F–REGULAR SEMIGROUPS

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Abstract. A semigroup S is called F-regular if S is regular and if there exists a group congruence ρ on S such that every ρ-class contains a greatest element with respect to the natural partial order of S (see [16]). These semigroups were investigated in [5] wehre a description similar to the F-inverse case (see [13]) is given. Further characterizations of F-regular semigroups, including an axiomatic one, are provided. The main objective is to give a new representation of such semigroups by means of Szendrei triples (see [17]). The particular case of F-regular semigroups S satisfying the identity $(xy)^* = y^*x^*$, where $x^* \in S$ denotes the greatest element of the ρ-class containing $x \in S$, is considered. Also the F-inversive semigroups, for which this identity holds, are characterized.

1. Introduction and summary

V. Wagner [18] introduced the class of *F–inverse* semigroups, which are defined as inverse semigroups S such that for the least group congruence σ of S every σ–class admits a greatest element with respect to the natural partial order $\leq_S$ of S. McFadden and O'Carroll [13] showed that every such semigroup can be constructed by means of a group, a semilattice and certain endomorphisms of the latter. Another method of construction of F–inverse semigroups was given by McAlister [11], Theorem 2.8, by means of P–semigroups over semilattices (see also the textbooks [10] and [15]).

A first generalization to the regular case was provided by Edwards [5]. A regular semigroup S is called $F - regular$ if for the least group congruence ρ on S every ρ–class $a\rho \in G = S/\rho$ contains a greatest element with respect to the natural partial order $\leq_S$ of S:

$$a \leq_S b \text{ if and only if } a = eb = bf \text{ for some } e, f \in E_S \text{ (see[16])}.$$

This partial order defined for regular semigroups is a particular case of the natural partial order $\leq_S$ defined on any semigroup S by $a \leq_S b$ if and only if $a = xb = by$, $xa = a(a = ay)$ for some $x, y \in S^1$ (see [14]). Note that if $a \in S$, $e \in E_S$ and $a \leq_S e$ then $a \in E_S$. In general, the relation $\leq_S$ is not compatible with multiplication on either side, that is, $(S, \cdot, \leq_S)$ is *not* a partially ordered semigroup.

The construction of all F-regular semigroups given in [5] is based on the ideas of McFadden and O'Carroll in the inverse case mentioned above. Here we will provide another method for the construction of all F–regular semigroups. It follows the development performed for generalized F–semigroups in [6] and for F–semigroups in [7]. A *generalized F–semigroup* was defined as a semigroup S on which there exists a group–congruence ρ such that the identity ρ–class $1_G \in G = S/\rho$ contains a greatest element ξ with respect to the natural partial order $\leq_S$, the so-called *pivot* of S. If every ρ–class of S admits a maximum element then S is called an *F–semigroup*. Note that by [6], Theorem 3.3, ρ is the least group congruence of S. Furthermore, by [6], Corollary 3.14, and [7], Theorem 3.5, we have:

(1) A semigroup S is generalized F-regular (with pivot ξ) if and only if S is an E-unitary regular monoid (with $1_S = \xi$).

(2) A semigroup S is F-regular if and only if S is an E-unitary regular monoid whose identity is residuated, i.e., for any $a \in S$, $1_S : a = max\{x \in S | ax \leq_S 1_S\} = max\{x \in S | xa \leq_S 1_S\}$ exists in $(S, \leq_S)$.

Recently, F–semigroups were investigated in the class of *abundant* semigroups: a construction in the spirit of McFadden and O'Carroll was provided by Gou [9].

In Section 2 several examples are given which prove useful in the sequel. Furthermore, some properties of F–regular semigroups S are derived. In particular, it is shown that for every $a \in S$ there exists a greatest $a^* \in S$ such that $a = aa^*a$. Also for the set E_S of idempotents of S, $E_S = \{x \in S | x^2 \leq_S x\}$ which holds already for any E-unitary semigroup. Section 3 contains three further characterizations of F–regular semigroups. The first as regular monoids S such that for every $a \in S$ there exists a greatest $x \in S$ with $ax \in E_S$. The second as E–unitary regular monoids such that for every $a \in S$, max $\{x \in S | axa \leq_S a\}$ exists. The last one is axiomatic following from a characterization of general F-semigroups due to M. Petrich (see[7]), and consists of three axioms for a unary operation $*$, which can be defined on S. In Section 4 the unary operation $*$ and another one o are investigated, which for partially ordered semigroups were introduced in [1] and [4]. Several properties of $*$ and o are deduced. In particular, it is shown that in general the identity $(xy)^* = y^*x^*$ or $(xy)^o = y^o x^o$ does not hold in S (the latter if and only if S is F-inverse). Section 5 offers a new construction of all F–regular semigroups based on the representation of E–unitary regular semigroups given by Szendrei [17]. Section 6 contains two constructions of F–regular semigroups satisfying the identity $(xy)^* = y^*x^*$. First it is shown that this class of semigroups coincides with that of all uniquely unit orthodox semigroups investigated in [3]. It follows that every such semigroup is a semidirect product of a band with identity and a group. The second representation is given in terms of Szendrei triples as a special case of the general result given in Section 5. Finally, the particular case of F-*inverse* semigroups satisfying the above $*$-identity is dealt with providing a representation as particular McAlister P-semigroups resp. by means of certain Szendrei triples. It follows that every such semigroup is a semidirect product of a semilattice with identity by a group.

2. Examples and basic properties

In this Section we first list some examples of F–regular semigroups, which will prove useful in the sequel.

EXAMPLES.

(1) Every group is F–regular (see Example (1) in [7], Section 2).

(2) A band B is an F–regular semigroup if and only if B admits a greatest element (see Example (3) in [7], Section 2).

(3) The direct product $S = B \times G$ of a band B with identity and a group G is a F–regular semigroup (see Example (6) in [7], Section 2).

(4) Every Clifford–semigroup $S = \langle Y, G_\alpha; \varphi_{\alpha,\beta} \rangle$, where $(Y, \leq_Y)$ is a finite chain and each $\varphi_{\alpha,\beta}$ is injective, is an F–inverse semigroup. Indeed, let $a, b \in S$ be maximal

in $(S, \leq_S)$, $a \in G_\alpha, b \in G_\beta, a \neq b$. If $\alpha = \beta$ then $a\varphi_{\alpha,\gamma} \neq b\varphi_{\beta,\gamma}$ for any $\gamma \leq_Y \alpha$ (since $\varphi_{\alpha,\gamma}$ is injective). If $\alpha \neq \beta$, $\alpha >_Y \beta$ say, then $a\varphi_{\alpha,\beta} \neq b$ (otherwise $b <_S a$ and b is not maximal); thus $(a\varphi_{\alpha,\beta})\varphi_{\beta,\gamma} \neq b\varphi_{\beta,\gamma}$ for any $\gamma \leq_Y \beta <_Y \alpha$ (since $\varphi_{\beta,\gamma}$ is injective) so that $a\varphi_{\alpha,\gamma} \neq b\varphi_{\beta,\gamma}$. It follows by [7], Corollary 6.7, that S is an F-semigroup. More generally:

(5) Let S be a monoid, which is a strong semilattice of completely simple semigroups S_α, where $(Y, \leq_Y)$ is a chain satisfying the ascending chain condition (in particular, is finite) and every $\varphi_{\alpha,\beta}$ is injective. Then $S = \langle Y, S_\alpha, \varphi_{\alpha,\beta}\rangle$ is an F–regular semigroup. Indeed, every $S_\alpha (\alpha \in Y)$ is a trivially ordered, regular semigroup (thus also S is regular) and 1_S is the greatest element of $(Y, \leq_Y)$. Furthermore, if $a, b \in S$ are maximal in $(S, \leq_S)$, $a \in S_\alpha, b \in S_\beta$, say, then as in the proof of (4), $a\varphi_{\alpha,\gamma} \neq b\varphi_{\beta,\gamma}$ for any $\gamma \leq_Y \alpha, \beta$ (taking into account that $x \leq_S y$, $x \in S_\alpha, y \in S_\beta$, if and only if $\alpha \leq_Y \beta$ and $x = y\varphi_{\beta,\alpha}$, since $\leq_\alpha$ is trivial on S_α). It follows by [7], Corollary 6.6, that S is an F–regular semigroup.

