The number of zeros of unilateral polynomials over coquaternions revisited

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Information

Abstract

The literature on quaternionic polynomials and, in particular, on methods for finding and classifying their zero sets, is fast developing and reveals a growing interest in this subject. In contrast, polynomials defined over the algebra of coquaternions have received very little attention from researchers. One of the few exceptions is the very recent paper by Janovská and Opfer [*Electronic Transactions on Numerical Analysis*, Volume 46, pp. 55-70, 2017], where, among other results, we can find a first attempt to prove that a unilateral coquaternionic polynomial of degree n has, at most, $n(2n-1)$ zeros. In this paper we present a full proof of this result, using a totally different and, from our point of view, much simpler approach. Also, we give a complete characterization of the zero sets of such polynomials and present a new result giving conditions which guarantee the existence of a special type of zeros. An algorithm to compute and classify all the zeros of a coquaternionic polynomial is proposed and several numerical examples are carefully constructed.

1   Introduction

In 1941, I. Niven, in his pioneering work [18], proved that any unilateral polynomial defined over the real algebra $\mathbb{H}$ of quaternions always has a zero in $\mathbb{H}$, describing, simultaneously, a process to compute the roots of any such polynomial. The procedure proposed by Niven has two distinct parts: the first is a process to determine which similarity classes of $\mathbb{H}$ contain roots of the polynomial and the second is a computational procedure for finding the roots belonging to each class. The first part of Niven's scheme is not very practical and the problem of the determination of the roots of a quaternionic polynomial remained dormant for quite a while. In fact, it was only in the year 2001 that Serôdio, Pereira and Vitória [22] proposed an efficient procedure to replace the first part of Niven's method, presenting what can be considered as the first really usable algorithm for finding the zeros of quaternionic polynomials. After this first paper, the interest in the development of root-finding methods for quaternionic polynomials has called the attention of many researchers; see e.g. [3, 7, 17, 20, 23]. Most of the methods available to compute the roots of a given polynomial P make use of the so-called *companion polynomial* of P to replace the first part of Niven's procedure, i.e. to

determine which are the classes containing the roots. They differ in the way how the roots are computed once the classes are found. This is precisely the case of the method introduced by Serôdio and Siu [23] and of the closely related method later proposed in [11].

In contrast to the case of quaternionic polynomials, the literature on polynomials defined over the algebra $\mathbb{H}_{\mathrm{coq}}$ of coquaternions is very scarce: see, nevertheless, [6, 12, 13, 19, 21] and the very recent publication by Janovká and Opfer [14]. In this last reference, the authors aim at extending to coquaternionic polynomials their method for polynomials over $\mathbb{H}$ given in [11]. Of course, due to the nature of the coquaternionic algebra, in particular the fact that this is not a division algebra, significant differences occur. In [14] the authors also state an important result relating the degree of a coquaternionic polynomial to the maximum number of zeros that the polynomial can have.[1]

The main purpose of this paper is to present a complete and simpler proof of the aforementioned result on the maximum number of zeros of a coquaternionic polynomial and simultaneously to describe the zero structure of such polynomials. A new result giving conditions which guarantee the existence of a special type of zeros – which we call *linear zeros* – is also presented and a positive answer to a question posed in [14] is given.

The rest of the paper is organized as follows: Section 2 contains a revision of the main definitions and results on the algebra of coquaternions. Section 3 is dedicated to unilateral coquaternionic polynomials and contains the main results of the paper; in particular, a revised version of the root-finding algorithm proposed in [14] is given. Finally, Section 4 contains carefully chosen examples illustrating some of the conclusions contained in Section 3.

2   Some basic results on coquaternions

Let $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ be an orthonormal basis of the Euclidean vector space $\mathbb{R}^4$ with a product given according to the multiplication rules

$$\mathbf{i}^2 = -1, \ \mathbf{j}^2 = \mathbf{k}^2 = 1, \ \mathbf{ij} = -\mathbf{ji} = \mathbf{k}.$$

This non-commutative product generates the algebra of real coquaternions, also known as split quaternions, which we will denote by $\mathbb{H}_{\mathrm{coq}}$. We will identify the space $\mathbb{R}^4$ with $\mathbb{H}_{\mathrm{coq}}$ by associating the element $q = (q_0, q_1, q_2, q_3) \in \mathbb{R}^4$ with the element $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \in \mathbb{H}_{\mathrm{coq}}$. Thus, throughout the paper, we will not distinguish an element in $\mathbb{R}^4$ (sometimes written as a column vector, if convenient) from the corresponding coquaternion, unless we need to stress the context.

The explicit multiplication of two coquaternions $p = p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$ and $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ is given by

$$pq = p_0q_0 - p_1q_1 + p_2q_2 + p_3q_3 + (p_0q_1 + p_1q_0 - p_2q_3 + p_3q_2)\mathbf{i} \\ + (p_0q_2 - p_1q_3 + p_2q_0 + p_3q_1)\mathbf{j} + (p_0q_3 + p_1q_2 - p_2q_1 + p_3q_0)\mathbf{k}. \quad (2.1)$$

The expression (2.1) shows that the product pq can be computed using matrices as $M_p q$, where

$$M_p = \begin{pmatrix} p_0 & -p_1 & p_2 & p_3 \\ p_1 & p_0 & p_3 & -p_2 \\ p_2 & p_3 & p_0 & -p_1 \\ p_3 & -p_2 & p_1 & p_0 \end{pmatrix} \quad (2.2)$$

and where, when performing the matrix multiplication, we identify q with the column vector $(q_0, q_1, q_2, q_3)^{\mathrm{T}}$.

Given a coquaternion $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \in \mathbb{H}_{\mathrm{coq}}$, its *conjugate* $\bar{q}$ is defined as $\bar{q} = q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}$; the number q_0 is called the *real part* of q and denoted by $\mathrm{Re}(q)$ and the *vector part* of q, denoted by $\mathrm{Vec}(q)$, is given by $\mathrm{Vec}(q) = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$.

We will identify the set of coquaternions whose vector part is zero with the set $\mathbb{R}$ of real numbers. We will also consider three particularly important subspaces of dimension two of $\mathbb{H}_{\mathrm{coq}}$, usually called the *canonical planes* or *cycle planes*. The first is $\{q \in \mathbb{H}_{\mathrm{coq}} : q = a + b\,\mathbf{i}, a, b \in \mathbb{R}\}$ which, of course, we identify with the

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[1]We should remark that the results contained in [14] are obtained for polynomials defined not only over the algebra of coquaternions but also over two other algebras: the algebra of nectarines $\mathbb{H}_{\mathrm{nec}}$ and the algebra of conectarines $\mathbb{H}_{\mathrm{con}}$. Since these two algebras are isomorphic to the algebra of coquaternions, we decided to consider only the coquaternionic case.

complex plane $\mathbb{C}$; the second, which we denote by $\mathbb{P}$ and whose elements are usually called *perplex numbers*, is given by $\mathbb{P} = \{q \in \mathbb{H}_{\text{coq}} : q = a + b\,\mathbf{j}, a, b \in \mathbb{R}\}$ and corresponds to the classical *Minkowski plane*; the third, denoted by $\mathbb{D}$, is the subspace of the so-called *dual numbers*, $\mathbb{D} = \{q \in \mathbb{H}_{\text{coq}} : q = a + b\,(\mathbf{i} + \mathbf{j}), a, b \in \mathbb{R}\}$ and can be identified with the classical *Laguerre plane*. We call *determinant* of q and denote by $\det(q)$ the quantity given by

$$\det(q) = q\,\overline{q} = q_0^2 + q_1^2 - q_2^2 - q_3^2. \tag{2.3}$$

Remark 2.1. *Other authors use different notations and names for the value given by (2.3); see e.g. [1, 5, 13, 14]. We propose the use of the term determinant, since it is known that every coquaternion $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ can be represented by the real matrix*

$$\begin{pmatrix} q_0 + q_3 & q_1 + q_2 \\ q_2 - q_1 & q_0 - q_3 \end{pmatrix}$$

whose determinant is precisely the value given by (2.3).

