Investigating prospective science and mathematics teachers' meanings for and representations of functions: an international study

Elizabeth Oldham, Trinity College Dublin, the University of Dublin, Ireland,
eoldham@tcd.ie
Floriano Viseu, University of Minho, Portugal,
fviseu@ie.uminho.pt
Helena Martinho, University of Minho, Portugal,
mhm@ie.uminho.pt
Rook Doggen, Fontys University of Applied Sciences, the Netherlands,
r.doggen@fontys.nl
Elsa Price, Faulkner University, USA,
eprice@faulkner.edu
Laurinda Leite, University of Minho, Portugal,
lleite@ie.uminho.pt

Abstract

The concept of a *function* is crucial in both mathematics and science education. Appropriate teaching of the concept is therefore very important, and an understanding of the knowledge of functions possessed by prospective teachers of science and mathematics is highly desirable. The present study was instigated by the Association for Teacher Education in Europe's Research and Development Community (RDC) on Science and Mathematics Education in order to explore this knowledge across different countries. Using theoretical frameworks provided by research on teacher knowledge and representations in mathematics and science education, the study is investigating meanings that prospective teachers give for the term "function" and representations that they associate with it. An instrument was adapted from a previous RDC study (on ratio). For the initial phase, reported here, research team members in Portugal, Ireland and the Netherlands collected data from opportunity samples at their own institutions, focusing on prospective mathematics teachers. Analysis of the responses (N=145) started with team members examining their own data; they listed themes, guided by curricular traditions in their own countries as well as by research on knowledge of functions. Lists were then shared in order to identify commonalities and contrasts. The data reflected meanings and representations involving correspondence/dependence relationships and also those based the idea of a "machine"/formula/rule; the former were most prominent in the Portuguese data and least in the Dutch, in line with national curricula. The findings suggest that, in the ongoing study, the instrument will provide useful information for teacher educators.

Keywords: function, representation of functions, prospective teachers, curriculum

1. Introduction

The importance of teachers' – and hence prospective teachers' – knowledge of their subjects, especially in a form appropriate for teaching, has been well established by research ongoing since the 1980s. For developing students' understanding of individual topics, especially in mathematics, the use of multiple representations has emerged over the same period as a key area. Together, these fields of research provide a useful theoretical framework for investigations.

One such investigation is the study initiated in 2018 at the Annual Conference of the Association for Teacher Education in Europe (ATEE) by its Research and Development Community (RDC) "Science and Mathematics Education". The study focuses on the concept of a function – a major topic in mathematics education, and important also for science – and is intended to examine the knowledge possessed by *prospective teachers* of mathematics and science (that is, students in preservice teacher education programmes involving mathematics and science, and also those attending relevant courses or modules that attract students with an interest in teaching). The aims of the study are to address the following research questions:

a) What meanings do prospective teachers for primary and secondary levels, attending selected institutions in different countries, give to the term "function"?
b) What multiple representations do these prospective teachers associate with the term "function"?

c) What implications do the prospective teachers' descriptive meanings and representations have for teacher education courses with regard to functions?

An instrument intended to elicit the meanings and representations was adapted from one used in a previous RDC study (on ratio, another topic spanning mathematics and science). Originally written in English, it was translated into Portuguese, Dutch and German to facilitate use by RDC members from different countries. For a first exploration, the focus was on prospective teachers of mathematics. Members of the RDC from Portugal, the Netherlands and Ireland administered the instrument to selected groups of prospective mathematics teachers in institutions in their own countries. The data were analysed, first with RDC members considering their own data sets and then with their insights being combined. By using this approach, the RDC is seeking to find possible commonalities or contrasts that may reflect differing understandings of functions within different mathematical traditions, national curricula or contexts. These may illuminate a range of approaches to teaching and learning about functions and also highlight aspects of knowledge that may need to be addressed with prospective teachers.

In order to illustrate the work involved in the exploratory phase of the project, the curriculum with regard to functions and the sample data analysis for one country – Portugal – are presented in some detail. Those for Ireland and the Netherlands are described more briefly, with a focus on similarities and contrasts. While findings from the opportunity samples may not generalize, they may indicate the potential of the instrument to reveal useful information for teacher educators.

In section 2 of the paper, a review of literature is provided, addressing the notion of function and its occurrence in the mathematics curricula of the three countries considered in this paper, and also the use of representations in the teaching of functions. Section 3 describes the methodology for the study, in particular during its initial phase. Results for this phase are presented in section 4, with discussion and conclusions being provided in section 5.

2. Literature Review

Given the objective of this work, the literature review focuses on the notion of function (section 2.1) and its different representations in the teaching of mathematics (section 2.2).

