrics with a cosmological constant

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Abstract In this paper we complete the integration of the conformally flat pure radiation spacetimes with a non-zero cosmological constant Λ, and $\tau \neq 0$, by considering the case $\Lambda + \tau\bar{\tau} \neq 0$. This is a further demonstration of the power and suitability of the generalised invariant formalism (GIF) for spacetimes where only one null direction is picked out by the Riemann tensor. For these spacetimes, the GIF picks out a second null direction (from the second derivative of the Riemann tensor) and once this spinor has been identified the calculations are transferred to the simpler GHP formalism, where the tetrad and metric are determined. The whole class of conformally flat pure radiation spacetimes with a non-zero cosmological constant (those found in this paper, together with those found earlier for the case $\Lambda + \tau\bar{\tau} = 0$) have a rich variety of subclasses with zero, one, two, three, four or five Killing vectors.

1 Introduction

The method of integration within the Geroch–Held–Penrose (GHP) formalism using GHP operators [13] pioneered by Held [17,18] and developed by Edgar and Ludwig [4,6,7,23] has been shown to be particularly useful and efficient in spacetimes where two null directions are picked out by the geometry.

The generalised invariant formalism (GIF) of Machado Ramos and Vickers [28–30] generalises the GHP formalism by building the null rotation freedom of the second

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null direction into the formalism, which means that the GIF is built around only one spinor $\mathbf{o}_A$. An analogous integration method to that using GHP operators has been developed using operators of the GIF, [8,9,11].

The first investigation using the GIF integration method was for the class of conformally flat pure radiation spacetimes with zero cosmological constant [11]. The pure radiation component of the Ricci tensor immediately picks out one null direction $\mathbf{o}_A$, and a second intrinsic spinor $\mathbf{I}_A$ was obtained after a little manipulation in the GIF. After some more manipulation in the GIF, the investigations were transfered into, and completed in, the GHP formalism, by identifying the spinor $\mathbf{I}_A$ with the second dyad spinor $\boldsymbol{\iota}_A$ of the GHP formalism. In [11], the part of the investigation in the GIF which established the complete and involutive set of GIF tables was the most complicated; in particular because of the repeated use of the complicated GIF commutators.

In fact, with the benefit of hindsight, it is now clear that it was not necessary to carry out *all* of these GIF calculations in [11]: the crucial step in the GIF was to generate explicitly this second intrinsic spinor $\mathbf{I}_A$ from within the GIF formalism. As soon as this spinor was found and the GIF commutators applied to it, then $\mathbf{I}_A$ could have been identified as the second spinor $\boldsymbol{\iota}_A$ in the dyad for the GHP formalism; at this stage, the investigation could have been *immediately* transferred to the GHP formalism. Had this earlier transfer been made, the latter part of the complicated GIF calculations in [11] could have been replaced with simpler GHP calculations.

In this paper, we wish to further develop the GIF operator method by generalising this earlier derivation [11] of the metric for conformally flat pure radiation spaces to include the case of a non-zero cosmological constant. In [9] we looked at the subclass of these spacetimes for which we were unable to find a second unique *intrinsic* spinor due to the presence of one degree of null isotropy freedom; in this paper we look at the other subclass where there is no null isotropy freedom, and a second unique *intrinsic* spinor $\mathbf{I}_A$ is quickly generated within the GIF. This means that we can quickly transfer to the GHP formalism and so minimise the calculations.

These spacetimes will illustrate further refinements of our method, and they will also be shown to have a rich Killing vector structure.

In Sect. 2 we summarise the most relevant facts of the GIF equations, and discuss how to transfer from the GIF to the GHP formalism.

In the beginning of Sect. 3, we carry through the integration procedure, obtaining a table for the crucial second spinor $\mathbf{I}_A$. As soon as we obtain this second unique *intrinsic* spinor $\mathbf{I}_A$, and apply the GIF commutators to it, we translate all the results into the GHP formalism; in the latter part of this section we show, by a straightforward relabelling and rearranging of some of the coordinate candidates and unknown functions, that their GHP tables can be put into a much simpler form, so that we can more easily complete the application of the commutators to these tables. Finally, from these tables, we write down the tetrad and the metric explicitly.

The procedure in Sect. 3 is dependent on the condition that the four zero-weighted scalars, to which we assign the role of coordinate candidates, are functionally independent and hence can play the role of coordinates; indeed, if we make the assumption that none of these scalars are constants, then a check of the determinant formed from their four tables shows that all four scalars are in fact functionally independent. On the other hand, it is found that although three of the four coordinate candidates cannot be constant, the other one may be; in addition, we make some other assumptions in our calculations which exclude some other special cases. Hence the tetrad and metric obtained in Sect. 3 are not the most general that can be obtained for this class of spacetimes.

In Sect. 4, we extend our approach to include one of the special cases which were excluded in the analysis in Sect. 3. In Sect. 5 we consider the remaining special case and discuss in more detail the introduction and role of *complementary* coordinates, and how to *copy* their tables; also in that section we put together all the subclasses and present the most general form for the metric.

In Sect. 6 we summarise the methods and results of this paper together with those of [9].

2 GIF equations and transfering from GIF to the GHP formalism

A full explanation of GIF is given in [28–30]. For the purpose of this paper, the summaries given in [11] and especially [9] are sufficient.

In this section, we will list only those equations in GIF to which we will make direct reference. Although the definition of the GIF differential operators ($\mathbf{Þ}, \mathbf{Þ}, \boldsymbol{\eth}', \mathbf{Þ}'$) appears quite complicated, the fact that they take symmetric spinors to symmetric spinors means that one can write down the equations in a more compact and index free notation. In this compacted notation we have the following useful identities for scalars of weight $\{p, q\}$,

$$(\mathbf{Þ}'\eta)\cdot\bar{\mathbf{o}} = \frac{1}{2}\{(\boldsymbol{\eth}'\eta) - q\overline{\mathbf{T}}\eta\} \tag{1}$$

$$(\mathbf{Þ}'\eta)\cdot\mathbf{o} = \frac{1}{2}\{(\boldsymbol{\eth}\eta) - p\mathbf{T}\eta\} \tag{2}$$

$$(\boldsymbol{\eth}'\eta)\cdot\mathbf{o} = \frac{1}{2}\{(\mathbf{Þ}\eta) - p\mathbf{R}\eta\} \tag{3}$$

$$(\boldsymbol{\eth}\eta)\cdot\bar{\mathbf{o}} = \frac{1}{2}\{(\mathbf{Þ}\eta) - q\overline{\mathbf{R}}\eta\} \tag{4}$$

$$(\mathbf{Þ}'\eta)\cdot\mathbf{o}\cdot\bar{\mathbf{o}} = \frac{1}{4}\{(\mathbf{Þ}\eta) - p\mathbf{R}\eta - q\overline{\mathbf{R}}\eta\} \tag{5}$$

For a valence (1,0)-spinor $\boldsymbol{\eta}_A$ of weight $\{\mathbf{p}, \mathbf{q}\}$ we get

$$(\mathbf{Þ}'\boldsymbol{\eta})\cdot\mathbf{o} = \frac{1}{3}\{\mathbf{Þ}'(\boldsymbol{\eta}\cdot\mathbf{o}) + (\boldsymbol{\eth}\boldsymbol{\eta}) - (\mathbf{p}-1)\mathbf{T}\boldsymbol{\eta}\} \tag{6}$$

and

$$(\mathbf{Þ}'\boldsymbol{\eta})\cdot\bar{\mathbf{o}} = \frac{1}{3}\{\mathbf{Þ}'(\boldsymbol{\eta}\cdot\bar{\mathbf{o}}) + (\boldsymbol{\eth}'\boldsymbol{\eta}) - \mathbf{q}\overline{\mathbf{T}}\boldsymbol{\eta}\} \tag{7}$$

The GIF commutators (applied to a general symmetric spinor $\boldsymbol{\eta}$ of weight $\{\mathbf{p}, \mathbf{q}\}$ and with N unprimed and N' primed indices) are

$$(\boldsymbol{\text{þ}}\boldsymbol{\text{þ}}' - \boldsymbol{\text{þ}}'\boldsymbol{\text{þ}})\boldsymbol{\eta} = (\bar{\tau}\boldsymbol{\eth} + \tau\boldsymbol{\eth}')\boldsymbol{\eta} + (\mathbf{p} - N)\Lambda\boldsymbol{\eta} + (\mathbf{q} - N')\Lambda\boldsymbol{\eta} \tag{8}$$

$$(\boldsymbol{\text{þ}}\boldsymbol{\eth} - \boldsymbol{\eth}\boldsymbol{\text{þ}})\boldsymbol{\eta} = 2\Lambda(\boldsymbol{\eta}\cdot\mathbf{o}) \tag{9}$$

$$(\boldsymbol{\text{þ}}\boldsymbol{\eth}' - \boldsymbol{\eth}'\boldsymbol{\text{þ}})\boldsymbol{\eta} = 2\Lambda(\boldsymbol{\eta}\cdot\bar{\mathbf{o}}) \tag{10}$$

$$(\boldsymbol{\eth}\boldsymbol{\eth}' - \boldsymbol{\eth}'\boldsymbol{\eth})\boldsymbol{\eta} = -(\mathbf{p} - N)\Lambda\boldsymbol{\eta} + (\mathbf{q} - N')\Lambda\boldsymbol{\eta} \tag{11}$$