(6) Let $(B, *)$ be a band with identity ε, G be a group, and let $\Theta : G \to (Aut\ B, \circ)$ $\Theta(g) = \varphi_g$ (composition from the right) be an homomorphism. We write $\varphi_g(\alpha) = g\cdot\alpha$ for all $\alpha \in B$, $g \in G$. Then the semidirect product $S = B \times_\Theta G$ - with multiplication $(\alpha, g)(\beta, h) = (\alpha * g\cdot\beta, gh)$ - is an F–regular semigroup. Indeed, S is a regular monoid, in which for $(\alpha, g) \in S$ the element $(\varepsilon, g^{-1}) \in S$ is the greatest of all $(x, h) \in S$ such that $(\alpha, g)(x, h) \in E_S$:

$$(\alpha, g)(\varepsilon, g^{-1}) = (\alpha * g \cdot \varepsilon, 1_G) \in E_S;$$

$$(\alpha, g)(x, h) \in E_S \Rightarrow (\alpha * g \cdot x, gh) \in E_S \Rightarrow gh = 1_G \Rightarrow h = g^{-1} \text{ and}$$

$$(x, h) = (x, g^{-1}) \leq_S (\varepsilon, g^{-1}),$$

because $(x, g^{-1}) = (x, 1_G)(\varepsilon, g^{-1}) = (\varepsilon, g^{-1})(\delta, 1_G)$ with $(x, 1_G), (\delta, 1_G) \in E_S$, where $\delta \in B$ is such that $\varphi_{g^{-1}}(\delta) = x$ ($\varphi_{g^{-1}} \in Aut\ B$).

It follows by [7], Theorem 4.5 and its proof, that S is F–regular and that the pivot of S is 1_S. Note that by [7], Corollary 3.3, (ε, g^{-1}) is the greatest element of the ρ–class $[(\alpha, g)\rho]^{-1} \in S/\rho$, since $1_S : (\alpha, g) = (\varepsilon, g^{-1})$.

Next we deduce some properties of F–regular semigroups (see also [5]).
Let S be an F–semigroup and $a \in S$ be a regular element. Then $a = axa$ for some $x \in S$, whence $ax \in E_S$. If ξ is the pivot of S (i. e., the maximum of the identity class $1_G \in S/\rho$) then it follows by [6], Corollary 3.6, that $ax \leq_S \xi$. Since by [7], Theorem 3.5, the residual $\xi : a = max\{x \in S | ax \leq_S \xi\}$ exists in S we obtain that $x \leq_S \xi : a$. Therefore, $\xi : a \in S$ is an upper bound of the set $\{x \in S | axa = a\}$. We will show that $\xi : a$ belongs to this set, whence that $a(\xi : a)a = a$. In the theory of strong Dubreil–Jacotin regular semigroups $(S, \cdot, \leq)$ such an element $a \in S$ is called *perfect*, and it is shown in [2], Theorem 5, that every element of S is perfect if and only if $\leq$ on S extends the natural order of idempotents of S. Note that with respect to $\leq_S$, every F–regular semigroup S is ordered in this sense, but that $(S, \cdot, \leq_S)$ is not a partially ordered semigroup, in general.

In order to prove that nevertheless every element of an F–regular semigroup S is perfect, we first show the following

Lemma 2.1. *Let S be an F–semigroup with pivot ξ and let $a \in S$ be regular. Then (1) $a\xi = \xi a = a$; (2) $a(\xi : a)$, $(\xi : a)a \in E_S$.*

Proof. (1) $a = aa'a$ for some $a' \in S$ implies $aa', a'a \in E_S$ whence by ([6], Corollary 3.6), $\xi \cdot aa' = aa', a'a \cdot \xi = a'a$ so that

$$a\xi = aa'a \cdot \xi = a \cdot a'a\xi = a \cdot a'a = a, \text{and similary } \xi a = a.$$

(2) By [7], Theorem 3.5, $\xi : a$ exists in S. Thus by definition, $a(\xi : a) \leq_S \xi$ so that $a(\xi : a) = x\xi = \xi y$, $x \cdot a(\xi : a) = a(\xi : a)$ for some $x, y \in S^1$. Therefore by (1),

$$a(\xi : a) \cdot a(\xi : a) = x\xi \cdot a(\xi : a) = x \cdot \xi a \cdot (\xi : a) = x \cdot a \cdot (\xi : a) = a(\xi : a).$$

Similarly, $(\xi : a)a \leq_S \xi$ implies that $(\xi : a)a \in E_S$.

Theorem 2.2. *Let S be an F–semigroup with pivot ξ. Then every regular element $a \in S$ is perfect, i. e., $a = a(\xi : a)a$.*

Proof. Let $a \in S$ be regular, whence $a = aa'a$ for some $a' \in S$ and $aa' \in E_S$. Thus by [6], Corollary 3.6, $aa' \leq_S \xi$. By [7], Theorem 3.5, $\xi : a$ exists. Hence it follows by definition that $a' \leq_S \xi : a$ so that $a' = x(\xi : a) = (\xi : a)y$ for some $x, y \in S^1$. Therefore by Lemma 2.1(2),

$$a = aa'a = a \cdot x(\xi : a) \cdot a = ax \cdot (\xi : a)a = ax \cdot (\xi : a)a \cdot (\xi : a)a = ax(\xi : a) \cdot a(\xi : a)a,$$

where $ax(\xi : a) \cdot ax(\xi : a) = a \cdot a' \cdot ax(\xi : a) = a \cdot x(\xi : a) \in E_S$;

$$a = aa'a = a \cdot (\xi : a)y \cdot a = a(\xi : a) \cdot a(\xi : a) \cdot ya = a(\xi : a)a \cdot (\xi : a)ya.$$

It follows that $a \leq_S a(\xi : a)a$. Conversely, $a(\xi : a)a = a(\xi : a) \cdot a = a \cdot (\xi : a)a$, where $a(\xi : a) \in E_S$ by Lemma 2.1(2). Thus $a(\xi : a)a \leq_S a$ and equality prevails.

Given a semigroup S and $x \in S$, an element $x' \in S$ is called an *associate of x* if $x = xx'x$. We denote by $A(x)$ the set of all associates of $x \in S$.

Corollary 2.3. *Let S be an F–semigroup with pivot ξ. Then every regular element $a \in S$ has a greatest associate, namely $\xi : a$.*

Proof. This follows from Theorem 2.2 and its proof.

Corollary 2.4. *Let S be an F–regular semigroup. Then S has an identity and for every $a \in S$, $a = a(1_S : a)a$.*

Proof. By Lemma 2.1(1), the pivot ξ of S is the identity 1_S of S (note that S is regular). Hence the statement follows from Theorem 2.2.

Corollary 2.5. *Let S be an F–regular semigroup. Then every $a \in S$ has a greatest associate, namely $1_S : a$.*

Proof. This holds by Corollaries 2.3 and 2.4.

Remark. The necessary condition in Corollary 2.5 is *not* sufficient for a semigroup to be F–regular. For example, consider the Clifford–semigroup $S = \langle Y, G_\alpha, G_\beta; \varphi_{\alpha,\beta} \rangle$

such that Y is the two element chain $\alpha >_Y \beta, G_\alpha$ and G_β are two–element groups and $\varphi_{\alpha,\beta} : G_\alpha \to G_\beta$, $x\varphi_{\alpha,\beta} = 1_\beta$ for every $x \in G_\alpha$. Then the greatest associate of $1_\beta \in G_\beta$ is $1_\alpha \in G_\alpha$ and of any other element in S it is given by its inverse in the group to which it belongs. In spite of $(Y, \leq_Y)$ having a greatest element and of S being regular, S is not an F–semigroup since $\varphi_{\alpha,\beta}$ is not injective (see [6], Corollary 4.7).

For a property related to the above, see Section 3.

Next we shall describe the set of idempotents in an F–regular semigroup in a different way. To this end we show more generally:

Lemma 2.6. *Let S be an E–unitary semigroup. Then $E_S = \{x \in S | x^2 \leq_S x\}$.*

PROOF. Let $T = \{a \in S | a^2 \leq_S a\}$ and let $a \in T$. Then $a^2 \leq_S a$ and so $a^2 = xa = ay$, $xa^2 = a^2$ for some $x, y \in S^1$. Thus $a^3 = xa^2 = a^2$ and $a^4 = a^3 = a^2$, whence $a^2 \in E_S$. Therefore, $a^2 \cdot a = a^2 \in E_S$ implies that $a \in E_S$ (since S is E–unitary). It follows that $T \subseteq E_S$. Since evidently $E_S \subseteq T$ equality prevails.

Since by [6], Theorem 3.14, every F-regular semigroup is E-unitary, we obtain

Corollary 2.7. *Let S be an F–regular semigroup. Then $E_S = \{x \in S | x^2 \leq_S x\}$.*

3. Characterizations

We start with a more general result on F–semigroups S with regular pivot (this does not imply that S is regular - see [7], Remark (3) following Corollary 6.2). Recall that $\langle \xi . \cdot a \rangle = \{x \in S | ax \leq_S \xi\}$ for any $a \in S$.