2.1. The notion of function in mathematics curricula

The complexity of the construction of the notion of function causes many students and prospective teachers to manifest difficulties in expressing its meaning clearly (Martinho & Viseu, 2019; Viseu, Martins, & Rocha, 2019). A study by Vinner and Dreyfus (1989) shows that college students in a calculus course were often not able to apply the definition of a function correctly, even when they could give a correct explanation of it. Breidenbach, Dubinsky, Hawks, and Nichols (1992) point out that "college students, even those who have taken a fair number of mathematics courses, do not have much understanding of the function concept" (p. 247), confirming that this is a complex concept to understand and that, consequently, its conceptual development requires a longer period of time. According to Hansson (2004), the concept should be introduced dynamically as a kind of relationship, correspondence, or covariation, rather than through the static notion of a set of ordered pairs.

To set the study of functions in context, relevant features of the three education systems and curricula are described. As in many of the counties in the world, the *Portuguese* education system comprises twelve years of mandatory study to grant access to higher education levels. Of these years, the first nine correspond to basic education and the last three to secondary education. Basic education consists of three cycles: the first one, lasting four years – grades 1 to 4 – is also known as the primary school; the second cycle lasts two years – grades 5 to 6 – and the third one three years – grades 7 to 9. For all three cycles, the mathematics syllabus is the same for all students. Until the end of the third cycle, it is structured around the following themes: Numbers and Operations, Geometry and Measurement, Algebra, and Organization and Data Processing (Ministério da Educação e Ciência, 2013). Beyond their formative purpose, along with the acquisition of knowledge and procedures, the study of these themes aims at promoting the development of skills and attitudes. Capacities to be developed include problem solving, communication, and reasoning. As the educational process evolves, students' cognitive activities also gradually shift attention from concrete situations to more abstract ones. An example of activities on concrete situations are the arithmetic operations, within the theme Numbers and Operations, which can be translated into manipulatives. The topic "Function", within the Algebra theme, on the other hand requires activities involving abstract situations.

The notion of a function acquires an abstract nature as it results from a mental construction (Evangelidou, Spyrou, Elia, & Gagatsis, 2004). Such a construction, which in the Portuguese Mathematics syllabus is placed in the third cycle of basic education, highlights the relevance of mathematical communication, namely of a number of linguistic aspects that help students to build the correct intuitions. This includes, for example, the linguistic ability to distinguish functions from general correspondences, or to identify object, image, domain, range, and contradiction, as well as independent and dependent variables (Martinho & Viseu, 2019). In order to emphasize the meaning of this terminology, different representations of functions can be considered: for example arrow diagrams, tables and Cartesian graphs. The acquisition of such terminology plays a crucial role in the introduction of the notion of a linear function and, consequently, of a direct proportionality function.

Subsequently, the inverse proportionality function is studied through its different representations (tabular, analytical and graphical) and the quadratic function through the identification of the curve representing functions of type $f(x) = ax^2$ (with $a \neq 0$) and the solution set for the equation $f(x) = ax^2 + bx + c = 0$ (with $a \neq 0$) as the intersection of the parabola, $y = ax^2$, and a straight line, $y = -bx - c$.

In the transition to secondary education, the knowledge acquired in the third cycle acts as a prerequisite to the study of function composition and the inverse of a bijective function, as well as to identifying intervals of monotonic growth and the extremes of real-valued functions of real variables (Ministério da Educação e Ciência, 2013). The study of functions broadens: students are introduced to trigonometric functions, the Heine limits of real functions with real variables, derivatives of such functions and their applications. Finally, the study of limits and continuity of real-valued functions of a real variable is carried out and the study of derivatives of such functions deepened.

In *Ireland*, there are 14 years of school education, eight for primary school and six for post-primary school. The post-primary curriculum has two cycles, junior (for students typically aged 12-15) and senior (for students aged about 15 to 18); the mathematics courses contain strands on Number, Algebra, Geometry and Trigonometry, Statistics and Probability, and Functions (the latter including calculus in the senior cycle). Following the most recent revision, implemented in autumn 2018, the Functions strand in the junior cycle has been merged with the Algebra strand, reflecting intentions with regard to how functions might be taught (Department of Education and Skills, 2017). Similarly to the case for

Portugal, the school curricula address overarching key skills, and mathematics and other curricula move from a more concrete to a more abstract focus in the higher grades.

Prior to the 1960s, Irish students encountered functions determined by algebraic or trigonometric expressions, and typically represented by graphs. Consideration of limits underpinned the introduction of calculus for the more advanced students. From the time of "modern mathematics" in the 1960s, students were introduced to functions as special relations, hence as sets of couples displaying the uniqueness property (each first element being associated with just one second element), and the terminology of domain, codomain and range was emphasised. However, this approach coexisted with the more traditional one for dealing with equations, coordinate geometry, graphs and calculus. The two conceptions have continued, with varying degrees of prominence, to the present day. The junior cycle curriculum introduced from 2008 indicates that students should "engage with the concept of a function (that which involves a set of inputs, a set of possible outputs and a rule that assigns one output to each input)" (Department of Education and Skills, 2013, p. 30). The revised version currently being implemented stresses that learners should "represent and interpret functions in different ways – graphically…, diagrammatically, in words, and algebraically – using the language and notation of functions (domain, range, co-domain, f(x) = , f :x ↦, and y =)" (Department of Education and Skills, 2017, p. 19).