$$(\boldsymbol{\text{þ}}'\boldsymbol{\eth} - \boldsymbol{\eth}\boldsymbol{\text{þ}}')\boldsymbol{\eta} = -\tau\boldsymbol{\text{þ}}'\boldsymbol{\eta} - \Phi(\boldsymbol{\eta}\cdot\mathbf{o}) \tag{12}$$

$$(\boldsymbol{\text{þ}}'\boldsymbol{\eth}' - \boldsymbol{\eth}'\boldsymbol{\text{þ}}')\boldsymbol{\eta} = -\bar{\tau}\boldsymbol{\text{þ}}'\boldsymbol{\eta} - \Phi(\boldsymbol{\eta}\cdot\bar{\mathbf{o}}) \tag{13}$$

where $(\boldsymbol{\eta}\cdot\mathbf{o})$ is the $(N-1, N')$ -spinor $\boldsymbol{\eta}_{A_1....A_N A_1....A_{N'}}\mathbf{o}^{A_N}$, and $(\boldsymbol{\eta}\cdot\bar{\mathbf{o}})$ is the $(N, N'-1)$-spinor $\boldsymbol{\eta}_{A_1....A_N A_1....A_{N'}}\bar{\mathbf{o}}^{A_{N'}}$, and if the contraction is not possible then these terms are set to zero.

To perform the translation from GIF to the GHP formalism we use the links between the GIF operators and the GHP operators þ, $\eth$, $\eth'$, $\text{þ}'$, and in the case of a scalar field this gives

$$\begin{aligned}
(\boldsymbol{\text{þ}}'\eta)_{ABA'B'} &= (\text{þ}'\eta)\mathbf{o}_A\mathbf{o}_B\bar{\mathbf{o}}_{A'}\bar{\mathbf{o}}_{B'} - (\eth'\eta - q\bar{\tau}\eta)\mathbf{o}_A\mathbf{o}_B\bar{\mathbf{o}}_{(A'}\bar{\boldsymbol{\iota}}_{B')}\\
&\quad -(\eth\eta - p\tau\eta)\mathbf{o}_{(A}\boldsymbol{\iota}_{B)}\bar{\mathbf{o}}_{A'}\bar{\mathbf{o}}_{B'} + (\text{þ}\eta - p\rho\eta - q\bar{\rho}\eta)\mathbf{o}_{(A}\boldsymbol{\iota}_{B)}\bar{\mathbf{o}}_{(A'}\bar{\boldsymbol{\iota}}_{B')}\\
&\quad -p\sigma\boldsymbol{\iota}_A\boldsymbol{\iota}_B\bar{\mathbf{o}}_{A'}\bar{\mathbf{o}}_{B'} - q\bar{\sigma}\mathbf{o}_A\mathbf{o}_B\boldsymbol{\iota}_{A'}\boldsymbol{\iota}_{B'}\\
&\quad +p\kappa\boldsymbol{\iota}_A\boldsymbol{\iota}_B\bar{\mathbf{o}}_{(A'}\bar{\boldsymbol{\iota}}_{B')} + q\bar{\kappa}\mathbf{o}_{(A}\boldsymbol{\iota}_{B)}\bar{\boldsymbol{\iota}}_{A'}\bar{\boldsymbol{\iota}}_{B'}
\end{aligned} \tag{14}$$

$$\begin{aligned}
(\boldsymbol{\eth}'\eta)_{ABA'} &= (\eth'\eta)\mathbf{o}_A\mathbf{o}_B\bar{\mathbf{o}}_{A'} - (\text{þ}\eta - p\rho\eta)\mathbf{o}_{(A}\boldsymbol{\iota}_{B)}\bar{\mathbf{o}}_{A'}\\
&\quad +q\bar{\sigma}\mathbf{o}_A\mathbf{o}_B\boldsymbol{\iota}_{A'} - p\kappa\boldsymbol{\iota}_A\boldsymbol{\iota}_B\bar{\mathbf{o}}_{A'} - q\bar{\kappa}\mathbf{o}_{(A}\boldsymbol{\iota}_{B)}\bar{\boldsymbol{\iota}}_{A'}
\end{aligned} \tag{15}$$

$$\begin{aligned}
(\boldsymbol{\eth}\eta)_{AA'B'} &= (\eth\eta)\mathbf{o}_A\bar{\mathbf{o}}_{A'}\bar{\mathbf{o}}_{B'} - (\text{þ}\eta - q\bar{\rho}\eta)\mathbf{o}_A\bar{\mathbf{o}}_{(A'}\bar{\boldsymbol{\iota}}_{B')}\\
&\quad +p\sigma\boldsymbol{\iota}_A\bar{\mathbf{o}}_{A'}\bar{\mathbf{o}}_{B'} - p\kappa\boldsymbol{\iota}_A\bar{\mathbf{o}}_{(A'}\bar{\boldsymbol{\iota}}_{B')} - q\bar{\kappa}\mathbf{o}_A\bar{\boldsymbol{\iota}}_{A'}\bar{\boldsymbol{\iota}}_{B'}
\end{aligned} \tag{16}$$

$$(\boldsymbol{\text{þ}}\eta)_{AA'} = (\text{þ}\eta)\mathbf{o}_A\mathbf{o}_B + p\kappa\boldsymbol{\iota}_A\bar{\mathbf{o}}_{A'} - q\bar{\kappa}\mathbf{o}_A\bar{\boldsymbol{\iota}}_{A'}\ . \tag{17}$$

For the subsequent calculations we will need the GHP commutator equations, which can be obtained from [13] (specialised to this class of spacetimes),

$$\begin{aligned}
(\text{þ}\text{þ}' - \text{þ}'\text{þ})\eta &= \Big((\bar{\tau} - \tau')\eth + (\tau - \bar{\tau}')\eth' + p(\tau\tau' + \Lambda) + q(\bar{\tau}\bar{\tau}' + \Lambda)\Big)\eta\\
(\text{þ}\eth - \eth\text{þ})\eta &= -\bar{\tau}'\text{þ}\eta\\
(\eth\eth' - \eth'\eth)\eta &= \Big((\bar{\rho}' - \rho')\text{þ} - p\Lambda + q\Lambda\Big)\eta\\
(\text{þ}'\eth - \eth\text{þ}')\eta &= \Big(\rho'\eth + \bar{\sigma}'\eth' - \tau\text{þ}' - \bar{\kappa}'\text{þ} - q\bar{\tau}\bar{\sigma}' - p\rho'\tau\Big)\eta
\end{aligned} \tag{18}$$

where η is an arbitrary scalar of weight $\{p, q\}$.

Of course now we encounter the problem that the GHP formalism involves the spin coefficients τ', σ', μ', κ' which are missing from the GIF. However, assuming that we have obtained a table for $\mathbf{I}$ in our GIF analysis, once we have identified $\mathbf{I}$ with the second dyad spinor $\boldsymbol{\iota}$, we can use this table to obtain directly these additional four spin coefficients as follows

$$\begin{aligned}
\tau' &= -\iota^B D\,(\iota_B) = -\iota^B \text{Þ}\,(\iota_B) = -\iota^B \iota^C \bar{\iota}^{C'} \text{Þ}_{CC'}\,(\iota_B) \\
\rho' &= -\iota^B \iota^C \bar{\iota}^{C'} \bar{\iota}^{D'} \eth_{CC'D'}\,(\iota_B) \\
\sigma' &= -\iota^B \iota^C \iota^D \bar{\iota}^{C'} \eth'_{CDC'}\,(\iota_B) \\
\kappa' &= -\iota^B \iota^C \iota^D \bar{\iota}^{C'} \bar{\iota}^{D'} \text{Þ}'_{CDC'D'}\,(\iota_B)
\end{aligned} \tag{19}$$

3 The integration procedure: the generic case

3.1 Preliminary rearrangement

We are concerned with the Petrov type O pure radiation spaces with non-zero Ricci scalar, and since the equations for these spaces were given in [9], we will not repeat them here.

The Riemann tensor Φ and the spin coefficient τ supply three real scalars, and it will be convenient to rearrange slightly these three scalars, and use instead the real zero-weighted scalar

$$A = \frac{1}{\sqrt{2\tau\overline{\tau}}} \tag{20}$$

and the weighted scalars[1]

$$P = \sqrt{\frac{\tau}{2\overline{\tau}}}, \quad Q = \frac{\sqrt{\Phi}}{\sqrt[4]{2\tau\overline{\tau}}} \tag{21}$$

where P is a complex scalar of weight $\{1, -1\}$, with $P\overline{P} = \frac{1}{2}$; and Q is a real scalar of weight $\{-1, -1\}$. (As well as $\Phi = \frac{Q^2}{A} \neq 0 \neq \Lambda$, we are assuming $\tau = P/A \neq 0$, and so each of $A,\ P,\ Q$, will always be defined and different from zero.)