Proposition 3.1. *Let S be a semigroup. Then S is an F–semigroup with regular pivot ξ if and only if S is E–unitary, ξ is the greatest idempotent of S and for every $a \in S$ there exists a greatest $x \in S$ such that $ax \in E_S$.*

PROOF. *Necessity.* By [6], Proposition 3.13, S is E–unitary and contains a greatest idempotent, namely ξ. Let $a \in S$; then $ax \in E_S \Leftrightarrow ax \leq_S \xi \Leftrightarrow x \in \langle \xi . \cdot a \rangle$. Since by [7], Theorem 3.5, $\langle \xi . \cdot a \rangle$ has a greatest element, the assertion follows.
Sufficiency. This holds by [7], Theorem 4.3.

Theorem 3.2. *Let S be a regular semigroup. Then S is an F–semigroup if and only if S is a monoid and for every $a \in S$ there exists a greatest $x \in S$ such that $ax \in E_S$.*

PROOF. *Necessity.* By Lemma 2.1(1), S is a monoid. The second property of S holds by Proposition 3.1.
Sufficiency. Evidently, the identity $1_S \in S$ is the greatest idempotent of S. Let $e \in E_S$; then by hypothesis, there exists a greatest $x \in S$ such that $ex \in E_S$, that is, $ex \leq_S 1_S$. It follows by [7], Lemma 4.4, that S is E–unitary. Therefore by Proposition 3.1, S is an F–semigroup.

Remark. Note that the condition that for very $a \in S$ there exists a greatest $x \in S$ such that $ax \in E_S$, is a particular case of E–inversiveness (see [6]).

The next characterization will be given in terms of the sets

$$T(a) = \{x \in S | axa \leq_S a\}, a \in S.$$

These sets were used in [4] to define pincipally ordered semigroups: a partially ordered semigroup $(S, \cdot, \leq)$ is called principally ordered if for any $a \in S$, max $T(a)$ exists in S with respect to the given partial order $\leq$ on S. We shall prove this property for F–regular semigroups with respect to their natural partial order. First, we show more generally

Theorem 3.3. *Let S be a generalized F–semigroup with regular pivot ξ. Then S is an F–semigroup if and only if for every $a \in S$, the maximum of $T(a)$ exists.*

Proof. First note that by [6], Proposition 3.13 and Corollary 3.6, $\xi \in E_S$, $(\xi] = \{x \in S | x \leq_S \xi\} = E_S$ and S is E-unitary.

Necessity. Let $a \in S$ and $x \in T(a)$. Then $axa \leq_S a$, i.e., $axa = ya = az$ and $y \cdot axa = axa$ for some $y, z \in S^1$. Thus $(ax)^2 = y \cdot ax = ax \cdot ax$ and $y \cdot (ax)^2 = (ax)^2$, that is, $(ax)^2 \leq_S ax$. It follows by Lemma 2.6, that $ax \in E_S = (\xi]$, hence $ax \leq_S \xi$. Since by [7], Theorem 3.5, max $\langle \xi.\cdot a \rangle = \xi.\cdot a$ exists in S, it follows that $x \leq_S \xi.\cdot a$.

We show that $\xi.\cdot a \in T(a)$. By definition, $a(\xi.\cdot a) \leq_S \xi$, thus $a(\xi.\cdot a) \in (\xi] = E_S$. Therefore, $a(\xi.\cdot a)a = a(\xi.\cdot a) \cdot a = a \cdot (\xi.\cdot a)a$ implies that $a(\xi.\cdot a)a \leq_S a$, that is, $\xi.\cdot a \in T(a)$. Hence $\xi.\cdot a$ is the greatest element of $T(a)$.

Sufficiency. Let $a \in S$ and put $a^* = \max T(a)$. We show that $a^* = \max\langle \xi.\cdot a \rangle$. As above, we have that $(aa^*)^2 \leq_S aa^*$, so that by Lemma 2.6, $aa^* \in E_S = (\xi]$. Hence $aa^* \leq_S \xi$ and $a^* \in \langle \xi.\cdot a \rangle$.

Next let $x \in \langle \xi.\cdot a \rangle$; then $ax \leq_S \xi$ and $ax \in (\xi] = E_S$. Therefore, $axa = ax \cdot a = a \cdot xa$ implies that $axa \leq_S a$, i.e., $x \in T(a)$. It follows by definition of a^*, that $x \leq_S a^*$. Thus a^* is the greatest element of $\langle \xi.\cdot a \rangle$, that is, $\xi.\cdot a$ exists in S. Hence by [7], Theorem 3.5, S is an F-semigroup.

Corollary 3.4. *Let S be a regular semigroup. Then S is an F–semigroup if and only if S is an E–unitary monoid such that for every $a \in S$, max $T(a)$ exists.*

Proof. By [6], Theorem 3.14, S is a generalized F–semigroup if and only if S is an E–unitary monoid. Thus the result follows from Theorem 3.3.

Remarks. (1) There are non–regular F–semigroups S in which for every $a \in S$, max $T(a)$ exists. For example:

(i) $S = S_0 \cup S_1$, the inflation of the semilattice $Y = \{0, 1\}$ such that $S_0 = \{0\}, S_1 = \{1, \alpha\}$. S is not regular, since $\alpha \in S$ is not so, and max $T(0) =$ max $T(1) =$ max $T(\alpha) = \alpha$; S is an F-semigroup by [7], Example (2).

(ii) $S = \bigcup_{g \in G} S_g$, the inflation of the group G with $|S_g| = 2$ for every $g \in G.S$ is a non–regular F–semigroup (by [7], Corollary 6.2) and for every $a \in S$, max $T(a) = b$

if $a \in S_g$ and $S_{g^{-1}} = \{g^{-1}, b\}$. In fact, let $x \in S_h$; then since $g \leq_S a$ (see [7], Section 6) we have
$x \in T(a) \Leftrightarrow axa \leq_S a \Leftrightarrow ghg \leq_S a \Leftrightarrow ghg = g \Leftrightarrow h = g^{-1} \Leftrightarrow x \in S_{g^{-1}}$,
whence $T(a) = S_{g^{-1}} = \{g^{-1}, b\}$; since $g^{-1} <_S b$ we get $\max T(a) = b$.

Note that both are examples of F–semigroups whose pivot ξ is not regular: in (i) $\xi = \alpha$, in (ii) $\xi = c$ if $S_{1_G} = \{1_G, c\}$.

(2) Let S be an F-regular semigroup and let $a \in S$. Then $\xi = 1_S$, $1_S : a$ exists and $1_S : a = \max\{x \in S | ax \leq_S 1_S\} = \max\{x \in S | ax \in E_S\}$ (see Section 1). Furthermore, by the proof of Theorem 3.3, $\max T(a) = \xi : a = 1_S : a$. Finally by Corollary 2.5, $1_S : a = \max A(a)$. Therefore we have

$$1_S : a = \max\{x \in S | ax \in E_S\} = \max\{x \in S | axa \leq_S a\} = \max\{x \in S | axa = a\}.$$

Finally, we give an axiomatic description of F-regular semigroups, which follows from a characterization of general F-semigroup due to M. Petrich (see [7], Theorem 3.9). It consists of three axioms for a unary operation defined on an F-regular semigroup S, which reflect properties of the set of greatest elements in the different ρ-classes of S (see the next Section).

Theorem 3.5. *(M. Petrich) Let S be a regular semigroup. Then S is F-regular if and only if S has a unary operation $a \to a^*$ satisfying*

(1) $(ab)^* = (a^*b)^* = (ab^*)^*$ *for all* $a, b \in S$,
(2) $a \leq_S a^*$ *for any* $a \in S$,
(3) $e^* = f^*$ *for all* $e, f \in E_S$.

Proof. *Necessity.* Let ρ be the defining group congruence on S and for any $a \in S$, let a^* be the greatest element of $a\rho \in S/\rho$. Then for all $a, b \in S$, $a\rho a^*$ and $b\rho b^*$ imply that $ab\rho a^*b\rho ab^*$, whence (1) holds. (2) follows from the definition of a^*. If $e, f \in E_S$ then $e\rho f$, thus (3) holds.
Sufficiency. Define a relation ρ on S by: $a\rho b \Leftrightarrow a^* = b^*$. Then because of condition (1), ρ is a congruence on S. Since S is regular, so is S/ρ. Let $a\rho, b\rho \in E_{S/\rho}$; then by the Lemma of Lallement, $a\rho = e\rho$ and $b\rho = f\rho$ for some $e, f \in E_S$. It follows by condition (3), that $a^* = e^* = f^* = b^*$, whence $a\rho b$ and $a\rho = b\rho$. Therefore, S/ρ is a group. It remains to prove that S is an F-semigroup.
Let $a \in S$; we first show that $a^* \in a\rho$. Since by condition (2), $a \leq_S a^*$ we have $a = ea^* = ea$ for some $e \in E_S$. Hence in S/ρ, $a\rho = (e\rho)(a^*\rho) = (e\rho)(a\rho)$, so that $a^*\rho = a\rho$. Now, if $b \in a\rho$, then $b\rho a$ and $b^* = a^*$. Hence it follows by condition (2), that $b \leq_S a^*$. Therefore, a^* is the greatest element of the ρ-class $a\rho \in S/\rho$, and S is an F-semigroup.