The *Netherlands* provides up to 14 years of schooling. The number of different schooltypes in the system, and the tradition of relative curricular freedom, mean that it is hard to summarise the mathematical provision succinctly. Of more relevance here than the year-by-year details is the tradition of Realistic Mathematics Education. While both Portugal and Ireland adopted "modern" (hence, notably abstract and formal) curricula in the 1960s/1970s, the Netherlands introduced an approach that focused on being "real to the students," typically involving the solution of problems set in engaging contexts. With regard to functions, such a setting is suitable for a focus on the dynamic rather than the static (set-theoretic, "modern") conception, in line with the work of Hansson (2004), cited above. The curriculum duly reflects the approach (Creative Commons, n.d.).

2.2. Representations in the teaching of functions

Current methodological recommendations for the study of functions emphasise the use of different representations as a crucial element of the learning process (Chazan & Yerushalmy, 2003; National Council of Teachers of Mathematics, 2000, 2014), as each of them highlights a complementary aspect.

Tabular representation. This representation, also called numerical, depicts the function as a table relating objects and images. Therefore, it leads to the discovery of the general relationship underlying the function at hand. Checking that it is really a function requires analysis of the structure of the relationship represented in the table, to confirm that each object has a unique image. For Brown and Mehilos (2010), this representation promotes the passage from concrete to abstract, giving meaning to variables and algebraic expressions.

Graphical representation. This representation makes explicit the points corresponding to (object, image)-coordinates in a Cartesian plane. Therefore, it provides a quick way to detect typical properties of functions, such as zeros, sign change, and monotonicity. For Friedlander and Tabach (2001), the graphical representation is intuitive and appealing due to its visual character.

Analytical representation. In addition to the sets acting as the function's domain and range, this representation expresses the relationship between objects and image through an analytical expression. It paves the way to the use of algebraic laws to transform the analytical expression in the study of the function properties. Friedlander and Tabach (2001) consider this representation an accurate and general way to make explicit the underlying formal patterns and models. Moreover, algebraic manipulations are often the only way to justify general statements about the behaviour of a function.

The complementarity between aspects revealed by different representations of function entails the need for students to explore all of them (Santos & Barbosa, 2016). Nachlieli and Tabach (2012) argue that, in practice, such a multi-perspective view is not always present in the teaching process. Often only one representation is used, and, when more than one is considered, the way they relate to each other is not discussed. Tripathi (2008) argues that limiting the study of the notion of function to a single representation is "approach[ing] the concept blindfolded" (p. 438).

This assumption is the basis of some studies conducted in the field of mathematical education. For example, Evangelidou et al. (2004) carried out a study with prospective teachers, predominantly for primary education, seeking to understand how functions are interpreted in terms of their conceptual meaning, and recognized in multiple representations. The study highlighted three trends in the prospective teachers' notion of a function. Most of them identify the notion of a function with the more specific concept of a "one-to-one function". Although injective functions often arise in practice, this incorrect identification becomes a strong obstacle to understanding function as a broader concept. The second trend identifies a function with an analytical relationship between two variables. The third trend connects the notion of a function with a diagram type or a Cartesian graph.

The ways in which the notion of a function is conceptualized can be also be framed in terms of image (mental construction that represents the cognitive structure associated with the concept) and concept (Vinner & Dreyfus, 1989; Tall & Vinner, 1981). In particular, Vinner and Dreyfus (1989) identified different categories of definitions and conceptual images. Drawing on their work, Viirman, Attorps, and Tossavainen (2010) used the following classification:

Correspondence/dependency relationship. A function is any match or dependency relationship between two sets that assigns each element in the first set to exactly one element in the other.

Machine. A function is a "machine", i.e. one or more operations that transform variables into new variables. In this case, no explicit mention of domain and range is made.

Rule/formula. A function is a rule, a formula, or an algebraic expression. Compared to the second category, the difference is that a rule typically entails some form of regular behaviour, whereas the "machine" could perform totally different transformations of different elements.

Representation. The function is identified with one of its representations.

Meaningless. A meaningless answer or no answer.

For Vinner and Dreyfus (1989), students' common images of the concept of a function have direct implications for teaching, since they can be used as a starting point to the construction of the concept itself.

3. Methodology

The research questions for the study are listed in the Introduction. To address them, the RDC team developed an instrument and used it to collect data from groups in institutions in three countries, as described below.