These particular choices enable us to replace the Ricci equations with

$$\text{Þ}A = 0, \quad \eth A = -2P(\Lambda A^2 + 1/2) = -2P\text{k}, \quad \eth' A = -2\overline{P}(\Lambda A^2 + 1/2) = -2\overline{P}\text{k} \tag{22}$$

$$\text{Þ}(\overline{P}Q) = 0, \quad \eth(\overline{P}Q) = \frac{1}{2}Q\Lambda A, \quad \eth'(\overline{P}Q) = -3Q\overline{P}^2\Lambda A \tag{23}$$

[1] We have retained the notation P, Q which was used in [11] for these two weighted scalars; note the slightly different definitions compared with $\mathcal{P}, \mathcal{Q}$ used in [9] when considering the case $\Lambda + \tau\bar{\tau} = 0$. Care needs to be taken when comparing with the various quantities labelled with P, Q (sometimes p, q) in [1,2,8,14,32] and other references.

where we now have,[2]

$$\bar{k} = \Lambda A^2 + 1/2 \tag{24}$$

In [9] we considered the subclass $k = 0$; in the present paper we consider the remaining subclass $k \neq 0$

At various steps in the sequel it will be obvious that we are assuming $\bar{k} \neq 3/2$; however, this is not an additional restriction since we can deduce from the partial table (22) for A that this condition must always be satisfied.

3.2 Constructing a table for **I** and applying commutators to **I**

For our integration procedure we begin by completing the partial table (23) for the $\{-2, 0\}$ weighted scalar $\overline{P}Q$,

$$\text{þ}(\overline{P}Q) = 0, \quad \eth(\overline{P}Q) = \frac{1}{2}\Lambda A Q, \quad \eth'(\overline{P}Q) = -3\Lambda A Q\overline{P}^2, \quad \text{þ}'(\overline{P}Q) = \mathbf{J} \tag{25}$$

where we have completed the table with some spinor **J**, which is as yet undetermined.

We know from (1) and (2) that

$$\text{þ}'(\overline{P}Q)\cdot\bar{\mathbf{o}} = \eth'(\overline{P}Q) \tag{26}$$

$$\text{þ}'(\overline{P}Q)\cdot\mathbf{o} = \eth(\overline{P}Q) + 2\tau\overline{P}Q = \eth(\overline{P}Q) + \frac{Q}{A} \tag{27}$$

Substituting (25) we can then write

$$\mathbf{J} = -\Big(\frac{Q}{A} + \frac{1}{2}\Lambda A Q\Big)\mathbf{I} + 3\Lambda A Q\overline{P}^2\bar{\mathbf{I}} \tag{28}$$

where

$$\mathbf{I}\cdot\bar{\mathbf{o}} = 0, \quad \mathbf{I}\cdot\mathbf{o} = -1 \tag{29}$$

Hence **I** is a $(1, 0)$ valence spinor, and from

$$\Big(\text{þ}'(\overline{P}Q)\Big)_{ABA'B'} = -\Big(\frac{Q}{A} + \frac{1}{2}\Lambda A Q\Big)\mathbf{I}_{(A}\mathbf{o}_{B)}\bar{\mathbf{o}}_{A'}\bar{\mathbf{o}}_{B'} + 3\Lambda A Q\overline{P}^2\bar{\mathbf{I}}_{(A'}\bar{\mathbf{o}}_{B')}\mathbf{o}_A\mathbf{o}_B \tag{30}$$

we conclude that its weight is $\{-\mathbf{1}, \mathbf{0}\}$.

It is important to note two crucial properties of the new spinor **I**. Firstly, for this whole class of spaces, **I** can never be zero, nor parallel to **o**. Secondly, for the subclass under consideration in this paper, the spinor **I** is given *uniquely* in terms of the elements of the GIF formalism and so is an intrinsic spinor; this can be seen when we solve for **I** from (30) and its complex conjugate remembering that $\bar{k} \neq 0$ in this paper.

[2] This quantity $\bar{k}$ is closely related to the quantity κ in [32] and to k in [14]; any of these quantities can be used to classify the conformally flat pure radiation spaces (as well as more general Petrov types) into different subclasses.

It will be useful in the sequel to have separate tables for P and Q

$$\text{þ} P=0, \quad \eth P=-2\Lambda A P^2, \quad \eth' P=\Lambda A, \quad \text{þ}' P=\frac{2P^2 \bar{k}}{A}\mathbf{I}-\frac{\bar{k}}{A}\bar{\mathbf{I}} \tag{31}$$

$$\begin{aligned} &\text{þ} Q=0, \quad \eth Q=-\Lambda A Q P, \quad \eth' Q=-\Lambda A Q \overline{P}, \\ &\text{þ}' Q=-\frac{Q P(\frac{3}{2}-\bar{k})}{A}\mathbf{I}-\frac{Q\overline{P}(\frac{3}{2}-\bar{k})}{A}\bar{\mathbf{I}} \end{aligned} \tag{32}$$

Our first mission is to find the table for $\mathbf{I}$ which should follow from applying the GIF commutators to the table for $(\overline{P} Q)$; but first of all we will need to complete the partial table (22) for A. We obtain

$$\text{þ} A=0, \quad \eth A=-2P\bar{k}, \quad \eth' A=-2\overline{P}\bar{k}, \quad \text{þ}' A=\mathbf{C} \tag{33}$$

where we have completed the table with a spinor $\mathbf{C}$, which is as yet undetermined. It follows from (1) and (2) that

$$\mathbf{C}=\frac{Q}{A}C\bar{k}^2+2P\bar{k}\mathbf{I}+2\overline{P}\bar{k}\bar{\mathbf{I}} \tag{34}$$

and so $\mathbf{C}$ is a Hermitian (1, 1) type spinor of weight $\{\mathbf{2},\mathbf{2}\}$, with C a zero-weighted real scalar, as yet undetermined. We have introduced specific factors alongside the unknown scalar C for subsequent simplicity in presentation and to ensure that C is zero-weighted.

We are now able to apply the GIF commutators to the table for $(\overline{P} Q)$ which yields a partial table for the spinor$\mathbf{I}$; we obtain

$$\begin{aligned} \text{þ}\mathbf{I}&=\frac{3\Lambda A\overline{P}}{(\frac{3}{2}-\bar{k})}\\ \eth\mathbf{I}&=-\frac{\Lambda Q C\bar{k}(1-4\Lambda A^2)}{4\,(\frac{3}{2}-\bar{k})}-\frac{3\Lambda A\overline{P}}{(\frac{3}{2}-\bar{k})}\bar{\mathbf{I}}\\ \eth'\mathbf{I}&=\frac{3\Lambda C Q\bar{k}}{8P^2(\frac{3}{2}-\bar{k})}-\frac{3\Lambda A\overline{P}}{(\frac{3}{2}-\bar{k})}\mathbf{I}\\ \text{þ}'\mathbf{I}&=\mathbf{W} \end{aligned} \tag{35}$$

where we have completed the table with some spinor $\mathbf{W}$ as yet undetermined. In a similar manner as for previous tables, but this time, using (6) and (7) we find that

$$\mathbf{W}=\frac{\overline{P}Q^2}{A}W+\frac{\Lambda Q C\bar{k}(1-4\Lambda A^2)}{4\,(\frac{3}{2}-\bar{k})}\mathbf{I}\quad-\frac{3\Lambda Q C\bar{k}}{8P^2(\frac{3}{2}-\bar{k})}\bar{\mathbf{I}}-\frac{P}{A}\mathbf{I}^2+\frac{3\Lambda A\overline{P}}{(\frac{3}{2}-\bar{k})}\mathbf{I}\bar{\mathbf{I}} \tag{36}$$

where W is a zero-weighted *complex* scalar, as yet undetermined.

We next apply the GIF commutators to A and obtain the partial table for C,

$$
\begin{aligned}
\text{þ} C &= -\frac{4}{Q(\frac{3}{2}-\kappa)} \\
\eth C &= -\frac{P\Lambda AC(5\Lambda A^2-2)}{(\frac{3}{2}-\kappa)} + \frac{4}{Q(\frac{3}{2}-\kappa)}\bar{\mathbf{I}} \\
\eth' C &= -\frac{\overline{P}\Lambda AC(5\Lambda A^2-2)}{(\frac{3}{2}-\kappa)} + \frac{4}{Q(\frac{3}{2}-\kappa)}\mathbf{I} \\
\text{þ}' C &= \mathbf{L}
\end{aligned}
\tag{37}
$$

where $\mathbf{L}$ is a Hermitian (1, 1) type spinor of weight $\{\mathbf{2}, \mathbf{2}\}$ determined, from (1) and (2), to be:

$$
\begin{aligned}
\mathbf{L} = \frac{Q}{A}L + \Big(\frac{P\Lambda CA(5\Lambda A^2-2)}{(\frac{3}{2}-\kappa)}\Big)\mathbf{I} + \Big(\frac{\overline{P}\Lambda CA(5\Lambda A^2-2)}{(\frac{3}{2}-\kappa)}\Big)\bar{\mathbf{I}} \\
-\frac{1}{Q}\Big(\frac{4}{(\frac{3}{2}-\kappa)}\Big)\mathbf{I}\bar{\mathbf{I}}
\end{aligned}
\tag{38}
$$

where L is a zero-weighted real scalar, as yet undetermined.