4. Two unary operations

Let S be a F–regular semigroup. Then by Remark (2) of Section 3, for each $a \in S$, $1_S : a \in S$ exists and $a(1_S : a)a = a$. Putting

$$a^* = 1_S : a$$

the asignment $a \to a^*$ defines a unary operation on S (see Theorem 3.5). Note that $aa^*a = a$ implies that $aa^*, a^*a \in E_S$. Also by [7], Corollary 3.3(1), $a^* \in S$ is the greatest element of the ρ-class $(a\rho)^{-1} \in S/\rho = G$, where ρ is the least group congruence on S.

For partially ordered semigroups $(S, \cdot, \leq)$ the $*$–operation was defined as $a^* = \max\{x \in S | axa \leq a\}$ in [1] studying perfect strong Dubreil–Jacotin semigroups and was used in [4] to define principally ordered regular semigroups (compare with Remark (2) of Section 3).

We collect some basic properties of this operation writing $a^{**} = (a^*)^*$. Recall that $V(a) = \{x \in S|\ a = axa, x = xax\}$ and $A(a) = \{x \in S | a = axa\}$ for any $a \in S$.

Proposition 4.1. *Let S be an F–regular semigroup. Then for all $a, b \in S$, $e \in E_S$ the following hold:*
(i) $a \leq_S a^{**}$; (ii) $a \leq_S b \Rightarrow a^* = b^*$, $aa^* \leq_S bb^*$, $a^*a \leq_S b^*b$; (iii) $a^{***} = a^*$;
(iv) $a^{**} = \max V(a^*)$; (v) $a^* \rho\ a'$ for every $a' \in A(a)$; (vi) $e^* = 1_S$;
(vii) $(ea)^* = a^*e^* = a^* = e^*a^* = (ae)^*$; (viii) $aa^* \leq_S a^{**}a^*, a^*a \leq_S a^*a^{**}$.

Proof. (i) By [7], Corollary 3.3 (2), $(a^*)^* = 1_S : a^* = 1_S : (1_S : a)$ is the greatest element of the ρ–class $a\rho \in S/\rho$, hence $a \leq_S a^{**}$.

(ii) Since by [7], Lemma 2.1, every ρ–class of S is a principal order ideal of $(S, \leq_S)$, $a \leq_S b$ implies that $a \in b\rho$, that is, $a\ \rho\ b$. Therefore the greatest element of the ρ–class $(a\rho)^{-1} = (b\rho)^{-1}$ is $a^* = b^*$. Furthermore, $a = eb = bf$ for some $e, f \in E_S$; hence

$$aa^* = eb \cdot a^* = e \cdot ba^* = e \cdot bb^*, aa^* = bf \cdot a^* = bb^*b \cdot fa^* = bb^* \cdot bfa^*;$$

since $aa^*, bb^* \in E_S$ it follows that $aa^* \leq_S bb^*$, and similarly $a^*a \leq_S b^*b$.
(iii) By (i), $a \leq_S a^{**}$, whence by (ii), $a^* = a^{***}$.
(iv) By Corollary 2.4, $a^*a^{**}a^* = a^*$, and $a^{**}a^{***}a^{**} = a^{**}$; hence by (iii), $a^{**}a^*a^{**} = a^{**}$. Thus $a^{**} \in V(a^*)$. Let $a' \in V(a^*)$; then $a^*a'a^* = a^*$ so that $a' \leq_S \max \{x \in S | a^*xa^* \leq_S a^*\} = a^{**}$ by Remark (2) of Section 3.
(v) Let $a' \in S$ be such that $aa'a = a$; then by Remark (2) of Section 3, $a' \leq_S a^*$. Since by [7], Lemma 2.1, every ρ–class is a principal order ideal of $(S, \leq_S)$, it follows that $a' \rho\ a^*$.
(vi) The greatest element of the ρ–class $(e\rho)^{-1} = 1_G^{-1} = 1_G \in G = S/\rho$ is by definition the pivot $\xi = 1_S$ of S. Thus $e^* = 1_S$.
(vii) The greatest element of $[(ea)\rho]^{-1} = [(e\rho)(a\rho)]^{-1} = (a\rho)^{-1} \in S/\rho$ is a^*. Thus by (vi), $(ea)^* = a^* = a^*e^* = e^*a^*$.
(viii) By (i) we have that $a \leq_S a^{**}$. Hence by (ii), $aa^* \leq_S a^{**}a^{***}$. Therefore $aa^* \leq_S a^{**}a^*$ by (iii), and similarly $a^*a \leq_S a^*a^{**}$.

Remark. The identity $(ab)^* = b^*a^*$ (compare with (vii)) does not hold for F–regular semigroups, in general. For example: let $G_\alpha = \{1_\alpha, a, b\}$ the three–element group, $G_\beta = \{1_\beta\}$ the one–element group, $Y = \{\alpha, \beta\}$ with $\alpha <_Y \beta$ be the two–element chain, and let $\varphi_{\beta,\alpha} : G_\beta \to G_\alpha$, $1_\beta\varphi_{\beta,\alpha} = 1_\alpha$. Then by Example (4) of

Section 2, the Clifford–semigroup $S = \langle Y; G_\alpha, G_\beta; \varphi_{\beta,\alpha}\rangle$ is an F–regular semigroup. Since $E_S = \{1_\alpha, 1_\beta\}$, where 1_β is the identity $1_S \in S$ and $1_\alpha <_S 1_\beta$, we have for $a, b \in G_\alpha$ (note that $ax \in G_\alpha$ for every $x \in S$):
$a^* = 1_S : a = \max\{x \in S | ax \leq_S 1_S\} = \max\{x \in S | ax \in E_S\}$
$\quad = \max\{x \in S | ax = 1_\alpha\} = \max\{a^{-1}\} = \max\{b\} = b,$
and similarly $b^* = a$. Hence $b^*a^* = ab = 1_\alpha$; but $(ab)^* = 1_\alpha^* = 1_S = 1_\beta$ (by Proposition 4.1 (vi)). Thus $(ab)^* \neq b^*a^*$.
Concerning F–regular semigroups satisfying the above identity see Section 6.

The second unary operation was introduced for partially ordered semigroups in [4]. We define for an F-regular semigroup S and any $a \in S$:

$$a^o = a^*aa^*, \text{ where } a^* = 1_S : a.$$

Again the asignment $a \to a^o$ gives a unary operation on S. Note that $aa^o, a^oa \in E_S$ for any $a \in S$, since aa^*, $a^*a \in E_S$.

Proposition 4.2. *Let S be an F–regular semigroup. Then we have for all $a, b \in S$, $e \in E_S$:*
(i) $a^o \in V(a) = \{x \in S |\ a = axa, x = xax\}$; (ii) $aa^o = aa^*$, $a^oa = a^*a$;
(iii) a^o is incomparable with every $a' \in V(a)$, $a' \neq a^o$, with respect to $\leq_S$;
(iv) $a^{*o} = a^{o*} = a^{**}$; (v) $e^o = e$; (vi) $aa^* = a^{**}a^o$, $a^*a = a^oa^{**}$;
(vii) $a^{oo} = a$; (viii) $a^{**}a^*a = a = aa^*a^{**}$; (ix) $a \leq_S b \Rightarrow aa^o \leq_S bb^o$, $a^oa \leq_S b^ob$.