3.1. Research instrument and administration

Since the international study involves groups speaking different languages, the RDC chose to use a short written questionnaire (hence, obviating the difficulties that might have arisen in sharing data from extended interviews). The instrument was based on one used in a similar RDC study, on the concept of ratio (Berenson, Oldham, Price, & Leite, 2013). It contained four open questions – one in two parts – allowing participants to describe their knowledge of the meanings and representations of ratio. The equivalent questions (or items) for the functions study are as follows:

1. What does the term "function" mean to you?

2a. When do you yourself use functions?

2b. Who else uses functions, and when do they use them?

3. Which mathematical symbol(s) do you use to represent functions? You may write expressions that include the symbols, rather than just the symbols themselves.

4. Show how you would explain the concept of "function" (not using words only!). Give a few examples if you can. Present your ideas here and/or overleaf as you wish.

Question 2 involves awareness of applications; question 4 allows participants to demonstrate some mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008). The four questions were set out on a single page of A4 paper, also providing introductory and classifying material, as shown in Figure 1.

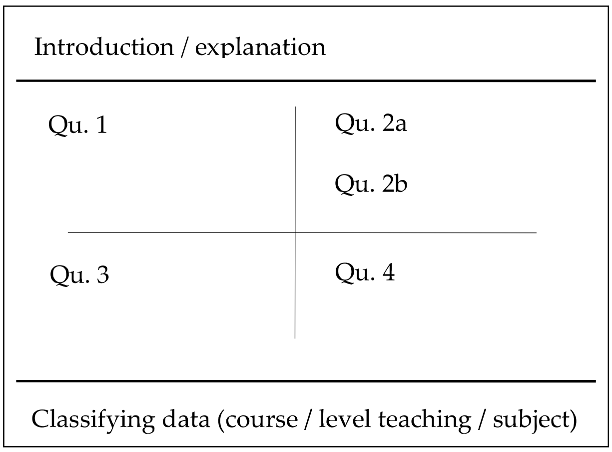


Figure 1 - Layout of the questionnaire

It was intended that data collection could be completed in about fifteen minutes, say at the end of a class period, making minimal demands on teaching time. As indicated above, the instrument was localised for use in different countries and institutions, with the original English-language version being translated into Dutch, German and Portuguese, and the classifying material (identifying the participating class groups) being formulated appropriately for each site.

For the exploratory phase of the study, appropriate ethical clearance was obtained for participation at institutions in Portugal, the Netherlands and Ireland. (Unfortunately, the team member from the USA, who had hoped to collect data, did not receive ethical clearance in time.) The instrument was administered to opportunity samples in these institutions during the period from autumn 2018 to spring 2019, at times that suited the schedules of the participating groups. The sets of data from each country were subjected to content analysis by the RDC members responsible for collecting them. Tentative classifications and themes were identified, using categories suggested by the data but also influenced by local curricular issues and literature on functions and representations, and the data were coded accordingly. This resulted in some differences in classification and interpretation, which have yet to be resolved. In this paper, the Portuguese codes are given priority, and the Irish and Dutch data are tentatively examined using the Portuguese categories where possible. Those categories are as follows:

Question 1: Correspondence/Dependency relationship; Machine; Formula/rule.

Question 2a: School context; Out-of-school context; School and out-of-school context.

Question 2b: Teachers and students; Specific professions; Any professional context.

Question 3: Isolated terminology; Analytical expression; Arrow diagram; Multiple representations.

Question 4: Everyday language; Algebraic expressions; Arrow diagram; Cartesian graph; Multiple representations.

3.2. Participants

The *Portuguese* study included 87 prospective teachers organized into three groups according to the year they attended. Group A consisted of 29 students from the first year of the bachelor's degree in basic education (S1 to S29); Group

B consisted of 40 students from the third year of the bachelor's degree in basic education (S30 to S69); Group C consisted of 18 students from the second year of the master's degree in teaching primary school and Mathematics and Natural Sciences at basic school (S70 to S87). The codification, from S1 to S87, followed the order of the students of each grade considered.

The *Irish* participants consisted of two groups in one university: 18 undergraduate mathematicians taking a Mathematics Education module that involves them in helping students in school or other classrooms (hence being considered as prospective mathematics teachers) and four preservice teachers of mathematics (in the first year of their two-year professional master's programme leading to national accreditation). The unusually small number in the preservice course obviates analysis of the two groups separately. The 22 students are coded as S88 to S109, with the undergraduate group being listed first.

The *Dutch* cohort consisted of 36 preservice mathematics teachers in their first year of teacher education. They are coded S110 to S145.

4. Results

As indicated above, the Portuguese results are given priority. Selected findings from Ireland and the Netherlands are then compared and contrasted with them.