The theory requires that we also apply the GIF commutators to the table for $\mathbf{I}$, which yields a partial table for complex W,

$$
\begin{aligned}
\text{þ} W &= \frac{\Lambda C(5\Lambda A^2+4)\,\kappa^2}{Q\,(\frac{3}{2}-\kappa)^2} \\
\eth W &= -2P + \frac{\Lambda^2 C^2 A\kappa^2(-8\Lambda^2 A^4 + 28\Lambda A^2 + 7)}{8\overline{P}\,(\frac{3}{2}-\kappa)^2} - \frac{3\Lambda A\overline{W}}{2\overline{P}(\frac{3}{2}-\kappa)} - \frac{\Lambda AW}{\overline{P}} \\
&\quad -\frac{\Lambda\kappa L(1-4\Lambda A^2)}{4\overline{P}(\frac{3}{2}-\kappa)} - \frac{\Lambda C\kappa^2(5\Lambda A^2+4)}{Q\,(\frac{3}{2}-\kappa)^2}\bar{\mathbf{I}} \\
\eth' W &= \frac{3\Lambda^2 C^2 A\kappa^2(4\Lambda A^2+5)}{8P(\frac{3}{2}-\kappa)^2} - \frac{\Lambda A\kappa W}{P(\frac{3}{2}-\kappa)} + \frac{3\Lambda L\kappa}{4P(\frac{3}{2}-\kappa)} - \frac{\Lambda C\kappa^2(5\Lambda A^2+4)}{Q\,(\frac{3}{2}-\kappa)^2}\mathbf{I}
\end{aligned}
\tag{39}
$$

So we have obtained the core element required in our GIF analysis: a new spinor $\mathbf{I}$, its table, and the results of applying the GIF commutators to this table. Since $\mathbf{I}$ is uniquely defined in terms of intrinsic elements of the GIF, we can now transfer these tables into the GHP formalism.

3.3 Transfering to the GHP formalism

We now identify this spinor **I** with the second dyad spinor $\boldsymbol{\iota}$ of the GHP formalism. Then the two tables for the zero weighted A, C can be immediately translated into the ordinary GHP scalar operators,

$$\text{þ} A = 0, \quad \eth A = -2P\text{ꝁ}, \quad \eth' A = -2\overline{P}\text{ꝁ}, \quad \text{þ}' A = \frac{Q}{A} C\text{ꝁ}^2 \tag{40}$$

$$\begin{aligned} \text{þ} C &= -\frac{4}{Q(\frac{3}{2} - \text{ꝁ})} \\ \eth C &= -\frac{P\Lambda AC(5\Lambda A^2 - 2)}{(\frac{3}{2} - \text{ꝁ})} \\ \eth' C &= -\frac{\overline{P}\Lambda AC(5\Lambda A^2 - 2)}{(\frac{3}{2} - \text{ꝁ})} \\ \text{þ}' C &= \frac{Q}{A} L \end{aligned} \tag{41}$$

This translation is carried out using (14), (15), (16), (17), and is especially simple since the operators are acting on scalars. The table for complex W can also be easily rewritten in GHP operators, but it will be more convenient to write down two tables for the real and imaginary parts of W by putting,

$$M = \frac{1}{2}(W + \overline{W}) - A, \quad B = \frac{i}{2}(W - \overline{W}) \tag{42}$$

which gives,

$$\begin{aligned} \text{þ} M &= \frac{\Lambda C\text{ꝁ}^2(5\Lambda A^2 + 4)}{Q\,(\frac{3}{2} - \text{ꝁ})^2} \\ \eth M &= -\frac{2\Lambda A^2 P\text{ꝁ}}{(\frac{3}{2} - \text{ꝁ})} + \frac{\Lambda^2 PAC^2\text{ꝁ}^3(-\Lambda A^2 + \frac{11}{2})}{(\frac{3}{2} - \text{ꝁ})^2} + \frac{\Lambda PL\text{ꝁ}^2}{(\frac{3}{2} - \text{ꝁ})} - \frac{3\Lambda APM}{(\frac{3}{2} - \text{ꝁ})} \\ &\quad - \frac{i2k\Lambda \text{Ⱥ} BP}{(\frac{3}{2} - \text{ꝁ})} \\ \eth' M &= -\frac{2\Lambda A^2 \overline{P}\text{ꝁ}}{(\frac{3}{2} - \text{ꝁ})} + \frac{\Lambda^2 \overline{P}AC^2\text{ꝁ}^3(-\Lambda A^2 + \frac{11}{2})}{(\frac{3}{2} - \text{ꝁ})^2} + \frac{\Lambda \overline{P}L\text{ꝁ}^2}{(\frac{3}{2} - \text{ꝁ})} - \frac{3\Lambda A\overline{P}M}{(\frac{3}{2} - \text{ꝁ})} \\ &\quad + \frac{i2k\Lambda \text{Ⱥ} B\overline{P}}{(\frac{3}{2} - \text{ꝁ})} \end{aligned} \tag{43}$$

and

$$\begin{aligned} þB &= 0 \\ ðB &= i\left(-2P\bar{k} - \Lambda^2 PAC^2\bar{k}^2 - \Lambda PL\bar{k} - 2\Lambda APM\right) \\ ð'B &= -i\left(-2\overline{P}\bar{k} - \Lambda^2\overline{P}AC^2\bar{k}^2 - \Lambda\overline{P}L\bar{k} - 2\Lambda A\overline{P}M\right) \end{aligned} \tag{44}$$

In the table (35) for **I** we will now make the substitution $W = A + M - iB$: this table is needed to calculate τ', ρ', σ', κ' at the end of this section. From (25) and (14), (15), (16), (17), the GHP tables for the weighted scalars P, Q, are

$$þP = 0, \quad ðP = -2P^2\Lambda A, \quad ð'\mathcal{P} = \Lambda A, \quad þ'P = 0 \tag{45}$$

$$þQ = 0, \quad ðQ = QP\Lambda A, \quad ð'\mathcal{Q} = -Q\overline{P}\Lambda A, \quad þ'Q = 0 \tag{46}$$

The zero-weighted scalars A, C, M, B suggest themselves as the four coordinate candidates. Then, *providing that these scalars are functionally independent*, they can be adopted as coordinates. It will be easier to check for this functional independence after we have simplified the structure of the tables and after we have also completed the calculation by applying the commutators to all four candidates.

For subsequent calculations we will require the GHP commutators, which in turn require the missing four GHP spin coefficients. These four GHP spin coefficients follow immediately from (20) and the table for **I** (35), and are given by

$$\begin{aligned} \tau' &= -\iota^B\iota^C\bar{\iota}^{C'}\mathbf{þ}_{CC'}(\iota_B) = \frac{3\Lambda A\overline{P}}{(\frac{3}{2} - \bar{k})} \\ \rho' &= -\iota^B\iota^C\bar{\iota}^{C'}\bar{\iota}^{D'}\mathbf{ð}_{CC'D'}(\iota_B) = \frac{\Lambda QC\bar{k}(1 - 4\Lambda A^2)}{4\,(\frac{3}{2} - \bar{k})} \\ \sigma' &= -\iota^B\iota^C\iota^D\bar{\iota}^{C'}\mathbf{ð}'_{CDD'}(\iota_B) = -\frac{3\Lambda CQ\bar{k}}{8P^2(\frac{3}{2} - \bar{k})} \\ \kappa' &= -\iota^B\iota^C\iota^D\bar{\iota}^{C'}\bar{\iota}^{D'}\mathbf{þ}'_{CDC'D'}(\iota_B) = \frac{\overline{P}Q^2}{A}(A + M - iB) \end{aligned} \tag{47}$$

and of course $\tau = P/A$; these should now be substituted into the GHP commutators (18).

3.4 Simplifying and completing the tables in the GHP operators

We have already obtained a GHP table (40) for the real zero-weighted scalar A, and via the commutators we have also obtained a GHP table (41) for the real zero-weighted scalar C; when we apply the GHP commutators we obtain the partial table for the real zero-weighted scalar L,

$$
\begin{aligned}
\text{Þ}L &= \frac{-18\Lambda^2 C A^3 \bar{k}}{Q\,(\frac{3}{2}-\bar{k})^2}\\
\eth L &= -\frac{\Lambda C^2 P \bar{k}^2 (11\Lambda A^2 - 2)}{(\frac{3}{2}-\bar{k})^2} - \frac{\Lambda L A P (4\Lambda A^2 - 1)}{(\frac{3}{2}-\bar{k})} + \frac{4P}{(\frac{3}{2}-\bar{k})}(A + M + iB)\\
\eth' L &= -\frac{\Lambda C^2 \overline{P} \bar{k}^2 (11\Lambda A^2 - 2)}{(\frac{3}{2}-\bar{k})^2} - \frac{\Lambda L A \overline{P} (4\Lambda A^2 - 1)}{(\frac{3}{2}-\bar{k})} + \frac{4\overline{P}}{(\frac{3}{2}-\bar{k})}(A + M - iB)
\end{aligned}
\tag{48}
$$

So we can adopt C as a second coordinate candidate and add the partial table for L to our equations.

We would next like to complete the partial tables for two of M, B, L, and then apply the commutators to each to exploit them as two more coordinate candidates. However, it will be easier if we first do a little rearranging and relabelling. The simpler the form which we can obtain for our tables for the four coordinate candidates, the simpler the form will be for the associated metric.