Contrary to what happens in the case of quaternions, not all non-zero coquaternions are invertible. It can be shown that a coquaternion q is invertible if and only if $\det(q) \neq 0$. In that case, we have $q^{-1} = \frac{\overline{q}}{\det(q)}$. A non-invertible element $q \in \mathbb{H}_{\text{coq}}$ is also called *singular*. It can also be shown that a coquaternion q is singular if and only if it is a *zero divisor*, i.e. there exist $r, s \in \mathbb{H}_{\text{coq}}, r, s \neq 0$ such that $rq = qs = 0$.

We now recall the concept of similarity in the set of coquaternions.

Definition 2.2. *We say that a coquaternion q is similar to a coquaternion p, and write $q \sim p$, if there exists an invertible coquaternion h such that $q = h^{-1}ph$.*

This is an equivalence relation, partitioning $\mathbb{H}_{\text{coq}}$ in the so-called *similarity classes*, defined, for each $q \in \mathbb{H}_{\text{coq}}$, by $[q] = \{p \in \mathbb{H}_{\text{coq}} : p \sim q\}$. It can easily be shown that $[q] = \{q\}$ if and only if $q \in \mathbb{R}$.

It is a well-known result that any Hamilton quaternion is similar to a complex number with non-negative real part – see e.g. [2, Lemma 3] – thus allowing the choice of that special form for the representative of any similarity class. The following result shows that the situation is different in the case of coquaternions, where three different types of representatives need to be used (for non-real coquaternions).

In what follows, given a coquaternion $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} = q_0 + \text{Vec}(q)$, we will use $\text{dv}(q)$ to denote the determinant of the vector part of q, i.e. $\text{dv}(q) := \det(\text{Vec}(q))$.

Theorem 2.3 ([8],[15]). *Let $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ be a non-real coquaternion. Then:*

(i) *If $\text{dv}(q) > 0$, q is similar to the complex number $q_0 + \sqrt{\text{dv}(q)}\,\mathbf{i}$, i.e. $[q] = [q_0 + \sqrt{\text{dv}(q)}\,\mathbf{i}]$;*

(ii) *if $\text{dv}(q) < 0$, q is similar to the perplex number $q_0 + \sqrt{-\text{dv}(q)}\,\mathbf{j}$, i.e. $[q] = [q_0 + \sqrt{-\text{dv}(q)}\,\mathbf{j}]$;*

(iii) *if $\text{dv}(q) = 0$, q is similar to the dual number $q_0 + \mathbf{i} + \mathbf{j}$, i.e. $[q] = [q_0 + \mathbf{i} + \mathbf{j}]$.*

The proof contained in [15] for the cases (i)-(ii) and in [8] for the case (iii) gives explicit expressions on how we may choose $h \in \mathbb{H}_{\text{coq}}$ such that $h^{-1}qh$ has the specified form. We simply recall here those expressions.

(i) Take $h = q_1 + \sqrt{\text{dv}(q)} - q_3\mathbf{j} + q_2\mathbf{k}$, if $q_2^2 + q_3^2 \neq 0$ and $h = \mathbf{j}$, if $q = q_0 + q_1\mathbf{i}$ with $q_1 < 0$.

(ii) If $q_1^2 + q_3^2 \neq 0$, take $h = q_1 - q_3\mathbf{j} + (q_2 - \sqrt{-\text{dv}(q)})\mathbf{k}$, if $q_2 \leq 0$ and $h = q_2 + \sqrt{-\text{dv}(q)} + q_3\mathbf{i} + q_1\mathbf{k}$, if $q_2 > 0$; if $q = q_0 + q_2\mathbf{j}$ with $q_2 < 0$, simply take $h = \mathbf{i}$.

(iii) Take $h = 1 + q_1 - q_3\mathbf{j} - (1 - q_2)\mathbf{k}$, if $q_1 + q_2 \neq 0$, and $h = (1 + q_1)\mathbf{i} + (1 - q_1)\mathbf{j}$, otherwise.

As an immediate consequence of the previous theorem, we have the result contained in the following corollary.

Corollary 2.4. *Two non-real coquaternions p and q are similar if and only if they satisfy the following conditions:*

$$\text{Re}(p) = \text{Re}(q) \quad and \quad \text{dv}(p) = \text{dv}(q). \tag{2.4}$$

Remark 2.5. *Since, for any coquaternion q, we have $\det(q) = (\operatorname{Re}(q))^2 + \operatorname{dv}(q)$, conditions (2.4) are equivalent to*

$$\operatorname{Re}(p) = \operatorname{Re}(q) \quad and \quad \det(p) = \det(q). \tag{2.5}$$

We should emphasize that, since any $q \in \mathbb{R}$ is never similar to any other coquaternion, the conditions (2.4) (or (2.5)) guarantee the similarity of p and q only if both these coquaternions are non-real. In connection to this, Janovská and Opfer [13] introduced the notion of *quasi-similarity* for any two coquaternions.

Definition 2.6. *We say that two coquaternions p and q are quasi-similar, and write $p \approx q$, if and only if they satisfy conditions (2.4).*

Quasi-similarity is an equivalence relation in $\mathbb{H}_{\mathrm{coq}}$; the equivalence class of q with respect to this relation is called the *quasi-similarity class* of q and will be denoted[2] by $[\![q]\!]$.

It is convenient to introduce the following definition.

Definition 2.7. *A coquaternion q is said to be of Type 1, Type 2 or Type 3, depending on whether $\operatorname{dv}(q) > 0$, $\operatorname{dv}(q) < 0$ or $\operatorname{dv}(q) = 0$, respectively.*[3]

We have

$$\begin{aligned}
[\![q]\!] &= \{p \in \mathbb{H}_{\mathrm{coq}} : \operatorname{Re}(p) = \operatorname{Re}(q) \ and \ \operatorname{dv}(p) = \operatorname{dv}(q)\} \\
&= \{p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k} : p_0 = q_0 \ and \ p_1^2 - p_2^2 - p_3^2 = \operatorname{dv}(q)\}.
\end{aligned} \tag{2.6}$$

Thus, the quasi-similarity class $[\![q]\!]$ can be identified with a hyperboloid in the hyperplane $\{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4 : x_0 = q_0\}$. This will be:

1. a hyperboloid of two sheets, if $\operatorname{dv}(q) > 0$, i.e. if q is of Type 1; in this case $[\![q]\!] = [q] = [q_0 + \sqrt{\operatorname{dv}(q)}\,\mathbf{i}]$;

2. a hyperboloid of one sheet, if $\operatorname{dv}(q) < 0$, i.e. if q is of Type 2; in this case $[\![q]\!] = [q] = [q_0 + \sqrt{-\operatorname{dv}(q)}\,\mathbf{j}]$;

3. a degenerate hyperboloid (i.e. a cone), if $\operatorname{dv}(q) = 0$, i.e. if q is of Type 3; in this case, $[\![q]\!] = [\![q_0]\!]$ and:

   (i) if $q \in \mathbb{R}$, $[q] = \{q_0\}$;

   (ii) if $q \notin \mathbb{R}$, $[q] = [q_0 + \mathbf{i} + \mathbf{j}] = [\![q_0]\!] \setminus \{q_0\}$.

Note that no quasi-similarity class reduces to a single point and also that any quasi-similarity class contains a non-real element.

3 Unilateral coquaternionic polynomials

In this section we study polynomials defined over the algebra of coquaternions with special interest on the number and nature of their zeros.

3.1 Definitions and basic results

Unlike the real or complex case, there are several possible ways to define coquaternionic polynomials, since the coefficients can be taken to be on the right, on the left or on both sides of the variable. In this paper, we will restrict our attention to polynomials whose coefficients are located on the left of the variable, i.e. we only consider the set of polynomials of the form

$$P(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0, \ c_i \in \mathbb{H}_{\mathrm{coq}}. \tag{3.1}$$

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[2] It is important to remark that we use different notations from the ones introduced in [13], where the symbol $\overset{q}{\sim}$ is used for the quasi-similarity relation and the quasi-similarity class of $u \in \mathbb{H}_{\mathrm{coq}}$ is denoted by $[u]_q$. Since we frequently use q for a coquaternion, we found convenient to adopt different notations.