4.1. Portuguese results

Regardless of the cycle of study in which they are enrolled, most future teachers in the Portuguese sample relate the notion of function to a correspondence between elements of two sets, or to the dependence of values of one variable on the values of another (72%) (Table 1).

Table 1. - Frequency of answers from different Portuguese groups to question 1

Meaning of the term function	Group A	Group B	Group C	Total
Correspondence/Dependency relationship	24	29	10	63
Machine	1	0	2	3
Formula/rule	0	1	1	2
Meaningless	3	9	5	17
No reply	1	1	0	2

Total	29	40	18	87

For some students, the existence of a correspondence between two sets is the only requirement they make explicit as characterising a function, as exemplified by the following answers: "a relationship between two sets" (S9); "Relates to sets, images and objects" (S11); "a relationship between variables" (S76). Such a conception translates into a fragile definition, which mixes up functional and non-functional correspondences.

The correspondence between elements of two sets is made more explicit in the answers mentioning a dependence between the values of variable, as illustrated for example in: "function means a transformation of an x by a y. When we have a dependent variable and an independent variable" (S80); "A function implies the existence of two variables (x, y). By organizing the regular data in a graph, we can predict the values of x or y, knowing other variables" (S55). It should be noted that the last answer was given by a student who was previously exposed to modelling tasks to search the curve that best fits points from a particular experience.

A few students, seven out of the 87, explain in their responses that a function is a relationship that forces each object to have one and only one image, as exemplified by the following answer: "A function is when one element of the starting set corresponds to one and only one element of the target set" (S52).

Some other students lean toward the operational perspective classified as a "machine" by Viirman et al. (2010). Examples are: "A function is a mathematical method used to find an unknown value" (S23); "A function is something that allows us to determine and correspond to an x or the opposite in a particular case" (S79); and again "Function means a transformation of an x by a y. When we have a dependent variable and an independent variable" (S80). Such responses somehow resort to the transformation metaphor, which is sometimes claimed to capture the uniqueness of image in a functional correspondence.

The association of a function with a rule or formula occurs only in the responses of two students: "It consists of an expression with at least 2 unknowns, where one can verify the relationship between them"; "a function is an expression that relates two variables, thus one being a dependent variable of another" (S76). Such answers reveal a symbolic conception of the notion of a function.

Along with the meaning students give to the notion of a function, situations in which they perceive that they use functions are investigated (question 2a). Most answers (59%) favour the school context. Only a few responses highlight their use in out-of-school contexts (20%) or in both school and out-of-school contexts (18%) (Table 2).

Table 2. - Frequency of answers from different Portuguese groups to question 2a

When using functions (by context)	Group A	Group B	Group C	Total
School context	11	29	11	51
Out-of-school context	9	4	4	17
School and out-of-school context	7	7	2	16
Meaningless	1	0	1	2
No reply	1	0	0	1
Total	29	40	18	87

The use of functions in the school context emerges within activities proposed by the teacher – "I only use functions when asked so by teachers" (S47) – or in the context of academic activities, i.e. when studying "school or university" subjects (S18). A curious answer stresses the fundamental role of functions in all mathematical activities: "mathematics as a way of getting a relationship between two sets" (S3). Some students argue that functions can be used in any context, such as "in mathematics classes and sometimes in everyday life" (S27).

With respect to the question "Who else uses functions, and when?", question 2b, responses highlight in a similar way academic situations, specific professional contexts, and even the professional context in a broad sense (Table 3).

Table 3. - Frequency of answers from different Portuguese groups to question 2b

Who uses functions (by context)	Group A	Group B	Group C	Total
Academic context	7	13	4	24
Specific professions (engineers, nurses, etc.)	7	12	8	27
Any professional context	9	11	6	26
Meaningless	1	2	0	3

No reply	5	2	0	7
Total	29	40	18	87

Within the academic context, students identify as typical users of functions ”mathematicians and learners” (S1), the “teachers in the class” (S43) or the “mathematicians and students of mathematics, as they are presented with a diverse number of problems in which you have to decipher or apply functions” (S53).

For professional contexts, there are students who consider that those who use functions are “traders and their buyers” (S3), “architects to carry out projects” (S21), the people “who work with money and quantities” (S15), and “Taxi drivers when calculating the total value of the fare; … any seller who wants to know the full value of a purchase” (S30). Some participants combined both academic and everyday contexts, as in: “mathematicians mainly, but everybody uses them, on a daily basis, to solve mathematical problems, to calculate unknowns that arise in everyday life” (S31).

Concerning the representation of functions, students were asked about which mathematical symbols they usually use. Their answers highlight the use of isolated terminology and analytical expressions (Table 4). Examples of the former include “x, y, ( )” (S2); “A lowercase letter, ex. f” (S6); or “f to represent a function and we have an image and an object” (S24).