Proof. (i) By Remark (2) of Section 3, $aa^*a = a$, thus

$$aa^oa = a \cdot a^*aa^* \cdot a = (aa^*)^2a = aa^*a = a,$$

$$a^oaa^o = a^*aa^* \cdot a \cdot a^*aa^* = (a^*a)^3a^* = a^*aa^* = a^o.$$

(ii) $a^o = a^*aa^* \Rightarrow aa^o = aa^*aa^* = aa^*$ and similarly, $a^oa = a^*a$.
(iii) Let $a' \in V(a)$ be such that $a' \leq_S a^o$ or $a^o \leq_S a'$. In the first case, $a' = ea^o = a^of$ for some $e, f \in E_S$. Hence by (i), $aa' = a \cdot ea^o = ae \cdot a^oaa^o = aea^o \cdot aa^o = aa' \cdot a \cdot a^o = aa^o$, and similarly $a'a = a^oa$. It follows that $a' = a' \cdot aa' = a' \cdot aa^0 = a'a \cdot a^o = a^oa \cdot a^o = a^o$ by (i). The second case is dealt with interchanging a' and a^o.
(iv)

$$a^{*o} = (a^*)^o = a^{**}a^*a^{**} = a^{**}a^{***}a^{**} = a^{**} \text{ by Proposition 4.1(iii)},$$
$$a^{o*} = (a^o)^* = (a^*a \cdot a^*)^* = a^{**} \text{ by Proposition 4.1(vii)}.$$

(v) By Proposition 4.1(vi), $e^* = 1_S$ whence $e^o = e^*ee^* = e$.
(vi) By Proposition 4.1(viii), $aa^* \leq_S a^{**}a^*$. Since both of these elements are idempotent it follows that $aa^* = a^{**}a^* \cdot aa^* = a^{**} \cdot a^*aa^* = a^{**}a^o$. Similarly, $a^*a = a^oa^{**}$.
(vii) By (iv) and (vi) we have

$$a^{oo} = (a^o)^o = a^{o*} \cdot a^o \cdot a^{o*} = a^{**} \cdot a^o \cdot a^{**} = a^{**} \cdot a^oa^{**} = a^{**} \cdot a^*a,$$

and similarly, $a^{oo} = aa^*a^{**}$. Thus $a^{oo} = a^{**}a^* \cdot a = a \cdot a^*a^*$ with $a^{**}a^*$, $a^*a^{**} \in E_S$, whence $a^{oo} \leq_S a$. For the opposite we have by Proposition 4.1(viii), that $aa^* \leq_S a^{**}a^*$ and $a^*a \leq_S a^*a^{**}$, and so

$$a = aa^*a = a \cdot a^*a^{**}a^* \cdot a = aa^* \cdot a^{**}a^*a = aa^* \cdot a^{oo}$$
$$a = aa^*a = a \cdot a^*a^{**}a^* \cdot a = aa^*a^{**} \cdot a^*a = a^{oo} \cdot a^*a.$$

Since aa^*, $a^*a \in E_S$ it follows that $a \leq_S a^{oo}$ and equality prevails.
(viii) By the proof of (vii), $a^{**}a^*a = a^{oo} = aa^*a^{**}$. Since by (vii), $a^{oo} = a$ the statement follows.
(ix) Let $a \leq_S b$; then by Proposition 4.1(ii), $aa^* \leq_S bb^*$. Since by (ii), $aa^* = aa^o$ and $bb^* = bb^o$ it follows that $aa^o \leq_S bb^o$, and similarly, $a^oa \leq_S b^ob$.

Remark. For F–regular semigroups, the identity $(ab)^o = b^oa^o$ is not satisfied, in general. For example, consider a two - element leftzero semigroup with an identity adjoined: $S = \{e, f, 1\}$. Then by Example (2) in Section 2, S is an F–regular semigroup. By Proposition 4.2(v), $e^o = e$ and $f^o = f$, whence $(ef)^o = e^o = e \neq f = fe = f^oe^o$.

Proposition 4.3. *Let S be an F–regular semigroup. Then $(ab)^o = b^oa^o$ holds for all $a, b \in S$ if and only if S is F–inverse.*

Proof. *Necessity.* Let $e, f \in E_S$; then by [6], Proposition 3.7, $ef \in E_S$. It follows by Proposition 4.2(v) and the hypothesis, that $ef = (ef)^o = f^oe^o = fe$. Hence S is an inverse semigroup.
Sufficiency. Let $a \in S$; we show that $a^o = a^{-1}$. By Proposition 4.2(i), $a^o \in V(a)$. Since S is inverse, $V(a) = \{a^{-1}\}$ whence $a^o = a^{-1}$. Now the statement follows from the identity $(ab)^{-1} = b^{-1}a^{-1}$, which holds in every inverse semigroup.

5. A representation

Let S be an F–regular semigroup. Then by Theorem 3.2, S can be characterized as a regular monoid such that for every $a \in S$ there exists a greatest $x \in S$ with $ax \in E_S$. Furthermore by Proposition 3.1, S is necessarily E–unitary. Hence we are dealing with particular E–unitary regular semigroups. A construction of all E–unitary regular semigroups was given in [17] in the following way (see also [8], Theorem IX.5.6):
Let G be a group, let $(X, *)$ be a strictly combinatorial semigroup, and let $Y \subseteq X$ be such that
T1) G acts on X on the left by automorphisms and $G \cdot Y = X$;
T2) Y is a right ideal of X and the nonzero idempotents of Y form a subsemigroup of X;
T3) for every $g \in G$ there exists $a \in Y$ such that $g \cdot a \in Z$, where

$$Z = \{x \in X | x \neq 0, x \text{ has an inverse in } Y\}.$$

On $P = PO(G, X, Y) = \{(a, g) \in Y \times G|\ g^{-1} \cdot a \in Z\}$ define a multiplication by

$$(a, g)(b, h) = (a * g \cdot b, gh).$$

Then $(P,\cdot)$ is an E–unitary regular semigroup with $E_P = \{(e,1_G) \in P | e \in E_Y\backslash 0\}$; conversely every such semigroup can be constructed in this way.
Specializing this construction we obtain the desired representation by Szendrei triples (G,X,Y). To this end we need some preliminary results.

Lemma 5.1. *Let $P = PO(G,X,Y)$. Then P has an identity if and only if Y contains a greatest idempotent.*

PROOF. *Necessity.* If $(\omega,h) \in P$ is the identitiy of P, then $(\omega,h) \in E_P$ so that $\omega \in E_Y\backslash 0$ and $h = 1_G$. Let $e \in E_Y, e \neq 0$; then $(e,1_G) \in P$ since $1_G^{-1} \cdot e = 1_G \cdot e = e \neq 0$ and $e \in E_Y$ has trivially an inverse in Y. Therefore

$$(\omega,1_G)(e,1_G) = (e,1_G) \Rightarrow (\omega * 1_G \cdot e, 1_G) = (e,1_G) \Rightarrow w * e = e;$$
$$(e,1_G)(\omega,1_G) = (e,1_G) \Rightarrow (e * 1_G \cdot \omega, 1_G) = (e,1_G) \Rightarrow e * \omega = e.$$

Hence $e \leq_Y \omega$ and $\omega \in E_Y$ is the greatest idempotent of Y.

Sufficiency. If $\omega \in E_Y$ denotes the greatest idempotent of Y then first $(\omega,1_G) \in P$ since $1_G^{-1} \cdot \omega = 1_G \cdot \omega = \omega \in E_Y$ has an inverse in Y. Note that $\omega \neq 0$, since the non zero idempotents of Y form a subsemigroup by condition T2) on Y, hence there exists a nonzero idempotent in Y.
Let $(a,g) \in P$; then $g^{-1} \cdot a \in Z$. Let $c \in Y$ be an inverse of $g^{-1} \cdot a \in X$ in Y, that is, $c * g^{-1} \cdot a * c = c$. Hence $c * g^{-1} \cdot a \in E_Y$ since Y is a right ideal of X, by condition T2). Furthermore, by [8], p. 367, we have that

$$(a,g)(c,g^{-1})(a,g) = (a,g),\ a * g \cdot c * a = a,\ a * g \cdot c \in E_Y.$$

Thus
$(a,g)(\omega,1_G) = (a,g)(c,g^{-1})(a,g)\cdot(\omega,1_G) = (a,g)\cdot(c * g^{-1}\cdot a, 1_G)\cdot(\omega,1_G)$
$= (a,g)\cdot(c * g^{-1} \cdot a * 1_G \cdot \omega, 1_G) = (a,g)(c * g^{-1} \cdot a, 1_G) = (a * g\cdot(c * g^{-1} \cdot a), g)$
$= (a * g \cdot c * g \cdot (g^{-1} \cdot a), g) = (a * g \cdot c * a, g) = (a,g);$

$(\omega,1_G)(a,g) = (\omega,1_G)\cdot(a,g)(c,g^{-1})(a,g) = (\omega,1_G)(a * g \cdot c, 1_G)\cdot(a,g)$
$= (\omega * a * g \cdot c, 1_G)(a,g) = (a * g \cdot c, 1_G)(a,g) = (a * g \cdot c * a, g) = (a,g).$
Therefore, $(\omega,1_G)$ is the identity of P.