[3] A coquaternion q is usually classified as *time-like*, *space-like* or *light-like* according to $\det(q) > 0$, $\det(q) < 0$ or $\det(q) = 0$, respectively. Hence, Type 1, Type 2 and Type 3 coquaternions can also be described as coquaternions whose vector part is time-like, space-like or light-like, respectively.

We define the addition and multiplication of such polynomials as in the commutative case where the variable commutes with the coefficients.

With these operations, this set becomes a ring, which we call the ring of (left) *unilateral*, *one-sided* or *simple* polynomials in $\mathbb{H}_{\mathrm{coq}}$ and denote[4] by $\mathbb{H}_{\mathrm{coq}}[x]$.

As usual, if $c_n \neq 0$, we say that the *degree* of the polynomial $P(x)$ is n and call c_n the leading coefficient of the polynomial. When $c_n = 1$, we say that $P(x)$ is *monic*.

Due to the non-commutativity of the product in $\mathbb{H}_{\mathrm{coq}}$, the product of polynomials is also non-commutative. However, as for the product of coquaternions, a polynomial with real coefficients commutes with any other polynomial.

For a given coquaternion q, let $\mathscr{E}_q : \mathbb{H}_{\mathrm{coq}}[x] \to \mathbb{H}_{\mathrm{coq}}$ be the *evaluation map* at q, defined, for the polynomial given by (3.1), by $\mathscr{E}_q(P(x)) = c_n q^n + c_{n-1} q^{n-1} + \cdots + c_1 q + c_0$. Due to the way we defined the product of polynomials, this map is not a ring homomorphism, i.e., in general, we do not have $\mathscr{E}_q(P(x)Q(x)) = \mathscr{E}_q(P(x))\mathscr{E}_q(Q(x))$.

Remark 3.1. *Since all the polynomials considered will be in the indeterminate x, we will usually omit the reference to this variable and write simply P when mentioning an element $P(x) \in \mathbb{H}_{\mathrm{coq}}[x]$; an expression of the form $P(q)$, with $q \neq x$, will be used to denote the evaluation of P at a specific value $q \in \mathbb{H}_{\mathrm{coq}}$, i.e. $P(q) = \mathscr{E}_q(P(x))$.*

We say that a polynomial $R \in \mathbb{H}_{\mathrm{coq}}[x]$ is a *right divisor* (*left divisor*) of the polynomial P if there exists a polynomial Q such that $P = QR$ ($P = RQ$). We say that a polynomial D is a divisor of the polynomial P if D is simultaneously a left and a right divisor of P.

A coquaternion q such that $P(q) = 0$ is called a *zero* or a *root* of P. We will use $Z(P)$ to denote the *zero set* of P, i.e. the set of all zeros of P.

Theorem 3.2 (Factor Theorem). *([16, Proposition (16.2)]) Let $P(x)$ be a given non-zero polynomial in $\mathbb{H}_{\mathrm{coq}}[x]$. An element $q \in \mathbb{H}_{\mathrm{coq}}$ is a zero of P if and only if $(x - q)$ is a right divisor of $P(x)$ in $\mathbb{H}_{\mathrm{coq}}[x]$, i.e. if and only if there exists a polynomial $Q(x) \in \mathbb{H}_{\mathrm{coq}}[x]$ such that $P(x) = Q(x)(x - q)$.*

This theorem has the following immediate corollary.

Corollary 3.3. *Let $P(x) = L(x)R(x)$ with $L(x), R(x) \in \mathbb{H}_{\mathrm{coq}}[x]$. Then, all the zeros of $R(x)$ are zeros of $P(x)$.*

3.2 Characteristic polynomial of a class

Given a coquaternion $q \in \mathbb{H}_{\mathrm{coq}}$, consider the following polynomial

$$(x - q)(x - \bar{q}) = x^2 - 2\mathrm{Re}(q)\, x + \det(q).$$

Since this polynomial depends only on $\mathrm{Re}(q)$ and $\det(q)$, we immediately conclude that this is an invariant of the quasi-similarity class of q. We will call it the characteristic polynomial of $[\![q]\!]$ and will denote it by $\Psi_{[\![q]\!]}$[5], i.e.

$$\Psi_{[\![q]\!]}(x) := (x - q)(x - \bar{q}) = x^2 - 2\mathrm{Re}(q)\, x + \det(q). \tag{3.2}$$

Note that the discriminant Δ of the characteristic polynomial (3.2) is given by

$$\Delta = 4(\mathrm{Re}(q))^2 - 4\det(q) = -4\mathrm{dv}(q).$$

This means that $\Psi_{[\![q]\!]}$ will be:

   (i) an irreducible polynomial (over the reals), if $\mathrm{dv}(q) > 0$, i.e. if q is of Type 1;

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[4]Right unilateral polynomials are defined in an analogous manner, by considering the coefficients on the right of the variable; all the results for left unilateral polynomials have corresponding results for right unilateral polynomials and hence we restrict our study to polynomials of the first type. The case of general polynomials is out of the scope of this paper. We refer the interested readers to [6], where the case of linear equations with terms of the form axb was considered.

[5]This polynomial is more commonly called the characteristic polynomial of the coquaternion q. We prefer to use our definition to emphasize the biunivocal relation between quasi-similarity classes and characteristic polynomials.

(ii) a polynomial of the form $(x - r_1)(x - r_2)$ with $r_1, r_2 \in \mathbb{R}, r_1 \neq r_2$, if $\mathrm{dv}(q) < 0$, i.e. if q is of Type 2;

(iii) a polynomial of the form $(x - r)^2$, with $r \in \mathbb{R}$, if $\mathrm{dv}(q) = 0$, i.e. if q is of Type 3.

On the other hand, any second degree monic polynomial with real coefficients is the characteristic polynomial of a (uniquely defined) quasi-similarity class. In fact, let $p_2(x) = x^2 + bx + c$ with $b, c \in \mathbb{R}$ and let $\Delta = b^2 - 4c$. Considering p_2 as a polynomial in $\mathbb{C}[x]$, we have:

(i) if $\Delta < 0$, then p_2 has two (distinct) complex conjugate roots, w and $\overline{w}$; hence, $p_2(x) = (x - w)(x - \overline{w})$ i.e. $p_2 = \Psi_{[\![w]\!]}$;

(ii) if $\Delta = 0$, then $p_2(x)$ has a double real root r, i.e. $p_2(x) = (x - r)^2$ and so $p_2 = \Psi_{[\![r]\!]}$;

(iii) if $\Delta > 0$, then p_2 has two distinct real roots r_1, r_2, i.e. $p_2(x) = (x - r_1)(x - r_2) = x^2 - (r_1 + r_2)x + r_1 r_2$ and it is easy to see that $p_2 = \Psi_{[\![p]\!]}$, with p the perplex number $p = \frac{r_1 + r_2}{2} + \frac{r_1 - r_2}{2}\mathbf{j}$.

The next theorem states an important property of the zero set of characteristic polynomials; see e.g. [8, Theorem 4] for this and other properties of $Z(\Psi_{[\![q]\!]})$.

Theorem 3.4. *Let $\Psi_{[\![q]\!]}$ be the characteristic polynomial of a given quasi-similarity class $[\![q]\!]$. Then $[\![q]\!] \subseteq Z(\Psi_{[\![q]\!]})$.*

3.3   Zeros of polynomials and the companion polynomial

Given a polynomial $P \in \mathbb{H}_{\mathrm{coq}}[x]$, the polynomial obtained from P by replacing each coefficient by its conjugate is called the *conjugate of P* and denoted by $\overline{P}$. The properties given in the following proposition are easily verified.