Table 4. - Frequency of answers from different Portuguese groups to question 3

Symbols used to represent functions	Group A	Group B	Group C	Total
Isolated terminology	12	22	5	39
Analytical expression	11	11	8	30
Arrow diagram	2	4	0	6
Multiple representations	0	1	3	4
Meaningless	4	1	2	7
No reply	0	1	0	1
Total	29	40	18	87

In the answers mentioning the use of analytical expressions, students tend to resort to the usual textbook symbolism, e.g. "x and $f(x)$, for example, $f(x) = 2x^2$ or $g(x) = 2x + 1$"(S27); "$Y = mx + a$; $y = x^2 + a$"(S53).

Finally, the last question asked students to indicate how they would explain the concept of a function. Most of them claim to resort to the use of an "Arrow diagram" (39%) or to multiple representations (16%) (Table 5).

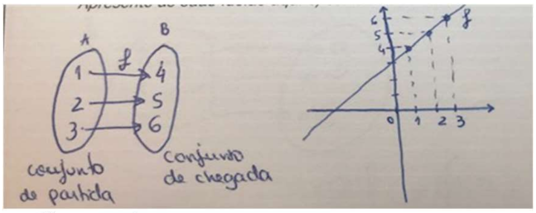
Table 5. - Frequency of answers from different Portuguese groups to question 4

Explanation of the function concept	Group A	Group B	Group C	Total
Everyday language	3	3	4	10
Algebraic expressions	1	4	0	5
Arrow diagram	12	18	4	34
Cartesian graph	3	2	4	9
Multiple representations	5	4	5	14
Meaningless	5	9	1	15
Total	29	40	18	87

A few students would resort to "everyday language" (12%), as illustrated by the following answer: "Something that is associated with something for a particular purpose. For example, in 2 weeks I complete 2 homeworks, in 4 weeks I complete 4. Another example: My brother is 6 years old, I am 9. When he will be 18, I will be 21" (S2).

In the illustration of representations to highlight ways of explaining the concept of a function, there are students who refer simultaneously to an "arrow diagram" and a "Cartesian graph", as exemplified by the answer of the student S20 depicted in Figure 2.

.



(Starter set) (Target set)

Figure 2. - Reply (S20) to question 4

The student starts from an example presented through an arrow diagram with a discrete set, and then states that "To explain the concept of a function, I would start by drawing two sets and match one to the other in terms of one and only one term. Then I would draw a graph and explain that the function is the line connecting the intersection points of x and y" (S44). This answer, like many others, reveals students' weakness in mastering the concept of a function. It should be added that the majority of answers to the question under consideration are based on examples only. Few students were concerned with providing any additional explanation.

A final point can be made about the Portuguese data; the patterns across the three participating groups are very similar. Analysis of differences in detail is outside the scope of the paper.

4.2. Irish results

Analysis of the Irish data was guided initially by the way in which responses reflected the curricular trends outlined in section 2.1 as well as the literature described in section 2.2. The Irish presentation of results for questions 1, 3 and 4 is available elsewhere in this volume (Oldham & Prendergast, 2020). For the present paper, therefore, most of the data was recoded using the Portuguese categories – though further work would be needed, with cross-coding between Irish and Portuguese team members, before comparability could be firmly established. A feature of the Irish data is that many students gave multiple responses (more than one example in a single category – not reflected in the tables below – or responses in more than one category), so the total number of responses can be more than 22.

The frequencies of occurrence of the Portuguese categories in the Irish responses to question 1 are shown in Table 6. Not all responses fitted easily into the classification, or at least into the one by Viirman et al. (2010) on which it is based; for instance, the (explicit or tacit) "machine" analogy was presented along with mention of domain and range (see section 2.2 above). This perhaps reflects the intertwining of different function concepts in the Irish curriculum. Some participants gave a formal definition, for example stating that a function is "a subset, for given [sets] A and B, of A×B such that [the uniqueness property holds]" (S88). Others appear to have taken literally the query regarding what function meant to them, and provided a looser or more personal statement; an instance is: "A function, for me, is a program which converts an input (or set of inputs) into an output (or outputs)" (S102). In contrast to the Portuguese case, most responses were in the machine/formula/rule categories, and seven of the students (almost one-third) mentioned the uniqueness property.

Table 6. - Frequency of answers from Irish participants to question 1

Meaning of the term function	Total
Correspondence/Dependency relationship	9
Machine	11
Formula/rule	4
Meaningless/other	1
No reply	0
Total	25

In responding to question 2, most students indicated where they would use functions (hence providing a context), but three – perhaps focusing on the word "when" – identified a task without giving such a context. An example is "When I want to see the outcome of a certain event which requires inputs" (S96). The classification is presented in Table 7.