A direct substitution of M by T via

$$
T = -\frac{1}{\sqrt{2}\bar{k}^{\frac{1}{2}}} M - \frac{\Lambda \bar{k}^{\frac{3}{2}} (9\bar{k} + 4)}{8\sqrt{2}} C^2 - \frac{\Lambda A \bar{k}^{\frac{1}{2}}}{2\sqrt{2}} L \tag{49}
$$

enables the complicated partial table for M to be replaced with

$$
\text{Þ}T = 0, \quad \eth T = 0, \quad \eth' T = 0, \quad \text{Þ}' T = \frac{Q \bar{k}^{\frac{1}{4}}}{A} F \tag{50}
$$

which we have completed in the usual way for the, as yet undetermined, zero-weighted scalar function F. So we decide to replace M with T as a third coordinate candidate.

It now remains to get a simpler replacement for the rather complicated tables (44) and (48), for B and L, respectively.

Making a direct substitution of L with S in (44) via

$$
S = (2\bar{k} + \Lambda^2 A C^2 \bar{k}^2 + \Lambda L \bar{k} + 2\Lambda A M)/\Lambda\, \bar{k}^{1/2} \tag{51}
$$

gives the simpler form

$$
\text{Þ}B = 0, \quad \eth B = -iP\Lambda \bar{k}^{1/2} S, \quad \eth' B = iP\Lambda \bar{k}^{1/2} S \tag{52}
$$

as well as replacing the complicated partial table for L with the simpler partial table for S

$$
\text{Þ}S = 0, \quad \eth S = 4i\mathcal{P}\bar{k}^{\frac{1}{2}} B, \quad \eth' S = -4i\overline{P}\bar{k}^{\frac{1}{2}} B \tag{53}
$$

The term in L in the table (41) for C will now be replaced, using (51) and (49), with

$$L = \left(S + 2\sqrt{2}AT - \frac{2\text{ꝁ}^{1/2}}{\Lambda} + \frac{9\Lambda^2 A^3 C^2 \text{ꝁ}^{3/2}}{4} \right) / \text{ꝁ}^{1/2} \left(\frac{3}{2} - \text{ꝁ} \right) \tag{54}$$

These two partial tables (52) and (53) are much simpler in appearance than (44) and (48), but unfortunately, because of the coupled nature of B and S in the two tables (52) and (53) the subsequent application of the commutators to such an arrangement gets very complicated; therefore it is more convenient to make one more rearrangement.

So we make a substitution of B with V by

$$V = 2B/S \tag{55}$$

and we shall assume $S \neq 0$ in the remainder of this section; we shall later have to look at the special case $S = 0$ separately.

This means that now a comparatively simple table for V replaces the partial table (52) for B,

$$\begin{aligned} &\text{þ} V = 0, \quad \eth V = -2iP\, k^{-\frac{1}{2}} (V^2 + \Lambda), \\ &\eth' V = 2i\overline{P}\, k^{-\frac{1}{2}} (V^2 + \Lambda), \quad \text{þ}' V = \frac{Q\, k^{-\frac{1}{4}}}{2A} (V^2 + \Lambda) H \end{aligned} \tag{56}$$

which we have completed in the usual way with the, as yet undetermined, real zero-weighted scalar function H.

In order to obtain a still simpler form for this table, we can now divide across the whole table by $(V^2 + \Lambda)$ and by integration define an alternative coordinate candidate to V with a simpler table.

However, it is important to note that in order to integrate with respect to V we have made the assumption that $V \neq$ constant; this assumption also ensures that $V^2 + \Lambda \neq 0$. Hence we will need to consider separately $V =$ constant as a special case.

So we define

$$X = \int \frac{dV}{V^2 + \Lambda} = \frac{1}{\sqrt{|\Lambda|}} \tan[\mathrm{h}]^{-1} \left(\frac{V}{\sqrt{|\Lambda|}} \right) \tag{57}$$

where we have introduced this compact notation

$$\tan[\mathrm{h}]^{-1}\left(\frac{V}{\sqrt{|\Lambda|}}\right) = \begin{cases} -\tanh^{-1}\left(\frac{V}{\sqrt{-\Lambda}}\right) & \text{for } \Lambda < 0 \\ \tan^{-1}\left(\frac{V}{\sqrt{\Lambda}}\right) & \text{for } \Lambda > 0 \end{cases} \tag{58}$$

and we have now the table

$$\text{þ} X = 0, \quad \eth X = -2iP\text{ꝁ}^{\frac{1}{2}}, \quad \eth' X = 2i\overline{P}\text{ꝁ}^{\frac{1}{2}}, \quad \text{þ}' X = \frac{Q\text{ꝁ}^{\frac{1}{4}}}{2A} H \tag{59}$$

Since this table turns out to be more manageable, we will adopt X as the fourth coordinate candidate.

The partial table for S is now modified to

$$
\begin{aligned}
\text{Þ} S &= 0 \\
\eth S &= 2i\, P \bar{k}^{\frac{1}{2}} S \sqrt{|\Lambda|} \tan[\mathrm{h}](\sqrt{|\Lambda|}\, X) \\
\eth' S &= -2i \overline{P} \bar{k}^{\frac{1}{2}} S \sqrt{|\Lambda|} \tan[\mathrm{h}](\sqrt{|\Lambda|}\, X).
\end{aligned}
\tag{60}
$$

where

$$
\tan[\mathrm{h}](\sqrt{|\Lambda|}\, x) = \begin{cases} -\tanh(\sqrt{-\Lambda}\, x) & \text{for} \quad \Lambda < 0 \\ \ \tan(\sqrt{\Lambda}\, x) & \text{for} \quad \Lambda > 0 \end{cases}
\tag{61}
$$

Earlier, we postponed applying the GIF commutators to the two real scalars B, M, so we need to apply the GHP commutators equivalently to their replacements, the two real zero-weighted scalars T, X: this gives the simple partial tables for F and H respectively,

$$
\text{Þ} F = 0, \quad \eth F = 0, \quad \eth' F = 0 \tag{62}
$$

$$
\text{Þ} H = 0, \quad \eth H = 0, \quad \eth' H = 0 \tag{63}
$$

The rather extensive relabelling and rearranging which we have just carried out was in order to obtain such simple and manageable forms. Clearly the gradient vector ∇F is parallel to ∇T; this means that the scalar function F is an arbitrary function of only the one coordinate candidate T. Similarly, from (63) the function H is also an arbitrary function of only the one coordinate candidate T. The function S in (60) has a more complicated structure; we shall find it as the solution of a partial differential equation when we translate into explicit coordinates.

So we have completed the formal integration procedure for these spaces; all the information has been extracted in the generic case, by which we mean the case where we have assumed that *the four zero-weighted real scalar functions,* A, C, T, X *are functionally independent*; these are our coordinate candidates which we intend to adopt as coordinates.

In the remainder of this section we will obtain the coordinate version of the tetrad vectors, and hence the metric.

As we emphasised in the last subsection, before we can adopt the coordinate candidates as coordinates, we must confirm that they are functionally independent. First of all we check on the possibility of these four scalars being constant: since we are assuming in this section that $\bar{k} \neq 0$, then none of A, C, X can be constant, but T may be. From the tables if follows that T is constant iff $F = 0$. Hence, in this section, the additional assumption that $F \neq 0$ is sufficient to ensure that none of the coordinate candidates are constant. Moreover, when we assume that none of the coordinate candidates are constant, a check of the determinant formed from their four tables (50), (41), (40), (59), confirms that the four coordinate candidates are indeed functionally independent — providing $F \neq 0$. Hence we will complete this section for the generic case with the additional assumption $F \neq 0$ ensuring that the coordinate candidates A, C, T, X can be adopted as explicit coordinates.

In addition, we must not forget that in order that X could be a coordinate candidate, we made the additional assumptions that $V \neq$ constant, and $S \neq 0$. We will look separately at the special cases $V =$ constant, and $F = 0 = S$ in the following sections.

3.5 Using coordinate candidates as coordinates

If we now make the obvious choice of the coordinate candidates as coordinates

$$t = T, \quad c = C, \quad a = A, \quad x = X \tag{64}$$

the above four tables for the zero-weighted scalars enable us to immediately write down the tetrad vectors in the coordinates t, c, a, x,

$$\begin{aligned} l^i &= \frac{1}{Q}\Big(0,\ -\frac{4}{(\frac{3}{2} - \bar{k})},\ 0,\ 0\Big) \\ m^i &= P\Big(0,\ -\frac{\Lambda(5\Lambda a^2 - 2)ac}{(\frac{3}{2} - \bar{k})},\ -2\bar{k},\ -2i\bar{k}^{\frac{1}{2}}\Big) \\ \overline{m}^i &= \overline{P}\Big(0,\ -\frac{\Lambda(5\Lambda a^2 - 2)ac}{(\frac{3}{2} - \bar{k})},\ -2\bar{k},\ 2i\bar{k}^{\frac{1}{2}}\Big) \\ n^i &= \frac{Q}{a}\Big(F\bar{k}^{\frac{1}{4}},\ L,\ \bar{k}^2 c,\ \frac{H\bar{k}^{\frac{1}{4}}}{2}\Big) \end{aligned} \tag{65}$$

where the function L is given in terms of S by (54), the functions $S,\ H,\ F$ are respectively solutions of the partial tables (60), (63), (62), $\bar{k}$ is given by (24).