Proposition 3.5. *Let $P, Q \in \mathbb{H}_{\mathrm{coq}}[x]$. Then:*

(i) $\overline{PQ} = \overline{Q}\,\overline{P}$;

(ii) $P\overline{P} = \overline{P}P$ *is a polynomial with real coefficients.*

Definition 3.6. *The polynomial*

$$\mathcal{C}_P = \overline{P}P = P\overline{P} \tag{3.3}$$

is called the companion polynomial of P.

Remark 3.7. *This polynomial already appears in the famous work [18] of Niven, published in 1941 – unfortunately with no name – and has been used by many authors under different designations. The name* companion polynomial *adopted here was introduced in [11].*

In what follows, we restrict our attention to the study of monic polynomials, i.e. we will consider only polynomials of the form

$$P(x) = x^n + c_{n-1}x^{n-1} + \ldots + c_1 x + c_0, \quad c_i \in \mathbb{H}_{\mathrm{coq}}. \tag{3.4}$$

Note that, in what concerns the zeros, the study of polynomials of this type is equivalent to the study of polynomials of the form (3.1) with a non-singular leading coefficient[6] c_n.

Niven [18] proved that every non-constant unilateral polynomial with quaternionic coefficients always has a quaternionic zero, thus establishing that the "Fundamental Theorem of Algebra" holds for unilateral quaternionic polynomials.[7] The situation in what concerns polynomials defined over the algebra of coquaternions is, however, different. In fact, as observed by Özdemir [19, Theorem 9-i.], any equation of the form $x^n - q = 0$, with n even and q a coquaternion with negative determinant, does not have a solution; other examples of coquaternionic polynomials with no roots can be found in [13].

The following theorem plays an important role in the root-finding procedure that we are going to propose.

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[6]The fact that P is monic (or with non-singular leading coefficient) guarantees that the companion polynomial has degree $2n$ and avoids pathological situations as having a non-zero polynomial whose companion polynomial is the zero polynomial; see [13] for such an example.

[7]This result was later extended to more general quaternionic polynomials in [4].

Theorem 3.8. *Let $P \in \mathbb{H}_{\mathrm{coq}}[x]$. If $z \in \mathbb{H}_{\mathrm{coq}}$ is a zero of P, then $\Psi_{[\![z]\!]}$ is a divisor of the companion polynomial of P.*

Proof. Let z be a zero of P; by Theorem 3.2, we know that $(x - z)$ is a right divisor of P, i.e. $P(x) = Q(x)(x - z)$ for a given polynomial $Q \in \mathbb{H}_{\mathrm{coq}}[x]$. Thus

$$\begin{aligned}
\mathcal{C}_P(x) = P(x)\overline{P}(x) &= Q(x)(x - z)\overline{(Q(x)(x - z))} \\
&= Q(x)(x - z)(x - \overline{z})\overline{Q}(x) = Q(x)\Psi_{[\![z]\!]}(x)\overline{Q}(x) \\
&= Q(x)\overline{Q}(x)\Psi_{[\![z]\!]}(x) = \mathcal{C}_Q(x)\Psi_{[\![z]\!]}(x) = \Psi_{[\![z]\!]}(x)\mathcal{C}_Q(x),
\end{aligned}$$

where we used the fact that a polynomial with real coefficients commutes with any other polynomial. Hence, $\Psi_{[\![z]\!]}$ is a divisor of $\mathcal{C}_P$, as we wished to prove. $\square$

The result of the previous theorem can also be stated as follows: if a certain quasi-similarity class $[\![z]\!]$ of $\mathbb{H}_{\mathrm{coq}}$ contains a zero of P, then its characteristic polynomial $\Psi_{[\![z]\!]}$ divides the companion polynomial $\mathcal{C}_P$. This means, in particular, that there is no point in searching for zeros of P in classes whose characteristic polynomial is not a factor of $\mathcal{C}_P$. We are thus led to introduce the following definition.

Definition 3.9. *Given a polynomial P, a quasi-similarity class $[\![z]\!]$ of $\mathbb{H}_{\mathrm{coq}}$ is called admissible (with respect to the zeros of P) if and only if the corresponding characteristic polynomial $\Psi_{[\![z]\!]}$ is a divisor of the companion polynomial of P.*

If P is a monic polynomial of degree n, its companion polynomial $\mathcal{C}_P$ is a polynomial with real coefficients of degree $2n$ and, as such, has $2n$ roots in $\mathbb{C}$. Let these roots be $w_1, \overline{w}_1, \ldots, w_m, \overline{w}_m \in \mathbb{C} \setminus \mathbb{R}$ and $r_1, \ldots, r_s \in \mathbb{R}$, where $s = 2n - 2m$, $0 \le m \le n$. Then,

$$\mathcal{C}_P(x) = \underbrace{(x - w_1)(x - \overline{w}_1)}_{\Psi_{[\![w_1]\!]}} \ldots \underbrace{(x - w_m)(x - \overline{w}_m)}_{\Psi_{[\![w_m]\!]}}(x - r_1) \ldots (x - r_s)$$

and it is clear that the characteristic polynomials (i.e. the real monic polynomials of degree two) which divide $\mathcal{C}_P$ are the m irreducible polynomials

$$\Psi_{[\![w_i]\!]}, i = 1, \ldots, m, \tag{3.5}$$

and the $\binom{s}{2}$ polynomials $(x - r_j)(x - r_k)$; $j = 1, \ldots, s - 1$, $k = j + 1, \ldots, s$, i.e. the polynomials

$$\Psi_{[\![p_{jk}]\!]} \quad \text{with} \quad p_{jk} = \frac{r_j + r_k}{2} + \frac{r_j - r_k}{2}\mathbf{j}; j = 1, \ldots, s - 1, \ k = j + 1, \ldots, s. \tag{3.6}$$

Note that, if r_j is a multiple root, a polynomial of the form $(x - r_j)^2 = \Psi_{[\![r_j]\!]}$ appears. The maximum number of such polynomials occurs when $m = 0$ and the $2n$ real roots of $\mathcal{C}_P$ are all distinct, and is equal to $\binom{2n}{2} = n(2n - 1)$. Having in mind the correspondence between characteristic polynomials and quasi-similarity classes, we can then state the following result.

Theorem 3.10. *If P is a polynomial of degree n in $\mathbb{H}_{\mathrm{coq}}[x]$, then the zeros of P belong to, at most, $n(2n - 1)$ quasi-similarity classes.*

Remark 3.11.

1. *This can be seen as the analogue, in the coquaternionic setting, of a result first established by Gordon and Motzkin [10] for quaternionic polynomials.*

2. *The result of the previous theorem was first stated, by using totally different arguments, in [14, Corollary 5.3]. We remark that the proof contained in [14] is not only much more elaborate than the proof presented here but also incomplete. In fact, Corollary 5.3. follows from the use of two theorems – Theorem 5.1 and Theorem 5.2. – which deal, respectively, with the case of zeros belonging to a similarity class of the form $[u + v\mathbf{i}], v > 0$, or to a similarity class $[u + v\mathbf{j}], v > 0$. So, the case of zeros in a class of the type $[u + \mathbf{i} + \mathbf{j}]$ (i.e. zeros whose vector part has zero determinant, but which are not real), is not considered.*

3.4 Computing the zeros from the admissible classes

We now explain how to compute the zeros belonging to each of the admissible classes. The process is an adaptation to the coquaternionic setting of the already mentioned process for quaternionic polynomials proposed by Niven in [18]. Of course, some differences occur due to the fact that $\mathbb{H}_{\text{coq}}$, contrary to the algebra of quaternions, is not a division algebra. The same idea (although using a different computational procedure) was used in [13] and [14].

For any monic polynomial of the first degree, $P(x) = x - q$, the only root of $P(x)$ is q, and so we will now consider only polynomials of degree $n \geq 2$.