Table 7. - Frequency of answers from Irish participants to question 2a

When using functions (by context)	Total
School/college/academic context only	9
Out-of-school context only	2
School and out-of-school context	7

No context given	3
Meaningless/other	1
No reply	0
Total	22

For question 2b, an explicit category has been added in Table 8, covering responses that highlighted the broad applicability of functions especially in everyday life (compare S31 above). For example, S90 wrote "… and also in everyday life, people use functions in some simple form," and S101 stated "Everyone, whether they realise it or not."

Table 8. - Frequency of answers from Irish participants to question 2b

Who uses functions (by context)	Total
Academic context	4
Specific professions (engineers, nurses, etc.)	18
Any professional context	6
Everyone/everyday life	7
Meaningless	0
No reply	0
Total	35

The original Irish coding for question 3 specifically reflected the notation used in the Irish curriculum, as described in section 2.1 above (see Oldham & Prendergast (2020)). It was not considered useful to recode the data for presentation here. For question 4, however, the Irish categories are sufficiently close to the Portuguese; the results are presented in analogous form in Table 9.

Table 9. - Frequency of answers from Irish participants to question 4

Explanation of the function concept	Total
Everyday language	15
Algebraic expressions/symbols	12

Arrow diagrams	6
Input/function-box/output diagrams or pictures and variants	12
Cartesian graph	2
Meaningless	0
Total	47

Again, in contrast with the Portuguese responses, over half of the students (12 out of 22) indicated that they would provide a machine-type explanation, hence focusing on a dynamic rather than a set-theoretic approach; just six (fewer than one-third) used arrow diagrams. Everyday language and symbols generally appeared alongside another type of response, and only five students gave a single example. Responses similar to the one in Figure 2 above are included in the Irish paper (Oldham & Prendergast, 2020).

A final point here is that the Irish coders may have been less rigorous than the Portuguese in judging responses as meaningless or referring to them as inadequate. Cross-country coding would resolve the issue.

4.3. Dutch results

The contrasts observed between the Portuguese and Irish responses are even more noticeable in the case of the Portuguese and Dutch data. This is especially clear with regard to question 1. As shown in Table 10, meanings referring to the correspondence/dependency relationship – predominant in Portuguese responses – were given by only three students. Half of the group gave a response broadly in the "formula" category. Almost as many gave responses referring to a "task", "problem solver" or "calculator"; these appear to refer to specific aspects of the approach in the Dutch curriculum. The machine-type analogy, prominent in the Irish data, is not reflected here. Language issues in translating from Dutch, for example in trying to capture approaches to problem-solving, may have exacerbated the contrasts; further discussion and shared coding might resolve some of the difficulties.

Table 10. - Frequency of answers from Dutch participants to question 1

Meaning of the term function	Total

Correspondence/Dependency relationship	3
Machine	0
Formula/rule	18
Task/problem solver/calculator	17
Meaningless/other	6
No reply	0
Total	44

Because of these difficulties, analysis of the Dutch data is still a work in progress, and data from other questions are not tabulated. However, it can be said that the responses to question 4 do not greatly feature the static, set-theoretic approaches that figure for the other two countries, nor do they make heavy use of the machine analogy popular with Irish respondents; in the Dutch sample, an approach via formulae is prominent. This appears consistent with the curricular traditions. The differences regarding question 2 are less marked.

Further examination is left for a later phase in the study. It could also lead to a separate investigation, in which the Dutch curriculum and approach to teaching could play the leading role.

5. Discussion and Conclusion

This paper has described the initial, exploratory phase of a small-scale cross-national project intended to investigate meanings that prospective teachers give for the term "function" and representations that they associate with it, hence hopefully contributing to enhancements in teacher education programmes. Data were collected by means of a short instrument from opportunity samples of prospective mathematics teachers in institutions in Portugal, Ireland and the Netherlands. The data sets from each country were subjected to content analysis by the research team members responsible for collecting them; coding was guided by the data and local curricular issues as well as by the literature. This led to some differences in approach. These would provide a serious problem if the aim of the study were to compare responses from representative samples in each country; cross-country coding and discussion would be needed, as would explication of some language differences. However, especially for the exploratory phase of the study, such an approach was not essential. Rather, even allowing for the difficulties, the work done has already revealed the capacity of the instrument to elicit meanings and representations of varying degrees of appropriateness and reflecting different curricular traditions. This can lead to within-country exploration of prospective teachers' relevant knowledge as well as providing some cross-country pointers to areas of interest.

Further work is planned, within and between countries. For example, as indicated above, more exploration of the Dutch responses is warranted. Also, it is intended that data be collected by team members in the USA and Germany, hopefully involving prospective science teachers. There is scope also for a deeper study of science and mathematics curricula and analysis of relevant textbooks pertaining to the school year in which the function concept is introduced. This could help to illuminate difficulties, different representations used and conceptions likely to be held by students, and hence by prospective teachers of science and mathematics.