As noted in the last section, F and H, respectively will be arbitrary functions of only the one coordinate t, so we will write $-4F = \alpha_2(t)$ and $-2H = \alpha_3(t)$ — subject to the restrictions made in the calculations in this section that $F \neq 0$ which implies that $\alpha_2(t) \neq 0$ (note there is no restriction on $\alpha_3(t)$, which is a completely arbitrary function of t, including the zero function).

The partial table (60) for S now becomes, via the tetrad, a system of partial differential equations in the chosen coordinates,

$$\begin{aligned} \frac{\partial S}{\partial c} &= 0 \\ 2\bar{k}\frac{\partial S}{\partial a} + 2i\bar{k}^{\frac{1}{2}}\frac{\partial S}{\partial x} &= -2i\bar{k}^{\frac{1}{2}} S\sqrt{|\Lambda|}\tan[\mathrm{h}](\sqrt{|\Lambda|}\,x) \end{aligned} \tag{66}$$

which shows that S is independent of the coordinates c and a, and we easily find the solution using (57)

$$S(t, x) = \alpha_1(t)\cos[\mathrm{h}](\sqrt{|\Lambda|}\,x) \tag{67}$$

where $\cos[\mathrm{h}](\sqrt{|\Lambda|}\,x)$ is given by

$$\cos[\mathrm{h}](\sqrt{|\Lambda|}\,x) = \begin{cases} \cosh(\sqrt{-\Lambda}\,x) & \text{for } \Lambda < 0 \\ \cos(\sqrt{\Lambda}\,x) & \text{for } \Lambda > 0 \end{cases} \tag{68}$$

and $\alpha_1(t)$ is an arbitrary function of t, excluding the zero function, since we are assuming $S \neq 0$ in this section.

It follows immediately from the equation

$$g^{ij} = 2l^{(i}n^{j)} - 2m^{(i}\overline{m}^{j)} \tag{69}$$

that the metric g^{ij}, in the coordinates t, c, a, x, is given by

$$g^{ij} = \begin{pmatrix} 0 & \frac{\bar{k}^{1/4}\alpha_2(t)}{a(\frac{3}{2}-\bar{k})} & 0 & 0 \\ \frac{\bar{k}^{1/4}\alpha_2(t)}{a(\frac{3}{2}-\bar{k})} & -\frac{8}{a\bar{k}^{1/2}(\frac{3}{2}-\bar{k})^2}Z & -\frac{2\bar{k}(5\Lambda^2a^4+1)c}{a(\frac{3}{2}-\bar{k})} & \frac{\bar{k}^{1/4}\alpha_3(t)}{a(\frac{3}{2}-\bar{k})} \\ 0 & -\frac{2\bar{k}(5\Lambda^2a^4+1)c}{a(\frac{3}{2}-\bar{k})} & -4\bar{k}^2 & 0 \\ 0 & \frac{\bar{k}^{1/4}\alpha_3(t)}{a(\frac{3}{2}-\bar{k})} & 0 & -4\bar{k} \end{pmatrix} \tag{70}$$

where $\bar{k}$ is given by (24) and Z is given in terms of S from (54) by,

$$\begin{aligned} Z &= S + 2\sqrt{2}at - \frac{2\bar{k}^{1/2}}{\Lambda} + \frac{9\Lambda^2a^3c^2\bar{k}^{3/2}}{4} + \frac{\bar{k}^{1/2}\Lambda^2(5\Lambda a^2-2)^2a^3c^2}{8} \\ &= \alpha_1(t)\cos[\mathrm{h}](\sqrt{|\Lambda|}x) + 2\sqrt{2}at - \frac{2\bar{k}^{1/2}}{\Lambda} + \frac{\Lambda^2\bar{k}^{1/2}a^3c^2(25\Lambda^2a^4-2\Lambda a^2+13)}{8} \end{aligned} \tag{71}$$

where $\cos[\mathrm{h}](\sqrt{|\Lambda|}x)$ is given by (68), and $\alpha_3(t)$ is completely arbitrary.

We must remember that we have assumed that $\alpha_2(t) \neq 0$, and $\alpha_1(t) \neq 0$; furthermore we have assumed at certain stages in our calculations that $V \neq$ constant. So this metric is not necessarily the most general form for this class of spacetimes.

In the following sections we will first look at the excluded cases separately, and then obtain a more general form of the metric which will include all such previously excluded cases.

4 The integration procedure: special case $V =$ constant, and combined case

4.1 The special case with $V =$ constant

When we substitute the condition $V =$ constant into (56) we find that this case can only occur for a *negative cosmological constant*. So if we write

$$\lambda = \pm\sqrt{-\Lambda}$$

then we find $V = \lambda$. The calculations in Sect. 3 up to (53) are still valid. Since neither X, nor constant V, can be a coordinate as in the last section, we must find a replacement coordinate candidate which is functionally independent of the other three A, C, T. We shall continue to assume in this section that $F \neq 0 \neq S$.

Substitution of $V = \lambda$ into (55) modifies (53) to give the table for B for this special case,

$$Þ B = 0, \quad ðB = 2i\mathcal{P}ƛ^{\frac{1}{2}}\lambda B, \quad ð' B = -2i\overline{\mathcal{P}}ƛ^{\frac{1}{2}}\lambda B, \quad Þ' B = -\frac{Q}{2A}ƛ^{\frac{1}{4}}\lambda BG \tag{72}$$

The real zero-weighted scalar G—as yet undetermined—has been chosen to complete the table in the usual manner.

This comparatively simple table suggests B as the replacement coordinate candidate; this of course will require that $B \neq$ constant, but from (72) we then see that the only possible constant value is $B = 0$. However, from (52) it follows that $S = 0$, and this special class has been excluded from this section.

But an even simpler table is obtained by the substitution

$$e^{-\lambda Y} = |B| \tag{73}$$

giving

$$Þ Y = 0, \quad ðY = -2i\mathcal{P}ƛ^{\frac{1}{2}}, \quad ð' Y = 2i\overline{\mathcal{P}}ƛ^{\frac{1}{2}}, \quad Þ' Y = \frac{Q}{2A}ƛ^{\frac{1}{4}}G \tag{74}$$

So preferring Y as our fourth coordinate candidate, we apply the commutators to get

$$Þ G = 0, \quad ðG = 0, \quad ð' G = 0 \tag{75}$$

The tables (40), (50), (41) respectively for the other three coordinate candidates A, T, C and the partial table (62) for the function F, are unchanged. L is replaced in (41) by S from (54), which in return is replaced by Y from

$$S = 2B/\lambda = \frac{2}{\lambda}e^{-\lambda Y}$$

from (73) (remembering there is a $\pm$ included in our definition of λ).

We have already noted that A and C cannot be constants, and although T may be, we are excluding that possibility in this section (since $F \neq 0$); furthermore, it is clear that Y cannot be constant (remembering $ƛ \neq 0 \neq \lambda$). Moreover, an examination of the determinant of the four tables (40), (41), (50) and (74) shows that the four scalars A, C, T and Y are functionally independent and therefore can be chosen as coordinates.

So we now make the obvious choice of the coordinate candidates as coordinates,

$$t = T, \qquad c = C, \qquad a = A, \qquad y = Y\,.$$

Since the function G is a solution of the partial table (75) we can write $-2G = \beta_3(t)$, and similarly $-4F = \beta_2(t)$; both are arbitrary functions of t, but the latter has the constraint that $\beta_2(t) \neq 0$.

The metric in t, c, a, y coordinates is therefore given by

$$g^{ij} = \begin{pmatrix} 0 & \frac{\bar{k}^{1/4}\beta_2(t)}{a(\frac{3}{2}-\bar{k})} & 0 & 0 \\ \frac{\bar{k}^{1/4}\beta_2(t)}{a(\frac{3}{2}-\bar{k})} & -\frac{8}{a\bar{k}^{1/2}(\frac{3}{2}-\bar{k})^2}Z & -\frac{2\bar{k}(5\Lambda^2a^4+1)c}{a(\frac{3}{2}-\bar{k})} & \frac{\bar{k}^{1/4}\beta_3(t)}{a(\frac{3}{2}-\bar{k})} \\ 0 & -\frac{2\bar{k}(5\Lambda^2a^4+1)c}{a(\frac{3}{2}-\bar{k})} & -4\bar{k}^2 & 0 \\ 0 & \frac{\bar{k}^{1/4}\beta_3(t)}{a(\frac{3}{2}-\bar{k})} & 0 & -4\bar{k} \end{pmatrix} \tag{76}$$

where

$$Z = \frac{2}{\lambda}e^{-\lambda y} + 2\sqrt{2}at - \frac{2\bar{k}^{1/2}}{\Lambda} + \frac{\Lambda^2\bar{k}^{1/2}a^3c^2(25\Lambda^2a^4 - 2\Lambda a^2 + 13)}{8} \tag{77}$$

and $\bar{k}$ given by (24).

We emphasise again that this case only exists for negative $\Lambda = -\lambda^2$.