In what follows, $[\![q]\!] = [\![q_0 + \text{Vec}(q)]\!]$ is an admissible class of P, i.e., $\Psi_{[\![q]\!]}(x)$ divides $\mathcal{C}_P$. This implies, in particular, that all the elements in $[\![q]\!]$ are zeros of $\mathcal{C}_P$; see Theorems 3.2 and 3.4.

We first note that, since the product of two polynomials in $\mathbb{H}_{\text{coq}}[x]$ is defined in the usual manner, we can always use the "Euclidean Division Algorithm" to perform the division of two polynomials, provided that the leading coefficient of the divisor is non-singular. In particular, we can use it to divide $P(x)$ by the characteristic polynomial of the quasi-similarity class $[\![q]\!]$, i.e. by the quadratic monic polynomial $x^2 - 2\text{Re}(q)x + \det(q)$. If we perform this division we will obtain

$$P(x) = Q(x)\Psi_{[\![q]\!]}(x) + A + Bx \tag{3.7}$$

for some polynomial $Q(x)$ and values A and B which depend only on the coefficients c_i of the given polynomial P and on the values $\text{Re}(q)$ and $\det(q)$. As in the case of complex or quaternionic polynomials, expanded synthetic division can be used here to obtain the expressions of A and B (see [24] and [9] for computational details on the use of this shortcut method for dividing any polynomial by the special quadratic polynomial $\Psi_{[\![q]\!]}$, in the complex and quaternionic context, respectively). Specifically, we have

$$A = c_0 - \det(q)\,\alpha_0 \qquad \text{and} \qquad B = c_1 + 2\text{Re}(q)\,\alpha_0 - \det(q)\,\alpha_1, \tag{3.8a}$$

where α_j satisfy the following recurrence relations:

$$\begin{aligned}
&\alpha_{n-1} = 0, \ \alpha_{n-2} = 1, \\
&\alpha_k = c_{k+2} + 2\text{Re}(q)\,\alpha_{k+1} - \det(q)\,\alpha_{k+2}; \ k = n-3, n-2, \ldots, 0.
\end{aligned} \tag{3.8b}$$

For all $z \in [\![q]\!]$, we have $\Psi_{[\![q]\!]}(z) = 0$ (see Theorem 3.4), and so we obtain[8]

$$P(z) = A + Bz. \tag{3.9}$$

Hence, we conclude that a coquaternion $z \in [\![q]\!]$ is a zero of the polynomial P if and only it satisfies

$$A + Bz = 0. \tag{3.10}$$

We now discuss several cases, depending on the values A and B.

Case 1 *B non-singular*

In this case, there is only one zero z_0 of P in the class $[\![q]\!]$, given by the formula

$$z_0 = -B^{-1}A = -\frac{\overline{B}A}{\det(B)}, \tag{3.11}$$

as we will now show. Clearly, z_0 given by (3.11) is the unique solution of (3.10) and so it remains to prove that it belongs to $[\![q]\!]$. From (3.7), we have that

$$\begin{aligned}
\mathcal{C}_P(x) = \mathcal{C}_Q(x)\Psi_{[\![q]\!]}(x)\Psi_{[\![q]\!]}(x) &+ (A + Bx)\overline{Q}(x)\Psi_{[\![q]\!]}(x) \\
&+ Q(x)(\overline{A} + \overline{B}x)\Psi_{[\![q]\!]}(x) + \det(A) + 2\text{Re}(A\overline{B})x + \det(B)x^2.
\end{aligned}$$

---

[8]This result is obtained in a different way in [21] and also in [13], where a different computational procedure for obtaining A and B is proposed; counting the number of arithmetic operations involved, one can conclude (cf. [9] for complexity and stability analysis) that, for $n > 3$, the process given by (3.8) involves less computational effort than the method proposed in [13] and [21].

Let z be an element in $[\![q]\!]$ with a non-zero vector part (as mentioned before, such an element always exists). Recalling that $\mathcal{C}_P(z) = 0$ and $\Psi_{[\![q]\!]}(z) = 0$, we have

$$\det(B)z^2 + 2\mathrm{Re}(A\overline{B})z + \det(A) = 0$$

and

$$z^2 = 2\mathrm{Re}(z)z - \det(z).$$

Hence, we obtain

$$\det(B)(2\mathrm{Re}(z)z - \det(z)) + 2\mathrm{Re}(A\overline{B})z + \det(A) = 0$$

or

$$\det(B)\big(2\mathrm{Re}(z)(\mathrm{Re}(z) + \mathrm{Vec}(z)) - \det(z)\big) + 2\mathrm{Re}(A\overline{B})\big(\mathrm{Re}(z) + \mathrm{Vec}(z)\big) + \det(A) = 0$$

which implies

$$\begin{cases} 2\mathrm{Re}(z)\big(\det(B)\mathrm{Re}(z) + \mathrm{Re}(A\overline{B})\big) - \det(B)\det(z) + \det(A) = 0, \\[2mm] 2\big(\det(B)\mathrm{Re}(z) + \mathrm{Re}(A\overline{B})\big)\mathrm{Vec}(z) = 0. \end{cases}$$

Since $\mathrm{Vec}(z) \neq 0$, we immediately conclude that

$$\det(B)\mathrm{Re}(z) + \mathrm{Re}(A\overline{B}) = 0 \quad \text{and} \quad \det(A) - \det(B)\det(z) = 0$$

or

$$\mathrm{Re}(z) = -\frac{\mathrm{Re}(A\overline{B})}{\det(B)} \quad \text{and} \quad \det(z) = \frac{\det(A)}{\det(B)}.$$

Taking into account well-known properties of $\mathrm{Re}(q)$ and $\det(q)$, we obtain from the expression (3.11) of z_0

$$\mathrm{Re}(z_0) = -\frac{\mathrm{Re}(A\overline{B})}{\det(B)} = \mathrm{Re}(z)$$

and

$$\det(z_0) = \left(\frac{1}{\det(B)}\right)^2 \det(\overline{B})\det(A) = \frac{\det(A)}{\det(B)} = \det(z)$$

which shows that $z_0 \in [\![z]\!] = [\![q]\!]$.

Case 2 $B = 0$

Case 2.1 $A \neq 0$

In this case, equation (3.10) cannot be satisfied, so there are no zeros of P in $[\![q]\!]$.

Case 2.2 $A = 0$

From (3.10) we have that $P(z) = 0$ for all $z \in [\![q]\!]$, i.e. the whole hyperboloid $[\![q]\!]$ is made up of zeros of P.

Case 3 B singular, $B \neq 0$

The roots of P belonging to $[\![q]\!]$ can be obtained by studying the solvability of the system

$$M_B z = -A \tag{3.12}$$

where $z = (z_0, z_1, z_2, z_3)^\mathrm{T}$ and M_B is the matrix defined by (2.2) (and A is seen as a column vector) and selecting the solutions for which $\mathrm{Re}(z) = q_0$ and $\mathrm{dv}(z) = \mathrm{dv}(q)$.

Case 3.1 $M_B z = -A$ has no solution

In this case, we conclude that there are no zeros of P in $[\![q]\!]$.