Lemma 5.2. *Every $E_S\backslash 0$–unitary semigroup S is $E_S\backslash 0$–reflexive (i.e., for $a,b \in S$, $ab \in E_S\backslash 0$ implies that $ba \in E_S\backslash 0$).*

PROOF. Let $a,b \in S$ be such that $ab \in E_S\backslash 0$. Then $c = ba \in S$ satisfies $c^3 = bababa = b(ab)^2a = baba = c^2$, whence $c^4 = c^3 = c^2 \in E_S$. Now $c^2 \neq 0$, otherwise $ab = (ab)^3 = ababab = ac^2b = 0$. Since S is $E_S\backslash 0$–unitary, $c^2 \cdot c = c^3 = c^2 \in E_S\backslash 0$ implies that $c = ba \in E_S\backslash 0$.

Let $P = PO(G,X,Y)$ and $(a,g) \in P$; we put

$$Y_{(a,g)} = \{y \in Y\backslash 0\ |(y,g^{-1}) \in P, a * g \cdot y \in E_Y\backslash 0\}.$$

Lemma 5.3. *Let $P = PO(G, X, Y)$ be such that $E_Y \backslash 0$ is a unitary subset of the semigroup $(E_X, *)$ (i.e., $ef \in E_Y \backslash 0$ or $fe \in E_Y \backslash 0$, $e \in E_Y \backslash 0$, $f \in E_X \backslash 0$ implies $f \in E_Y \backslash 0$). Then for any $(a, g) \in P$ and $y, z \in Y_{(a,g)}$:*

$$y \leq_X z \text{ if and only if } y = e * z = z * g^{-1} \cdot f \text{ for some } e, f \in E_Y \backslash 0.$$

Proof. *Sufficiency* holds since $(X, *)$ is regular and $e, g^{-1} \cdot f \in E_Y \backslash 0$.
Necessity. Let $y, z \in Y_{(a,g)}$ be such that $y \leq_X z$. Then $a * g \cdot y \in E_Y \backslash 0$, $a * g \cdot z \in E_Y \backslash 0$ and $y = e * z = z * f'$ for some $e, f' \in E_X \backslash 0$ ($y \neq 0$). Thus

$$a * g \cdot y = a * g \cdot (z * f') = (a * g \cdot z) * g \cdot f' \in E_Y \backslash 0.$$

Since $g \cdot f' \in E_X \backslash 0$ it follows by hypothesis, that $g \cdot f' \in E_Y \backslash 0$. Hence $g \cdot f' = f$ for some $f \in E_Y \backslash 0$, so that $g^{-1} \cdot (g \cdot f') = g^{-1} \cdot f$, i.e., $f' = g^{-1} \cdot f$ with $f \in E_Y \backslash 0$.

Furthermore, $y * g^{-1} \cdot a \in Y$ since Y is a right ideal of X. Also, $a * g \cdot y \in E_Y \backslash 0$ implies that $g^{-1} \cdot (a * g \cdot y) = g^{-1} \cdot a * y$ is a nonzero idempotent of $(X, *)$. It follows by Lemma 5.2 (for $S = (X, *)$), that $y * g^{-1} \cdot a \in E_Y \backslash 0$. Similarly, $z * g^{-1} \cdot a \in E_Y \backslash 0$. Therefore we have that

$$e * (z * g^{-1} \cdot a) = (e * z) * g^{-1} \cdot a = y * g^{-1} \cdot a \in E_Y \backslash 0.$$

Hence it follows by hypothesis, that $e \in E_Y \backslash 0$. Thus $y = e * z = z * g^{-1} \cdot f$ for some $e, f \in E_Y \backslash 0$.

The announced representation theorem for F-regular semigroups follows.

Theorem 5.4. *A semigroup S is F-regular if and only if S is isomorphic with some $P = PO(G, X, Y)$ such that*

1. *Y contains a greatest idempotent,*
2. *$E_Y \backslash 0$ is a unitary subset of $(E_X, *)$,*
3. *for every $(a, g) \in P$, $m = \max Y_{(a,g)}$ exists with respect to $\leq_X$.*

Proof. *Sufficiency.* First by [17] (see also [8]), P is a regular (E-unitary) semigroup. By condition (1), P has an identity (see Lemma 5.1). Let $(a, g) \in P$; then by condition (3), $(m, g^{-1}) \in P$ and $a * g \cdot m \in E_Y \backslash 0$. Therefore

$$(a, g)(m, g^{-1}) = (a * g \cdot m, 1_G) \in E_P.$$

Let $(y, h) \in P$ be such that $(a, g)(y, h) \in E_P$. Then $a * g \cdot y \in E_Y \backslash 0$ and $gh = 1_G$, i.e., $h = g^{-1}$ and $(y, g^{-1}) \in P$. Hence $y \in Y_{(a,g)}$, and by definition of $m \in Y \backslash 0$ in condition (3), $y \leq_X m$. Thus by Lemma 5.3 (using condition (2)), $y = e * m = m * g^{-1} \cdot f$ for some $e, f \in E_Y \backslash 0$, so that

$$(y, g^{-1}) = (e, 1_G)(m, g^{-1}) = (m, g^{-1})(f, 1_G) \text{ where } (e, 1_G), (f, 1_G) \in E_P.$$

Therefore $(y, g^{-1}) \leq_P (m, g^{-1})$, which shows that $(m, g^{-1}) \in P$ is the greatest of all elements $(y, h) \in P$ such that $(a, g)(y, h) \in E_P$. It follows by Theorem 3.2, that P is F-regular. Therefore, so is S.

Necessity. By Proposition 3.1 and Theorem 3.2, S is a regular E-unitary monoid such that for every $a \in S$ there exists a greatest $x \in S$ with $ax \in E_S$. In particular, it follows by [17] ([8], Theorem IX.5.6) that S is isomorphic with the semigroup $P = PO(G, X, Y)$ where G is the greatest group which is an homomorphic image of S, $X = (G \times S) \cup \{0\}$ with operation $(g, s)(h, t) = (g, st)$ if $g\gamma(s) = h$, and $= 0$ otherwise, and $Y = (\{1_G\} \times S) \cup \{0\}$.
We will show that P satisfies the conditions (1), (2), and (3).

ad(1): Since S has an identity, also P has one. Thus by Lemma 5.1, Y has a greatest idempotent.

ad(2): Let $e \in E_Y \backslash 0$, $f \in E_X \backslash 0$ be such that $ef \in E_Y \backslash 0$. Then $e = (1_G, e')$ for some $e' \in E_S$, $f = (g, f')$ for some $g \in G, f' \in E_S$. Since $ef \in E_Y \backslash 0$ we have $(g, e'f') = (1_G, t)$ for some $t \in E_S$. Thus $g = 1_G$ and so $f = (1_G, f') \in E_Y \backslash 0$. Similarly for $fe \in E_Y \backslash 0$.

ad(3): Let $(a, g) \in P$; then by hypothesis, there exists a greatest $(m, h) \in P$ such that $(a, g)(m, h) \in E_P$. Hence $a * g \cdot m \in E_Y \backslash 0$ and $gh = 1_G, i.e., h = g^{-1}$ and $(m, g^{-1}) \in E_P$. Thus $m \in Y_{(a,g)}$. Let $y \in Y_{(a,g)}$; then $(y, g^{-1}) \in P$ and $a * g \cdot y \in E_Y \backslash 0$. Therefore

$$(a, g)(y, g^{-1}) = (a * g \cdot y, 1_G) \in E_P.$$

It follows that $(y, g^{-1}) \leq_P (m, g^{-1})$. Hence $y = e * m = m * g^{-1} \cdot f$ for some $e, f \in E_Y \backslash 0$ (note that P is regular and $E_P = \{(e, 1_G) \in P \mid e \in E_Y \backslash 0\}$). Thus it follows by Lemma 5.3, that $y \leq_X m$. Therefore $m = \max Y_{(a,g)}$.

Corollary 5.5. *A semigroup S is generalized F-regular if and only if S is isomorphic with a semigroup $P = PO(G, X, Y)$ such that Y contains a greatest idempotent.*

Proof. By [6], Corollary 3.14, a regular semigroup S is a generalized F-regular semigroup if and only if S is an E-unitary monoid, that is, if and only if S is isomorphic with some $P = PO(G, X, Y)$ where Y contains a greatest idempotent (see Lemma 5.1).

Remark. The particular case, where S is generalized F- inverse, was dealt with in [6], Section 4, using McAlister's P- semigroups $P(G, X, Y)$. In particular, if S is an F-inverse semigroup then by [11], Theorem 2.8, S is isomorphic with $P(G, X, Y)$ where X is a semilattice and Y is principal ideal of X.

6. A particular case

In this Section, a representation theorem for a particular class of F–regular semigroups is proved, which simplifies the general representation given in Section 5 considerably.