4.2 Generic case combined with special case, $V =$ constant

We now combine the result in the previous section (with the cosmetic changes $y \to x$, and $\beta_2(t) \to \alpha_2(t)$, $\beta_3(t) \to \alpha_3(t)$) with the generic result in Sect. 3 to present the metric in the coordinates t, c, a, x, given by

$$g^{ij} = \begin{pmatrix} 0 & \frac{\bar{k}^{1/4}\alpha_2(t)}{a(\frac{3}{2}-\bar{k})} & 0 & 0 \\ \frac{\bar{k}^{1/4}\alpha_2(t)}{a(\frac{3}{2}-\bar{k})} & -\frac{8}{a\bar{k}^{1/2}(\frac{3}{2}-\bar{k})^2}Z & -\frac{2\bar{k}(5\Lambda^2a^4+1)c}{a(\frac{3}{2}-\bar{k})} & \frac{\bar{k}^{1/4}\alpha_3(t)}{a(\frac{3}{2}-\bar{k})} \\ 0 & -\frac{2\bar{k}(5\Lambda^2a^4+1)c}{a(\frac{3}{2}-\bar{k})} & -4\bar{k}^2 & 0 \\ 0 & \frac{\bar{k}^{1/4}\alpha_3(t)}{a(\frac{3}{2}-\bar{k})} & 0 & -4\bar{k} \end{pmatrix} \tag{78}$$

where $\alpha_3(t)$ is an arbitrary function of t including the zero function, whereas $\alpha_2(t)$ is an arbitrary function of t excluding the zero function, and $\bar{k}$ is given by (24).

There are two possibilities for Z:

$$\text{(i) } Z = \alpha_1(t)\cos[\text{h}](\sqrt{|\Lambda|}x) + 2\sqrt{2}at - \frac{2\bar{k}^{1/2}}{\Lambda} + \frac{\Lambda^2\bar{k}^{1/2}a^3c^2(25\Lambda^2a^4 - 2\Lambda a^2 + 13)}{8} \tag{79}$$

from (71) where $\alpha_1(t) \neq 0$ is an arbitrary function of t excluding the zero function, and $\cos[\text{h}](\sqrt{|\Lambda|}x)$ is given by (68).

$$\text{(ii) } Z = \frac{2}{\lambda}e^{-\lambda x} + 2\sqrt{2}at - \frac{2\bar{k}^{1/2}}{\Lambda} + \frac{\Lambda^2\bar{k}^{1/2}a^3c^2(25\Lambda^2a^4 - 2\Lambda a^2 + 13)}{8} \quad (80)$$

from (77).

Note that case (i) exists for positive and negative cosmological constant, but case (ii) only exists for negative Λ, with $\lambda = \pm\sqrt{-\Lambda}$.

5 The most general form for the metric

5.1 Preliminaries to generalisations

We have not yet got the most general version of the metric because in Sect. 3 we assumed that T was not a constant and that $S \neq 0$.

We begin with the excluded case where T is a constant. In such a situation, clearly $F = 0$ so we cannot instead use F as a coordinate candidate, but we still have the possibility of choosing H or S as a coordinate candidate. However, if *neither* of the other functions H, S is functionally independent of the original three coordinates, then it will *not* be possible to find a replacement candidate *directly*. In such a situation we still need a replacement candidate in order to extract the remaining information from the commutators. So rather than treating the special case $F = 0$ separately, we will extend the generic result to include this special case as well.

We shall now show, instead, that a *complementary coordinate candidate* to replace T can easily be found, and then, using this coordinate, we will obtain a generalisation of the metric (78) which includes all possible values for T, including a constant.

Secondly we consider the excluded case $S = 0$, and for this case we find that not only can we not construct X (or V) as a coordinate candidate, but that we cannot generate *directly* any replacement coordinate candidate. We shall now show, instead, that a *complementary coordinate candidate* to replace X can easily be found, and then, using this coordinate, we will first obtain this excluded case $S = 0$ separately; we will then obtain a generalisation of the metric (78) which includes this additional special case, $S = 0$.

5.2 Finding a complementary coordinate candidate to replace T

The results in Sect. 3 up to the end of Sect. 3.4 apply as before; the only difference here is that we *interpret* them differently. When we are interpreting our tables and choosing our explicit coordinate candidates we will now consider only the three zero-weighted real scalars A, C, X as coordinate candidates while the zero-weighted scalar T is not now included as a coordinate candidate, and so there is now no hindrance to it acquiring a constant value, even zero. A related change is that since T is no longer a coordinate candidate, we no longer need its *complete* table, nor the resulting partial table for F; however we still need the *partial* table for T since it is a result of applying the commutators to $\mathbf{I}$, and so is still a crucial component of the analysis,

$$\text{Þ}T = 0, \quad \eth T = 0, \quad \eth' T = 0 \quad (81)$$

So, clearly we do not have our full quota of *four* coordinate candidates, but we do not wish to use any of the remaining intrinsic quantities from the tables.

It is now very important to note that all the *direct* information which can be obtained from the intrinsic elements of the GHP formalism is in these tables. On the otherhand, we require a fourth zero-weighted scalar — functionally independent of the other three A, C, X — which will be the fourth coordinate candidate. Since there is no such intrinsic zero-weighted scalar which we can generate directly in the GHP formalism, we introduce it *indirectly* via its table, which will have to be consistent with all the explicit equations in the GHP formalism, and in particular with the GHP commutators.

In fact, we get a strong hint from Sect. 3.4, by looking at the table (50) for the coordinate T (which is the missing coordinate candidate in this case); so we consider the possibility of the existence of a real zero-weighted scalar $\tilde{T}$, which satisfies the table[3]

$$þ\tilde{T} = 0, \quad ð\tilde{T} = 0, \quad ð'\tilde{T} = 0, \quad þ'\tilde{T} = -4\frac{Q}{A}\bar{k}^{\frac{1}{4}} \tag{82}$$

So we have chosen a zero-weighted real scalar $\tilde{T}$ defined by its table (82), whose structure we have 'copied' from the table structure (50) of T.

It is important to appreciate the different natures of T and $\tilde{T}$. In Sect. 3, T was defined directly in terms of intrinsic elements of the formalism, and so was itself an *intrinsic* coordinate candidate, and the table (50) was a consequence of its definition; on the otherhand, the *complementary* coordinate candidate $\tilde{T}$ is not defined in terms of intrinsic quantities of the formalism, but rather as the integral of the table (82). Hence, the introduction of the coordinate candidate $\tilde{T}$, via the table (82), is structurally different from the usual direct identification of coordinates with elements of the formalism: C, A, X are intrinsic coordinate candidates, while $\tilde{T}$ is a complementary coordinate candidate.

It is straightforward to confirm that this choice of table (82) is consistent with the GHP commutators (18) and creates no inconsistency with the other tables.

So, compared to Sect. 3, we have simply replaced the fourth *intrinsic* coordinate candidate T with the *complementary* coordinate candidate $\tilde{T}$ defined via its table (82) whose structure was 'copied' from the table (50) for T; in addition we remember that the real zero-weighted quantity T now satisfies (81). Clearly T now is a function of only the one coordinate candidate $\tilde{T}$, i.e., $T(\tilde{T})$. The remaining tables are unchanged.

5.3 Finding a complementary coordinate candidate to replace X

The results in Sect. 3 up to the end of Sect. 3.3 apply as before; and we shall also assume the results up to Eq. (54).

When we make the substitution $S = 0$ into (53) we find that the table collapses giving $B = 0$. This means that the table for B, (44) also collapses. At this stage we are left with only the GHP tables for the three coordinate candidates A, C, T and the GHP

[3] For easy reference, in an extended case, we will label by $\tilde{T}$ a *complementary* coordinate candidate which replaces an intrinsic coordinate candidate T in the corresponding generic case; but we emphasise this is not to imply any *direct* link between the two quantities, it simply points us to the source of the hint which suggested the table for the complementary coordinate candidate.

tables for the weighted scalars, P, Q. So, in a similar manner to the last subsection, we introduce a *complementary* coordinate candidate *indirectly* via its table.

Also, as in last section, we get a strong hint from Sect. 3.4, by looking at the table (59) for the coordinate X (which is the missing coordinate candidate in this case); so we consider the possibility of the existence of a real zero-weighted scalar $\tilde{X}$, which satisfies the table

$$þ\tilde{X} = 0, \quad ð\tilde{X} = -2iP\hbar^{\frac{1}{2}}, \quad ð'\tilde{X} = 2i\overline{P}\hbar^{\frac{1}{2}}, \quad þ'\tilde{X} = \frac{Q\hbar^{\frac{1}{4}}}{2A}H \tag{83}$$

where we also assume $H(t)$.

Again we have adopted the convention of labelling by $\tilde{X}$ a *complementary* coordinate candidate which replaces an intrinsic coordinate candidate X in the corresponding generic case.

It is straightforward to confirm that this choice of table (83) is consistent with the GHP commutators (18) and creates no inconsistency with the other tables.

Furthermore, we note since $\tilde{X}$ is a complementary coordinate candidate which does not occur except in its own table, that we could have made an even simpler choice of table, by choosing $H = 0$ (which can easily be confirmed by a coordinate transformation $\tilde{X} \to \tilde{X} + \int (H(t)/4)dt$.) However, we shall not make that simplification, for presentation reasons.