Case 3.2 $M_B z = -A$ is solvable

Let $\delta = (\delta_0, \delta_1, \delta_2, \delta_3)^\mathrm{T}$ be a solution of the system (3.12), i.e. let $M_B\delta = -A$. As is well known, the general solution of the system is the sum of a particular solution with the general solution of the associated homogeneous system, i.e. it is given by $z = u + \delta$, with $u = (u_0, u_1, u_2, u_3)^\mathrm{T}$ such that $M_B u = 0$. It can

easily be shown that, when $B = b_0 + b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ is singular (but $B \neq 0$), the matrix M_B has rank two and the general solution u of the homogeneous system $M_B u = 0$ is given by

$$\begin{cases} u_0 = \alpha, \\ u_1 = \beta, \\ u_2 = k_1\alpha + k_2\beta, \\ u_3 = k_2\alpha - k_1\beta; \ \alpha, \beta \in \mathbb{R}, \end{cases} \tag{3.13a}$$

with

$$k_1 = -\frac{b_0 b_2 + b_1 b_3}{b_0^2 + b_1^2} \quad \text{and} \quad k_2 = \frac{b_1 b_2 - b_0 b_3}{b_0^2 + b_1^2}. \tag{3.13b}$$

So, the solutions of (3.10) are $z = z_0 + z_1\mathbf{i} + z_2\mathbf{j} + z_3\mathbf{k}$ with

$$\begin{cases} z_0 = \alpha + \delta_0, \\ z_1 = \beta + \delta_1, \\ z_2 = k_1\alpha + k_2\beta + \delta_2, \\ z_3 = k_2\alpha - k_1\beta + \delta_3; \ \alpha, \beta \in \mathbb{R}. \end{cases} \tag{3.14}$$

There is no loss in generality in considering a particular solution γ of the system $M_B z = -A$ with the form $\gamma = (\gamma_0, \gamma_1, 0, 0)^{\mathrm{T}}$. This follows immediately by taking $\alpha = -k_1\delta_2 - k_2\delta_3$ and $\beta = -k_2\delta_2 + k_1\delta_3$ in (3.14) and observing that the values k_1 and k_2 given by (3.13) satisfy $k_1^2 + k_2^2 = 1$. So, we can state that the solutions of (3.10) are given by

$$\begin{cases} z_0 = \alpha + \gamma_0, \\ z_1 = \beta + \gamma_1, \\ z_2 = k_1\alpha + k_2\beta, \\ z_3 = k_2\alpha - k_1\beta; \ \alpha, \beta \in \mathbb{R}. \end{cases} \tag{3.15}$$

We now need to select solutions such that $\mathrm{Re}(z) = q_0$ and $\mathrm{dv}(z) = \mathrm{dv}(q)$. The first condition can always be satisfied provided we take $\alpha = q_0 - \gamma_0$. On the other hand, the expression for the determinant of the vector part of z is given by

$$\mathrm{dv}(z) = (\beta + \gamma_1)^2 - (k_1\alpha + k_2\beta)^2 - (k_2\alpha - k_1\beta)^2 = -\alpha^2 + \gamma_1^2 + 2\gamma_1\beta$$

which, with the choice $\alpha = (q_0 - \gamma_0)$, becomes

$$\mathrm{dv}(z) = -(q_0 - \gamma_0)^2 + \gamma_1^2 + 2\gamma_1\beta. \tag{3.16}$$

Case 3.2 (a) *There exists $\gamma \in \mathbb{R}$ such that $A + B\gamma = 0$*

In the special case where $\gamma_1 = 0$, i.e. when $\gamma = \gamma_0 \in \mathbb{R}$, the expression for $\mathrm{dv}(z)$ simplifies to $-(q_0 - \gamma_0)^2$ and so the condition $\mathrm{dv}(z) = \mathrm{dv}(q)$ simply reads as

$$-(q_0 - \gamma_0)^2 = \mathrm{dv}(q). \tag{3.17}$$

Thus, there will be zeros in the class $[\![q]\!]$ if and only if this condition is satisfied; in such a case, the zeros will form the set

$$\mathcal{L} = \left\{ q_0 + \beta\mathbf{i} + \left(k_2\beta + k_1(q_0 - \gamma_0) \right)\mathbf{j} + \left(-k_1\beta + k_2(q_0 - \gamma_0) \right)\mathbf{k} : \beta \in \mathbb{R} \right\}, \tag{3.18}$$

with k_1, k_2 given by (3.13). This set of points (considered as points in $\mathbb{R}^4$) can be seen as a line in the hyperplane $x_0 = q_0$: the line through the point $\left(0, k_1(q_0 - \gamma_0), k_2(q_0 - \gamma_0) \right)$ with the direction of the vector $(1, k_2, -k_1)$. Note also that this is an infinite set, but a strict subset of $[\![q]\!]$.

Case 3.2 (b) *There is no $\gamma \in \mathbb{R}$ such that $A + B\gamma = 0$*

When $\gamma_1 \neq 0$, the expression (3.16) for the determinant of the vector part of z shows that there is a unique value of β for which $\mathrm{dv}(z) = \mathrm{dv}(q)$:

$$\beta = \frac{\mathrm{dv}(q) + (q_0 - \gamma_0)^2 - \gamma_1^2}{2\gamma_1}. \tag{3.19}$$

Hence, in this case, we have the following unique zero in the class $[\![q]\!]$:

$$z_0 = q_0 + (\beta + \gamma_1)\mathbf{i} + \big(k_2\beta + k_1(q_0 - \gamma_0)\big)\mathbf{j} + \big(-k_1\beta + k_2(q_0 - \gamma_0)\big)\mathbf{k},$$

with β given by (3.19).

The previous discussion – see also [8] and [14] – shows that coquaternionic polynomials may have three different types of zeros, motivating us to introduce the following definition.

Definition 3.12. *Let z be a zero of a given coquaternionic polynomial P.*

    1. *z is said to be an isolated zero of P if $[\![z]\!]$ contains no other zeros of P;*

    2. *z is said to be a hyperboloidal zero of P if $[\![z]\!] \subseteq Z(P)$;*

    3. *z is said to be a linear zero of P if z is neither an isolated zero nor a hyperboloidal zero of P.*

Remark 3.13. *As far as we are aware, the first authors to note the appearance of zeros which are neither isolated nor hyperboloidal were Janovská and Opfer, in [14], giving an example of a special family of quadratic polynomials with such zeros, which they called* unexpected zeros*. However, we have to point out that there are linear zeros which are not unexpected zeros in the sense of [14], as the examples in Section 4 show.*

In what follows, we will treat all the zeros belonging to the same quasi-similarity class as forming a single zero, i.e., we will call to a whole hyperboloid of zeros or a line of zeros simply a hyperboloidal zero or a linear zero, respectively.

We may now summarize the results of our previous discussion in the following theorem.

Theorem 3.14. *Let $[\![q]\!]$ be an admissible class of a given polynomial $P \in \mathbb{H}_{\text{coq}}[x]$ and let $A + Bx$ be the remainder of the right division of $P(x)$ by the characteristic polynomial of $[\![q]\!]$. Also, denote by $Z_{[\![q]\!]}$ the set of the zeros of P belonging to $[\![q]\!]$. The set $Z_{[\![q]\!]}$ can be completely characterized in terms of A and B, as follows:*

1. *If B is non-singular, then P has an isolated zero in $[\![q]\!]$ and*

$$Z_{[\![q]\!]} = \Big\{ - \frac{\overline{B}A}{\det(B)} \Big\}.$$

2. *If $B = 0$ and*

  (a) *$A \neq 0$, then $Z_{[\![q]\!]} = \emptyset$;*

  (b) *$A = 0$, then $Z_{[\![q]\!]} = [\![q]\!]$, i.e. $[\![q]\!]$ is a hyperboloidal zero of P.*

3. *If $B \neq 0$ is singular and the equation $A + Bx = 0$ has*

  (a) *no solution, then $Z_{[\![q]\!]} = \emptyset$;*

  (b) *a real solution γ_0, then:*

     (i) *if $(q_0 - \gamma_0)^2 = -\text{dv}(q)$, then*

$$Z_{[\![q]\!]} = \big\{q_0 + \beta\mathbf{i} + (k_2\beta + k_1(q_0 - \gamma_0))\mathbf{j} + (-k_1\beta + k_2(q_0 - \gamma_0))\,\mathbf{k} : \beta \in \mathbb{R}\big\},$$

    with $k_1 = -\frac{b_0 b_2 + b_1 b_3}{b_0^2 + b_1^2}$ and $k_2 = \frac{b_1 b_2 - b_0 b_3}{b_0^2 + b_1^2}$, i.e. $Z_{[\![q]\!]}$ is a linear zero of P;

     (ii) *if $(q_0 - \gamma_0)^2 \neq -\text{dv}(q)$, then $Z_{[\![q]\!]} = \emptyset$;*

  (c) *a nonreal solution $\gamma = \gamma_0 + \gamma_1\mathbf{i}$, then P has an isolated zero in $[\![q]\!]$ and*

$$Z_{[\![q]\!]} = \big\{q_0 + (\beta + \gamma_1)\mathbf{i} + (k_2\beta + k_1(q_0 - \gamma_0))\mathbf{j} + (-k_1\beta + k_2(q_0 - \gamma_0))\,\mathbf{k}\big\},$$

    with $\beta = \frac{\text{dv}(q) + (q_0 - \gamma_0)^2 - \gamma_1^2}{2\gamma_1}$.