Let S be an F–regular semigroup satisfying $(ab)^* = b^*a^*$ for all $a, b \in S$. Recall that for $a \in S$, $a^* = 1_S : a$ (see Section 4). We will call such an S an *F–regular $*$–semigroup.* Putting $S^* = \{a^* | a \in S\}$ we first have

Lemma 6.1. *Let S be an F–regular semigroup. Then $S^* = \{x \in S | x$ is the greatest element of a ρ–class $\} = \{m \in S | m$ is maximal in $(S, \leq_S)\}$.*

Proof. Let $a^* \in S^*$; then by [7], Corollary 3.3, a^* is the greatest element of the ρ–class $(a\rho)^{-1} \in S/\rho$. Hence by [7], Lemma 5.1, a^* is a maximal element in $(S, \leq_S)$. Conversely, let m be a maximal element of $(S, \leq_S)$. Then by [7], Lemma 5.1, m is the greatest element of its ρ–class. By [7], Corollary 3.3 (2), the greatest element of the ρ–class $m\rho \in S/\rho$ is given by $1_S : (1_S : m) = 1_S : m^* = (m^*)^*$, whence $m = (m^*)^* \in S^*$.

Proposition 6.2. *Let S be an F–regular semigroup. Then $(ab)^* = b^*a^*$ for all $a, b \in S$ if and only if S^* is a subsemigroup of S.*

Proof. *Sufficiency.* Let $a, b \in S$; then $b^*a^* \in (b\rho)^{-1}(a\rho)^{-1} = [(a\rho)(b\rho)]^{-1} = [(ab)\rho]^{-1}$, whose greatest element is $(ab)^*$ (see Section 4). Hence $b^*a^* \leq_S (ab)^*$. Now by hypothesis, $b^*a^* \in S^*$; hence b^*a^* is a maximal element in $(S, \leq_S)$, by Lemma 6.1. It follows that $b^*a^* = (ab)^*$.

Necessity. Let $a^*, b^* \in S^*$; then by hypothesis, $a^*b^* = (ba)^* \in S^*$ so that S^* forms a subsemigroup of S.

Lemma 6.3. *Let S be an F–regular $*$–semigroup. Then $S^* = H_1$, the group of units of S. In particular, $(a^*)^{-1} = a^{**}$ for every $a^* \in S^*$.*

Proof. Consider $\varphi : S^* \to S/\rho, a^*\varphi = a\rho$; then it is easy to see that φ is an antiisomorphism (see Section 4). Therefore also $(S^*, \cdot)$ is a group. Next we show that $S^* = H_1$. Note first that S is a monoid by [6], Corollary 3.14, and that $1_S \in S^*$ since $1_S^* = 1_S : 1_S = 1_S$. Evidently, $S^* \subseteq H_1$ since S^* is a group with identity 1_S. Conversely, let $x \in H_1$; then by Corollary 2.4,
$(x^{-1})^*x^{-1} = 1_S \cdot (x^{-1})^*x^{-1} = xx^{-1} \cdot (x^{-1})^*x^{-1} = x \cdot x^{-1}(x^{-1})^*x^{-1} = x \cdot x^{-1} = 1_S$.
Therefore $x = (x^{-1})^* \in S^*$ so that $H_1 \subseteq S^*$ and equality prevails.
Finally let $a^* \in S^*$; then $a^*a^{**}a^* = a^*$ (see Section 4). It follows by cancellation in the group S^* that $a^*a^{**} = 1_S$, thus $(a^*)^{-1} = a^{**}$.

Examples of F–regular $*$–semigroups (see Proposition 6.2):

(1) Every group G, since for the identity relation $\rho = \varepsilon$ every ρ–class is a singleton, whence $G^* = G$ (which is a subsemigroup of G).

(2) Every band B with identity, since for the universal relation $\rho = \omega$ the unique ρ–class is B which has 1_B as greatest element, whence $B^* = \{1_B\}$ (which is a subsemigroup of B).

(3) The direct product $S = B \times G$ of a band with identity and a group G. More generally:

(4) The semidirect product $S = B \times_\Theta G$ of a band B with identity ε and a group G with respect to an homomorphism $\Theta : G \to \mathrm{Aut}B$, is an F– regular semigroup. The greatest element of the ρ–class $[(\alpha, g)\rho]^{-1} \in S/\rho$ is given by $(\varepsilon, g^{-1}) \in S$ (see Example (6) of Section 2). Thus by Lemma 6.1, $S^* = \{(\varepsilon, g) | g \in G\}$. Hence if $(\varepsilon, g), (\varepsilon, h) \in S^*$, then $(\varepsilon, g)(\varepsilon, h) = (\varepsilon * g \cdot \varepsilon, gh) = (\varepsilon, gh) \in S^*$ since every automorphism of B maps the identity $\varepsilon \in B$ onto itself. Thus S^* forms a subsemigroup of S.

Concerning an example of an F–regular semigroup S, for which S is *not* a subsemigroup, see the Remark following Proposition 4.1.

We proceed to give two representations for F–regular $*$–semigroups. The first will follow from [3] where a construction of all *uniquely unit orthodox semigroups* is provided. These are defined as regular monoids S such that E_S forms a subsemigroup and $|A(x) \cap H_1| = 1$ for every $x \in S$ (where $A(x) = \{x' \in S | x = xx'x\}$ is the set of associates of $x \in S$, and H_1 is the group of units of S). In fact, we will show that this class of semigroups coincides with the class of semigroups under consideration here. The second representation follows as a particular case from Theorem 5.4 in terms of Szendrei triples.

Theorem 6.4. *Let S be a semigroup. Then the following are equivalent.*

(i) *S is an F–regular $*$–semigroup.*
(ii) *S is uniquely unit orthodox.*
(iii) *S is isomorphic to a semidirect product of a band with identity by a group.*
(iv) *S is isomorphic to some $PO(G, X, X)$, where X is a band with identity and a zero adjoined.*

Proof. $(i) \Rightarrow (ii)$: First by [6], Corollary 3.6 and Proposition 3.7, S is a regular monoid such that E_S forms a subsemigroup. Furthermore by Lemma 6.3, $S^* = H_1$. Let $x \in S$ and $a^* \in A(x) \cap H_1 = A(x) \cap S^*$. Since by Corollary 2.5, $x^* = 1_S : x = \max(A(x) \cap S^*)$ we have that $a^* \leq_S x^*$. Now by Lemma 6.1, $a^* \in S^*$ is a maximal element of $(S, \leq_S)$. Thus it follows that $a^* = x^*$, that is, $|(A(x) \cap H_1| = 1$.
$(ii) \Rightarrow (iii)$: This holds by [3], Theorem 5.
$(iii) \Rightarrow (i)$: This is Example (4) above, in this Section.
$(i) \Rightarrow (iv)$: Let $G = S^*$ and $X = E_S^o$. Then G is a group (by Lemma 6.3) and X is a band with identity 1_S (by [6], Proposition 3.7 and Corollary 3.6) and a zero adjoined. Define a left action of G on X by: $g \cdot e = geg^{-1}$ for every $g \in G$, $e \in X$. We show that (G, X, X) is a Szendrei triple.

T1) Let $g \in G$, $e \in X$; then $\varphi_g : X \to X$, $\varphi_g(e) = g \cdot e$, is an automorphism of X. Indeed, $g \cdot e \in X$ since $geg^{-1}geg^{-1} = geg$; φ_g is injective, since $geg^{-1} = gfg^{-1}(e, f \in X)$ implies $e = f$; φ_g is surjective, since for $e \in X$, $\varphi_g(g^{-1}eg) = e$ where $g^{-1}eg \in X$; φ_g is an homomorphism, since for all $e, f \in X$

$$\varphi_g(ef) = gefg^{-1} = geg^{-1} \cdot gfg^{-1} = \varphi_g(e)\varphi_g(f).$$

Also $G \cdot X = X$, since $1_S \in G$ implies that $1_S \cdot X = X$ (note that φ_{1_S} is the identity function on X).
T2)Evidently, X is a right ideal of X and $E_X \backslash 0 = E_S$ forms a subsemigroup of X (by [6], Proposition 3.7).
T3) Let $g \in G$; then for $e \in X$, $e \neq 0$, we have that $g \cdot e \neq 0$ (φ_g is an automorphism) and $g \cdot e \in E_S$; hence $g \cdot e \in X$ has an inverse in X.