We can therefore present this special case in the coordinates $t, c, a, \tilde{x}$, as

$$g^{ij} = \begin{pmatrix} 0 & \frac{\hbar^{1/4}\alpha_2(t)}{a(\frac{3}{2}-\hbar)} & 0 & 0 \\ \frac{\hbar^{1/4}\alpha_2(t)}{a(\frac{3}{2}-\hbar)} & -\frac{8}{a\hbar^{1/2}(\frac{3}{2}-\hbar)^2}Z & -\frac{2\hbar(5\Lambda^2a^4+1)c}{a(\frac{3}{2}-\hbar)} & \frac{\hbar^{1/4}\alpha_3(t)}{a(\frac{3}{2}-\hbar)} \\ 0 & -\frac{2\hbar(5\Lambda^2a^4+1)c}{a(\frac{3}{2}-\hbar)} & -4\hbar^2 & 0 \\ 0 & \frac{\hbar^{1/4}\alpha_3(t)}{a(\frac{3}{2}-\hbar)} & 0 & -4\hbar \end{pmatrix} \tag{84}$$

where $\hbar$ is given by (24), and Z is given by,

$$Z = 2\sqrt{2}at - \frac{2\hbar^{1/2}}{\Lambda} + \frac{\Lambda^2\hbar^{1/2}a^3c^2(25\Lambda^2a^4 - 2\Lambda a^2 + 13)}{8} \tag{85}$$

and $\alpha_3(t)$ is completely arbitrary, while $\alpha_2(t)$ is arbitrary, except for the zero function.

It is clear that this special case simply fills the gap in our original case (70), (71) by now including the case $\alpha_1(t) = 0$ which was excluded there.

5.4 The most general metric

The metric (78) gives the most general form of the metric for this class of spaces — under the additional restrictions that no Killing vectors are present. This follows from the existence of *four intrinsic coordinates*; this is also confirmed in [10] where we consider the detailed invariant Karlhede classification of this class of metrics. In Sect. 5.2 we saw how to generalise (78) to include the possibility of the coordinate $\tilde{t}$ being a complementary coordinate, so that this more general class also permits the existence of a Killing vector. The special case (84) just deduced in Sect. 5.3 can also easily be generalised in the same manner by replacing t with a complementary coordinate $\tilde{t}$; this special case could then be listed alongside the generalisation of (78). However it is more convenient to simply incorporate (84) into the generalisation of (78) discussed in Sect. 5.2, by just removing the restriction $\alpha_1(t) \neq 0$. It is easy to confirm that the tables for the respective complementary candidates $\tilde{T}$ and $\tilde{X}$ are consistent with all the other tables, and with each other.

Hence we generalise the combined metric form (78) given in the last section by replacing the intrinsic coordinate candidate T and its table with the complementary coordinate $\tilde{T}$ and its table, and the intrinsic coordinate candidate X and its table with the complementary coordinate $\tilde{X}$ and its table, and finally obtaining the metric in the coordinates

$$\tilde{t} = \tilde{T}, \qquad c = C, \qquad a = A, \qquad \tilde{x} = \tilde{X},$$

given by

$$g^{ij} = \begin{pmatrix} 0 & \frac{\bar{k}^{1/4}}{a(\frac{3}{2}-\bar{k})} & 0 & 0 \\ \frac{\bar{k}^{1/4}}{a(\frac{3}{2}-\bar{k})} & -\frac{8}{a\bar{k}^{1/2}(\frac{3}{2}-\bar{k})^2} Z & -\frac{2\bar{k}(5\Lambda^2a^4+1)c}{a(\frac{3}{2}-\bar{k})} & \frac{\bar{k}^{1/4}\gamma_3(\tilde{t})}{a(\frac{3}{2}-\bar{k})} \\ 0 & -\frac{2\bar{k}(5\Lambda^2a^4+1)c}{a(\frac{3}{2}-\bar{k})} & -4\bar{k}^2 & 0 \\ 0 & \frac{\bar{k}^{1/4}\gamma_3(\tilde{t})}{a(\frac{3}{2}-\bar{k})} & 0 & -4\bar{k} \end{pmatrix} \tag{86}$$

where $\gamma_3(\tilde{t})$ is an arbitrary function of $\tilde{t}$ including the zero function, and $\bar{k}$ is given by (24). There are two possibilities for Z,

$$\text{(i) } Z = \gamma_1(\tilde{t})\cos[\text{h}](\sqrt{|\Lambda|}\tilde{x}) + 2\sqrt{2}a\gamma_2(\tilde{t}) - \frac{2\bar{k}^{1/2}}{\Lambda} + \frac{\Lambda^2\bar{k}^{1/2}a^3c^2(25\Lambda^2a^4 - 2\Lambda a^2 + 13)}{8} \tag{87}$$

where $\gamma_1(\tilde{t})$ and $\gamma_2(\tilde{t})$ are arbitrary functions of $\tilde{t}$ including the zero function.

$$\text{(ii) } Z = \frac{2}{\lambda}e^{-\lambda\tilde{x}} + 2\sqrt{2}a\gamma_2(\tilde{t}) - \frac{2\bar{k}^{1/2}}{\Lambda} + \frac{\Lambda^2\bar{k}^{1/2}a^3c^2(25\Lambda^2a^4 - 2\Lambda a^2 + 13)}{8} \tag{88}$$

where $\gamma_2(\tilde{t})$ is an arbitrary function of $\tilde{t}$ including the zero function. The changes $\alpha_1(t) \to \gamma_1(t),\ \alpha_2(t) \to \gamma_2(t),\ \alpha_3(t) \to \gamma_3(t)$ are simply cosmetic.

Note that case (i) exists for positive and negative cosmological constant, but case (ii) only exists for negative Λ, with $\lambda = \pm\sqrt{-\Lambda}$.

When we compare the metric (78) where Z is given by (79) or (80) with the above metric (86) where Z is given by (87) or (88), we can easily demonstrate that the former is a special case of the latter, by making the coordinate transformation $t = \gamma_2(\tilde{t})/2\sqrt{2}$, and identifying $\gamma_1(\tilde{t}) = \gamma_1\big(\gamma_2^{-1}(2\sqrt{2}t)\big) = \alpha_1(t)$ and $\gamma_3(\tilde{t}) = \gamma_3\big(\gamma_2^{-1}(2\sqrt{2}t)\big) = \alpha_3(t)$, we confirm that the former case is included in the latter. However the latter also permits $\gamma_2(\tilde{t})$ to be constant, even zero; this is a possibility missing from the former.

It is trivial to confirm that the special subclass (84) is simply the special case of (i) given by $\gamma_1(\tilde{t}) = 0$. We note that we have used the notation $\tilde{x}$ in this general form, although it is obvious that this coordinate is in fact an intrinsic coordinate—except in this very special case $\gamma_1(\tilde{t}) = 0$. Finally, we note again that in this very special case $\gamma_1(\tilde{t}) = 0$ a simple coordinate transformation gives $\gamma_3(\tilde{t}) = 0$, but leaves everything else unchanged.

6 Summary and discussion

The study of the class of conformally flat pure radiation spacetimes with a non-zero cosmological constant which began in [9] and concluded in this paper has provided a very good laboratory for developing techniques and increasing our experience in the GIF formalism.

An important new development in this paper is the realisation that we do not need to work the whole integration procedure in the GIF, but rather we can change to the simpler GHP formalism once the GIF has generated a second unique intrinsic spinor and its table.

This integration procedure within the GIF/GHP formalism is particularly suited to spaces with four intrinsic coordinates; spaces with less than four intrinsic coordinates may appear to pose more difficulties. Another important development in this paper is a fuller understanding of how ‘generic’ results help to suggest additional special cases; in the case where it is suspected that there exists additional special cases to the generic case, the structure of tables for complementary coordinates can be ‘copied’ from the corresponding intrinsic coordinates.

The actual metrics which we have obtained have been confirmed with Maple.

It is clear from the most general form of the metric, and the fact that it is—as much as possible—presented in essential coordinates, that there will be subclasses with zero, one and two Killing vectors. There is in fact a rich symmetry structure in the whole class of conformally flat pure radiation spacetimes with non-zero cosmological constant, and the full details are presented in [10].

As well as increasing our experience and expertise in the GIF operator integration method, this particular class of spaces is interesting in its own right. The analogous spacetimes with zero cosmological constant investigated in [11] revealed some complications and subtleties in the computer classification programmes [16,35]; it will be interesting to see how the computer programmes handle these new spacetimes. It will also be interesting to explore the physical interpretation of the spacetimes in this paper and in [9], along the lines investigated in [14] for the spaces with zero cosmological constant; the wide variety of individual subclasses with a range from zero to five Killing vectors give a rich area of investigation.

It may be suspected that the various examples of Type II, III and N spaces recently investigated in [1,2,14,32,33] will specialise in the conformally flat limit to the spaces under consideration in this paper. However, that is not necessarily so, since, in at least some of those investigations, properties of a non-zero Weyl tensor were built into the analysis. Furthermore, even if the conformally flat limit does exist in some of the investigations, the form of the metric may be much more complicated than in our version where we have built the structure around the conformally flat properties from the beginning. It remains to investigate the whole class of these spacetimes found via GIF, considering in more detail the coordinate systems, and comparing with the conformally flat limits of these various other investigations.

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