6. Acknowledgment

This work was co-funded by CIEd - Research Centre on Education, Institute of Education, University of Minho, projects UIDB/01661/2020 and UIDP/01661/2020, through national funds of FCT/MCTES-PT.

7. References

Ball, D., Thames, M., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389–407.

Berenson, S., Oldham, E., Price, E., & Leite, L. (2013). Investigating representations of ratio among prospective mathematics and science teachers: An international study. In E. Agaoglu, C. Terzi, D. Kavrayici, D. Aydug, & B. Himmetoglu (Eds.), *Proceedings of the 37th Annual Conference of the Association for Teacher Education in Europe* (pp. 78–92). Brussels, Belgium: ATEE.

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics, 23*(3), 247–285.

Brown, A. S., & Mehilos, M. (2010). Using tables to bridge arithmetic and algebra. *Mathematics Teaching in the Middle School, 15*(9), 532–538.

Chazan, D., & Yerushalmy, M. (2003). On appreciating the cognitive complexity of school algebra: research on algebra learning and directions of curricular change. In J. Kilpatrick, W. Martin, & D. Schifter (Eds.) *A research companion to Principles and standards for school mathematics* (pp. 123–135). Reston VA: NCTM.

Creative Commons (n.d.). The Dutch curriculum of mathematics. Retrieved from https://www.scribd.com/doc/184253335/The-whole-curriculum-of-mathematics-in-the-Netherlands-docx

Department of Education and Skills (2013). *Junior Certificate Mathematics syllabus: Higher, Ordinary & Foundation level.* Retrieved from https://www.curriculumonline.ie/getmedia/4f6cba68-ac41-485c-85a0-32ae6c3559a7/JCSEC18_Maths_Examination-in-2016.pdf

Department of Education and Skills (2017). *Junior Cycle Mathematics.* Retrieved from https://curriculumonline.ie/getmedia/6a7f1ff5-9b9e-4d71-8e1f-6d4f932191db/JC_Mathematics_Specification.pdf

Evangelidou, A., Spyrou, P., Elia, I., & Gagatsis, A. (2004). University students' conceptions of function. In *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2 (pp. 351–358). Bergen, Norway: PME.

Friedlander, A., & Tabach, M. (2001). Promoting multiple representations in algebra. In A. Cuoco (Ed.), *The roles of representation in school mathematics* (pp. 173-185). Reston VA: NCTM.

Hansson, Ö. (2004). *An unorthodox utilization of concept maps for mathematical statements: Preservice teachers' response to a diagnostic tool.* Research report, Kristianstad University/Luleå University of Technology.

Martinho, M. H., & Viseu, F. (2019). The concept of a function among prospective teachers. In L. Leite, E. Oldham, L. Carvalho, A. S. Afonso, F. Viseu, L. Dourado, & M. H. Martinho (Eds.), *Proceedings of the ATEE Winter Conference "Science and mathematics education in the 21st century"* (pp. 131–140). Brussels, Belgium: ATEE and CIEd.

Ministério da Educação e Ciência (2013). *Programa e Metas Curriculares: Matemática, Ensino Básico.* Lisboa: MEC.

Nachlieli, T., & Tabach, M. (2012). Growing mathematical objects in the classroom – the case of function. *International Journal of Educational Research, 51/52*, 10–27.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

National Council of Teachers of Mathematics (2014). *Principles to actions*. Reston, VA: NCTM.

Oldham, E., & Prendergast, M. (2020). Investigating prospective mathematics teachers' meanings for and representations of functions: A study of pre-service teachers and of students of mathematics in an Irish university. In L. Shagrir & D. Parmigiani (Eds.), *Teacher education in a changing context: ATEE 44th Annual Conference (2019) Conference Proceedings* (pp. 38–61). Brussels, Belgium: ATEE.

Santos, G., & Barbosa, J. (2016). Um modelo teórico de matemática para o ensino do conceito de função a partir de um estudo com professores. *UNIÓN, 48,* 143–167.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*, 151–169.

Tripathi, P. N. (2008). Developing mathematical understanding through multiple representations. *Mathematics Teaching in Middle School, 13*(8), 438–445.

Viirman, O., Attorps, I., & Tossavainen, T. (2010). Different views – Some Swedish mathematics students' concept images of the function concept. *Nordic Studies in Mathematics Education*, *15*(4), 5–24.

Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education, 20*(4), 266–356.

Viseu, F., Martins, P. M., & Rocha, H. (2019). The notion of function held by basic education pre-service teachers. In L. Leite, E. Oldham, L. Carvalho, A. S. Afonso, F. Viseu, L. Dourado, & M. H. Martinho (Eds.), *Proceedings of the ATEE Winter Conference "Science and mathematics education in the 21st century"* (pp. 120–130). Brussels: ATEE and CIEd.