Thus we can form

$$PO(G, X, X) = \{(e, g) \in X \times G | g^{-1} \cdot e \neq 0\} = \{(e, g) \in X \times G | e \in X \backslash 0, g \in G\} = E_S \times G$$

with the operation

$$(e,g)(f,h) = (e(g \cdot f), gh).$$

We show that S is isomorphic with $P = PO(G,X,X)$. Consider

$$\varphi : S \to P,\ \varphi(a) = (aa^*, a^{**}).$$

We have $aa^* \in E_S$, since $aa^*a = a$ (see Remark (2) in Section 3), and $a^{**} = (a^*)^* \in S^* = G$, so that indeed $\varphi(a) \in P$.
φ is injective:
$\varphi(a) = \varphi(b) \Rightarrow aa^* = bb^*,\ a^{**} = b^{**} \Rightarrow aa^* \cdot a^{**} = bb^* \cdot b^{**} \Rightarrow a = b$ (by Lemma 6.3).
φ is surjective:
let $(e,g) \in P$; then $(e,g) = (e,s^*)$ for some $s \in S$; for $a = es^* \in S$ we obtain that $a^* = (es^*)^* = s^{**}e^* = s^{**}$ (by Proposition 4.1 (vi)), $aa^* = es^*s^{**} = e$ (by Lemma 6.3) and $a^{**} = s^{***} = s^* = g$ (by Proposition 4.1(iii)); therefore $\varphi(a) = (e,g)$.
φ is an homomorphism:
$\varphi(ab) = ((ab)(ab)^*\ , (ab)^{**}) = (abb^*a^*, a^{**}b^{**})$
$\varphi(a)\varphi(b) = (aa^*, a^{**})(bb^*, b^{**}) = ((aa^*)(a^{**} \cdot bb^*), a^{**}b^{**}) = (abb^*a^*, a^{**}b^{**})$

since by Lemma 6.3,

$$a^{**} \cdot bb^* = a^{**}bb^*(a^{**})^{-1} = (a^*)^{-1}bb^*a^*,$$

thus

$$(aa^*)(a^{**} \cdot bb^*) = aa^*(a^*)^{-1}bb^*a^* = abb^*a^*.$$

(iv)$\Rightarrow$(iii): Let $P = PO(G,X,X)$ be such that $(X,*)$ is a band B with identity and a zero adjoined: $X = B^o$. Hence $P = \{(e,g) \in X \times G|\ g^{-1} \cdot e \neq 0\} = \{(e,g) \in X \times G|\ e \neq 0\} = (X \backslash 0) \times G = B \times G$. The multiplication on P is defined by

$$(e,g)(f,e) = (e * (g \cdot f), gh)$$

where $g \cdot f$ is given by the left action of G on X. By condition T1) above, for any $g \in G$, $\varphi_g : X\backslash 0 \to X\backslash 0$ defined by $\varphi_g(e) = g \cdot e$, is an automorphism of $B = X\backslash 0$. Furthermore, by definition of left action, $\varphi_{gh} = \varphi_g \circ \varphi_h$ for all $g,h \in G$ (with composition from the right). Thus $\Theta : G \to \text{ Aut } B$, $\Theta(g) = \varphi_g$, is an homomorphism of G into $(\text{Aut}B, \circ)$. Hence P is the semidirect product of the band $B = X\backslash 0$ with identity by the group G with respect to the homomorphism Θ.

In the particular case of an *F-inverse* semigroup satisfying the identity $(xy)^* = y^*x^*$ the above representations take the same form with "band" replaced by "semilattice". For the proof we will use the representation of an arbitrary E–unitary inverse semigroup in terms of P–semigroups $P = P(G,Y,X)$ given in [11] (but see Theorem 6.5 below): let $(X, \leq)$ be a down–directed partially ordered set, Y be an order–ideal and subsemilattice of $(X, \leq)$, G be a group acting on $(X, \leq)$ on the left by order automorphisms such that $G \cdot Y = X$; then $P = \{(\alpha, g) \in Y \times G |\ g^{-1}.\alpha \in Y\}$ together with the operation

$$(\alpha, g)(\beta, h) = (\alpha \wedge g \cdot \beta, gh)$$

(where $\wedge$ denotes the meet in Y) forms an E–unitary inverse semigroup; conversely, every such semigroup can be constructed in this way. Note in particular, that for $\alpha, \beta \in Y$, $\alpha \wedge \beta$ exists in Y and $g \cdot (\alpha \wedge \beta) = g \cdot \alpha \wedge g \cdot \beta$ for every $g \in G$ (see [15], Lemma VII.1.2).

Theorem 6.5. *Let S be a semigroup. Then the following are equivalent.*

(i) *S is an F–inverse $*$–semigroup;*
(ii) *S is isomorphic to some $P(G, X, X)$, where X is a semilattice with identity;*
(iii) *S is isomorphic to some $PO(G, X^o, X^o)$, where X is a semilattice with identity;*
(iv) *S is a semidirect product of a semilattice with identity by a group.*

Proof. *(i)* $\Rightarrow$ *(ii)*: Since S is F–inverse, S is isomorphic to some semigroup $P = P(G, Y, X)$, where X is a semilattice and Y is a principal order ideal of $(X, \leq_X)$, i. e., $Y = (\mu]$ for some $\mu \in X$ (by [11], Theorem 2.8). Furthermore by the proof of Theorem 2.3 in [12], the greatest element of the σ–class $g \in P/\sigma = G$ (σ the least group–congruence on P) is given by: $(\mu \wedge g \cdot \mu, g) \in P$. Hence $P^* = \{(\alpha, g) \in P | \alpha = \mu \wedge g \cdot \mu, g \in G\}$. Since by hypothesis, P^* forms a subsemigroup of P we have for every $g \in G$:

$$(\mu \wedge g \cdot \mu, g)(\mu \wedge g^{-1} \cdot \mu, g^{-1}) \in P^* \Rightarrow (\mu \wedge g \cdot \mu \wedge g \cdot (\mu \wedge g^{-1} \cdot \mu), 1_G) \in P^* \Rightarrow$$
$$\Rightarrow (\mu \wedge g \cdot \mu \wedge 1_G \cdot \mu, 1_G) = (\mu \wedge 1_G \cdot \mu, 1_G) \Rightarrow (\mu \wedge g \cdot \mu, 1_G) = (\mu, 1_G) \Rightarrow \mu \wedge g \cdot \mu = \mu;$$

thus $g^{-1} \cdot (\mu \wedge g \cdot \mu) = g^{-1} \cdot \mu \Rightarrow g^{-1} \cdot \mu \wedge \mu = g^{-1} \cdot \mu \Rightarrow g \cdot \mu \wedge \mu = g \cdot \mu$;

hence $g \cdot \mu = \mu$. It follows that $Y = X$; indeed, we have for every $g \in G$:

$$\alpha \in Y \Rightarrow \alpha \in (\mu] \Rightarrow \alpha \leq_X \mu \Rightarrow g \cdot \alpha \leq_X g \cdot \mu = \mu \Rightarrow g \cdot \alpha \in (\mu] = Y;$$

hence $G \cdot Y \subseteq Y$, so $X \subseteq Y$ and $Y = X$. Therefore $P = P(G, X, X)$, where X has an identity (since $X = Y$ has as greatest element μ).

(ii) $\Leftrightarrow$ *(iii)*: If $(X, \wedge)$ is a semilattice (with identity) then it easy to see that adjoining a zero, $(X^0, \wedge)$ is a strictly combinatorial semigroup (note that $S/\mathcal{P}$ consists of $\{0\}$ and one further class only: see [8], Proposition IX.5.2). For $Y = X^0$ we have $Z = X$ and (G, X^0, X^0) is a Szendrei triple. Furthermore,

$$PO(G, X^0, X^0) = \{(a, g) \in X^0 \times G \mid g^{-1} \cdot a \in X\} = X \times G = P(G, X, X)$$

(note that $g \cdot 0 = 0$ for any $g \in G$) and the operations on $P(G, X, X)$ and $PO(G, X^0, X^0)$ coincide. Therefore (iii) holds. The converse implication is shown similarly.

(ii) $\Rightarrow$ *(iv)*: Since $P = P(G, X, X) = \{(\alpha, g) \in X \times G |\ g^{-1}.\alpha \in X\} = X \times G$, where multiplication is given by: $(\alpha, g)(\beta, h) = (\alpha \wedge g \cdot \beta, gh)$, we have that P is a semidirect product of the semilattice $(X, \wedge)$ with identity by the group G.

(iv) $\Rightarrow$ *(i)*: Let S be a semidirect product of the semilattice X with identity by the group G. Then by Example (4) in this Section, S is an F–regular $*$–semigroup. Since $E_S = \{(\alpha, 1_G) | \alpha \in X\}$, it follows that the idempotents of S commute. Thus S is an inverse semigroup.

REMARK. A representation of *general* F–inverse semigroups S similar to that given in [11], Theorem 2.8, is provided in [15], Theorem VII.5.16. The particular case that S is a $*$-semigroup, can also be deduced from this construction using Exercise VII.5.27(v) in [15].

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