Remark 3.15. *We remark that the discussion contained in [14] for the case B singular, $B \neq 0$ is incomplete. In fact, there are two problems associated with the results in that paper. The first has to do with the fact that the authors do not consider case 3(c) in the previous theorem, assuming implicitly that, when B is singular and $B \neq 0$, roots may only appear if there exists $\gamma \in \mathbb{R}$ such that $A + B\gamma = 0$ (see the assumption in [14, Theorem 3.2] which is invoked in both Theorems 4.2 and 4.3 of [14] as a process for computing the roots). Second, even when the hypotheses of Theorem 3.2 in [14] are verified (i.e. we are in case 3(b) of Theorem 3.14), the formula*

$$z_0 = \alpha \overline{B} + \gamma, \quad \alpha \in \mathbb{R},$$

which is proposed in [14] for obtaining the solutions of equation $A + Bz = 0$ may not give all the solutions of this equation; as we noted, the general solution of $A + Bz$ is given by $z = u + \gamma$, with u any coquaternion satisfying $Bu = 0$, u not necessarily of the special form $\alpha\overline{B}$.

For the above reasons, the use of the algorithm proposed by the authors of [14] to compute the roots of a given polynomial – Algorithm 6.1 – may fail to produce all the roots. Examples 4.1–4.2 given later illustrate our remarks. A revised version of the Algorithm 6.1 of [14] to compute all the roots of a given polynomial will be presented in the end of this section.

We now give a theorem with a special case where we know that a polynomial has linear zeros.

Theorem 3.16. *Let $P(x)$ be a polynomial of degree n whose companion polynomial has m real simple zeros $r_1, \ldots, r_m$, $m \leq 2n$, and let $P_r(x) = P(x)(x - r)$ with $r \in \mathbb{R}$, $r \neq r_i; i = 1, \ldots, m$. Then, $P_r(x)$ has (at least) m linear zeros.*

Proof. The roots of the companion polynomial of P_r are the previous roots of the companion polynomial of P together with the double root r. So $r, r, r_1, \ldots, r_m$ are roots of $\mathcal{C}_{P_r}$. This means that the classes $[\![p_k]\!]$, with $p_k = \frac{r+r_k}{2} + \frac{r-r_k}{2}\mathbf{j}; k = 1, \ldots, m$, are admissible classes for P_r. We now show that each of these classes contains a linear zero. The characteristic polynomial of the class $[\![p_k]\!]$ is $\Psi_{[\![p_k]\!]}(x) = (x - r)(x - r_k)$. Dividing $P_r(x)$ by $\Psi_{[\![p_k]\!]}$ will give us

$$P_r(x) = Q(x)(x - r_k)(x - r) + Bx + A.$$

But, since $P_r(r) = 0$, we obtain $Br + A = 0$. We cannot have B nonsingular, since this would imply $z_0 = r \in [\![p_k]\!]$, which is false, since $\mathrm{Re}(p_k) \neq r$. Also, we cannot have $A = B = 0$: this would mean that $P_r(x) = Q(x)(x - r)(x - r_k)$ would have the real root r_k, implying that $\mathcal{C}_{P_r}$ would have r_k as a double root, which is contrary to the hypotheses of the theorem. Hence, we are in the case 3(b) of Theorem 3.14, with $\gamma_0 = r$; since

$$(\mathrm{Re}(p_k) - r)^2 = \left(\tfrac{r+r_k}{2} - r\right)^2 = \left(\tfrac{r-r_k}{2}\right)^2 = -\mathrm{dv}(p_k),$$

we may conclude that $[\![p_k]\!]$ contains a line of zeros. $\qquad\square$

The following observations clarify some characteristics of the zero sets of coquaternionic polynomials.

O1. We first note that classes of the type $[\![q_0 + \sqrt{\mathrm{dv}(q)}\,\mathbf{i}]\!]$, $\mathrm{dv}(q) > 0$, will never contain a linear zero. In fact, as shown in Theorem 3.14, linear zeros are only obtained in case 3(b)-(i), requiring $(q_0 - \gamma_0)^2 = -\mathrm{dv}(q)$ to be satisfied; this condition is, of course, impossible to verify if $\mathrm{dv}(q) > 0$. (The same conclusion could be reached invoking geometric arguments, by observing that a hyperboloid of two-sheets does not contain any straight line.) So, we conclude that linear zeros will always be either Type 2 or Type 3 coquaternions.

O2. We have seen that a polynomial P of degree n can attain the maximum number $n(2n-1)$ of zeros only if the roots of its companion polynomial are all real and simple. However, we must point out that this is not a sufficient condition for the existence of $n(2n-1)$ zeros, as illustrated in Example 4.4.

O3. A coquaternion z is a hyperboloidal zero of a given polynomial P if and only if $\Psi_{[\![z]\!]}$ divides P. This implies that $(\Psi_{[\![z]\!]})^2$ has to divide the companion polynomial, meaning that the roots in $\mathbb{C}$ of $\Psi_{[\![z]\!]}$ will appear as roots of $\mathcal{C}_P$ with a multiplicity greater or equal to two if z is of Type 1 or Type 2 or greater or equal to 4 if z is of Type 3. However, not all multiple roots of $\mathcal{C}_P$ correspond to hyperboloidal zeros of P; see Example 4.5.

We end this section by giving an algorithm to compute the roots of a coquaternionic polynomial which is based on the discussion given previously.

Algorithm 3.1 Compute the roots of a coquaternionic polynomial P

INPUT: Coefficients of P

OUTPUT: Lists ℓ_I, ℓ_H and ℓ_L with the isolated, hyperboloidal and linear roots of P

1: Initialize lists ℓ_I, ℓ_H and ℓ_L as empty lists
2: Compute $\mathcal{C}_P$ - formula (3.3)   ▷ $\mathcal{C}_P$ companion pol. of P
3: Determine roots in $\mathbb{C}$ of $\mathcal{C}_P$ ▷ Use a numerical method, if necessary
4: Identify admissible classes of P - formulas (3.5) and (3.6)
5: **for** each admissible class **do**
6:     Compute A and B - formulas (3.8)
7:     **if** $B = 0$ **then**
8:         **if** $A = 0$ **then**
9:             Add representative of class to list ℓ_H
10:    **else**
11:        Compute $\det(B)$ - formula (2.3)
12:        **if** $\det(B) \neq 0$ **then**
13:            Compute z_0 by formula (3.11) and add it to list ℓ_I
14:        **else**
15:            **if** system $Bx = -A$ has a solution γ **then**
16:                **if** $\gamma \in \mathbb{R}$ **then**
17:                    **if** condition (3.17) holds **then**
18:                        Determine k_1, k_2 by formulas (3.13)
19:                        Add $\{q_0, \gamma, k_1, k_2\}$ to ℓ_L ▷ q_0 real part of a representative of the class
20:                **else**
21:                    Determine solution z_0 by formula (3.20)
22:                    Add z_0 to list ℓ_I

---

4 Examples

In this section, we present several examples of application of Algorithm 3.1 which illustrate some of the results and remarks given in Section 3.

Example 4.1. *Let $P_1(x) = x^2 + (1 + \mathbf{i} + \mathbf{j} + \mathbf{k})x + 2 - \mathbf{i} - \mathbf{j} + \mathbf{k}$. The roots of the companion polynomial of P_1 are $\mathbf{i}, -\mathbf{i}, -1 - \sqrt{2}\mathbf{i}, -1 + \sqrt{2}\mathbf{i}$ and so we have the following two admissible classes: $[\![\mathbf{i}]\!], [\![-1 + \sqrt{2}\mathbf{i}]\!]$.*
  Consider the determination of the roots lying in class $[\![\mathbf{i}]\!]$. Dividing the polynomial P_1 by $\Psi_{[\![\mathbf{i}]\!]}$ leads to

$$A = 1 - \mathbf{i} - \mathbf{j} + \mathbf{k} \quad and \quad B = 1 + \mathbf{i} + \mathbf{j} + \mathbf{k}.$$

So, we have that B is singular, but there is no $\gamma \in \mathbb{R}$ such that $A + B\gamma = 0$. However, it can be easily verified that the system $M_B z = -A$ is consistent and so we are in case 3 (c) of Theorem 3.14, obtaining that $z_0 = \mathbf{i}$ is the only root of P_1 in $[\![\mathbf{i}]\!]$. This shows that the root $z = \mathbf{i}$ will not be computed if we use Algorithm 6.1 in [14].

Example 4.2. *We now consider the polynomial $P_2(x) = x^2 - (3 + \mathbf{j})x + 2 + \mathbf{j}$, whose corresponding companion polynomial has the triple root 1 and the simple root 3. Hence, P_2 has two admissible classes: $[\![1]\!]$ and $[\![2 + \mathbf{j}]\!]$. If we divide P_2 by the characteristic polynomial of the class $[\![1]\!]$ we obtain*

$$A = 1 + \mathbf{j} \quad and \quad B = -1 - \mathbf{j}.$$

Hence, there exists $\gamma = 1 \in \mathbb{R}$ such that $A + B\gamma = 0$; also, it is easily seen that the condition (3.17) is verified and so we are in the presence of a linear zero. More precisely, we conclude that all the elements in the set $\{1 + \beta\mathbf{i} + \beta\mathbf{k} : \beta \in \mathbb{R}\}$ are zeros of P_2.
  In a total analogous manner we find that P_2 has a linear zero in the class $[\![2 + \mathbf{j}]\!]$, given by $\{2 + \beta\mathbf{i} + \mathbf{j} - \beta\mathbf{k} : \beta \in \mathbb{R}\}$. The application of [14, Algorithm 6.1] would only find two isolated zeros, namely $z_0 = 1$ and $z_0 = 2 + \mathbf{j}$.

Example 4.3. *We consider now the polynomial $P_3(x) = P(x)(x-1)$, where $P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ with $a_3 = 2 + 2\mathbf{i} - \mathbf{j}$, $a_2 = -1 - 5\mathbf{j} - \mathbf{k}$, $a_1 = -4 - 5\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $a_0 = 2 - 2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ which was studied in [14, Example 7.2]. The companion polynomial of P has six simple real roots, as noted in that paper; hence, by applying Theorem 3.16, we know that P_3 has six linear roots which, however, are not found in [14]. As an example of a line of zeros of P_3, we have*

$$\mathcal{L} = \left\{ \beta\mathbf{i} + (-\tfrac{3}{5}\beta - \tfrac{4}{5})\mathbf{j} + (-\tfrac{4}{5}\beta + \tfrac{3}{5})\mathbf{k} : \beta \in \mathbb{R} \right\} \subsetneq [\![\mathbf{j}]\!],$$

as can be easily verified.

Example 4.4. *Let $P_4(x) = x^2 + (-5 - \mathbf{j})x + \frac{11}{2} + \frac{5}{2}\mathbf{j}$. This second degree polynomial has a companion polynomial with four simple real roots $1, 2, 3, 4$ and hence has six admissible classes, which is the maximum number allowed for a second degree polynomial. However, as we will now show, not all the admissible classes contain roots of P_4. If we consider, for example, the class $[\![2 + \mathbf{j}]\!]$, we obtain*

$$A = \tfrac{5}{2} + \tfrac{5}{2}\mathbf{j} \quad and \quad B = -1 - \mathbf{j}.$$

Hence, B is singular, $B \neq 0$ and there exists $\gamma = \frac{5}{2} \in \mathbb{R}$ such that $A + B\gamma = 0$; in this case, we have $(\frac{5}{2} - 2)^2 = \frac{1}{4} \neq 1 = -\mathrm{dv}(2 + \mathbf{j})$, and so we are in case 3 (b)-(ii) of Theorem 3.14, leading us to conclude that there are no roots in this class. In the same manner, we show that the class $[\![3 + \mathbf{j}]\!]$ does not contain any root of P_4. This illustrates our observation O2 made in Section 3.

Example 4.5. *We now consider three very simple polynomials that illustrate very clearly the observation O3 made in Section 3:*

$$
\begin{aligned}
P_5(x) &= x^2 - 2x + 1, \\
Q_5(x) &= x^2 - (2 + \mathbf{i} + \mathbf{j})x + 1 + \mathbf{i} + \mathbf{j}, \\
R_5(x) &= x^2 + (-2 - 6\mathbf{i} - 5\mathbf{j} - 3\mathbf{k})x + 3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}.
\end{aligned}
$$

All these polynomials have $(x-1)^4$ as companion polynomial (i.e., the companion polynomial has the real root 1 with multiplicity four). However, in what concerns the zeros in the (unique) class $[\![1]\!]$, they behave differently: the polynomial P_5 has $[\![1]\!]$ as a hyperboloidal zero, the polynomial Q_5 has the linear zero $\{1 + \beta\mathbf{i} + \beta\mathbf{j} : \beta \in \mathbb{R}\}$ and the polynomial R_5 has the isolated zero $z_0 = 1 + 5\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$.

Note also that, in the case of R_5, the real multiple root 1 of the companion polynomial is not a real root of R_5, contradicting the last assertion in Theorem 4.3 of [14].

Our last example addresses the problem posed by Janovská and Opfer in [14]: "*Given $n > 4$, can we find a coquaternionic polynomial of degree n with the maximal number $\binom{2n}{2}$ of zeros?*".

Example 4.6. *Let $P_6(x) = x^5 + c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0$ with*

$$c_4 = \tfrac{1}{2} + \mathbf{i} + 7\mathbf{j} + \tfrac{13}{2}\mathbf{k}, \ c_3 = 24 + \tfrac{13}{2}\mathbf{i} + \tfrac{11}{2}\mathbf{j} - 6\mathbf{k}, \ c_2 = -\tfrac{47}{2} - 32\mathbf{i} - 18\mathbf{j} + \tfrac{33}{2}\mathbf{k},$$
$$c_1 = -51 + \tfrac{81}{2}\mathbf{i} - \tfrac{57}{2}\mathbf{j} - 52\mathbf{k}, \ c_0 = -9 - 12\mathbf{i} - 18\mathbf{j} + 9\mathbf{k}.$$

The companion polynomial of this fifth degree polynomial has 10 real simple roots, $-6, -3, -1, 1, 2, 5, 1 \pm \sqrt{2}, \frac{1}{2}(-1 \pm \sqrt{5})$, and it can be verified that each of the 45 admissible classes contains an isolated root of P_6. For example, in the class $[\![1 + \sqrt{2}\mathbf{j}]\!]$ we have the root $z_0 = 1 + \frac{3}{8}\mathbf{i} - \frac{11}{8}\mathbf{j} - \frac{1}{2}\mathbf{k}$ and in the class $[\![\mathbf{j}]\!]$ the isolated root $z_1 = \frac{17}{84}\mathbf{i} - \frac{27}{28}\mathbf{j} - \frac{1}{3}\mathbf{k}$. Examples of polynomials of degrees 6, 7, 8 and 9 achieving the maximum possible number of roots, 66, 91, 120 and 153, respectively, were also constructed; all these polynomials were obtained as products of appropriately chosen linear factors and we believe that this type of process can be used to compute examples of polynomials of any prescribed degree with the maximum number of roots.

Acknowledgments

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