Generalized inverses and their relations with clean decompositions

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Abstract

An element a in a ring R is called clean if it is the sum of an idempotent e and a unit u. Such a clean decomposition $a = e + u$ is said to be strongly clean if $eu = ue$ and special clean if $aR \cap eR = (0)$. In this paper, we prove that a is Drazin invertible if and only if there exists an idempotent e and a unit u such that $a^n = e + u$ is both a strongly clean decomposition and a special clean decomposition, for some positive integer n. Also, the existence of the Moore-Penrose and group inverses is related to the existence of certain $*$-clean decompositions.

Keywords:
Drazin inverse, group inverse, Moore-Penrose inverse, strongly clean decomposition, special clean decomposition
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1. Introduction

Throughout this paper, we assume that R is a ring with unity 1. Recall that an element a of R is called clean if it is the sum of an idempotent $e \in R$ and a unit $u \in R$. Such a clean decomposition $a = e + u$ in a ring R is called strongly clean [10] if $eu = ue$ and special clean [2] if $aR \cap eR = (0)$. A ring

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is said to be clean if all its elements are clean. This definition dates back to the paper of Nicholson [9].

An involution $a \mapsto a^*$ in R is an anti-isomorphism of degree 2, that is, $(a^*)^* = a$, $(ab)^* = b^*a^*$ and $(a+b)^* = a^*+b^*$ for all $a, b \in R$. We say that R is a $*$-ring if there is an involution on R. An element p in a $*$-ring R is called a projection if $p^2 = p = p^*$. In 2010, Vaš [15] introduced $*$-clean elements and strongly $*$-clean elements by substituting "a projection" for "an idempotent" in the appropriate concepts. Also, a $*$-clean decomposition $a = p + u$ in a $*$-ring R is called special $*$-clean if $aR \cap pR = (0)$.

Let us now recall some notions of generalized inverses. An element $a \in R$ is (von Neumann) regular if there exists an element $x \in R$ such that $a = axa$. Such an x is called an inner inverse of a, and is denoted by a^-. By the symbol $a\{1\}$ we denote the set of all inner inverses of a.

We say that $a \in R$ is Drazin invertible [7] if there exist $b \in R$ and a nonnegative integer k such that

$$\text{(i) } a^k = a^kba, \text{ (ii) } bab = b, \text{ (iii) } ab = ba.$$

Any b satisfying (i)-(iii) is called a Drazin inverse of a. It is unique if it exists, and is denoted by a^D. The smallest nonnegative integer k is called the Drazin index of a and is denoted by $\mathrm{ind}(a)$. It is well-known that $a \in R$ is Drazin invertible if and only if there exists some positive integer m such that $a^m \in a^{m+1}R \cap Ra^{m+1}$. We call a group invertible if it is Drazin invertible and $\mathrm{ind}(a) = 1$. The group inverse of a is denoted by $a^{\#}$.

Following Penrose [11], an element a of a $*$-ring R is Moore-Penrose invertible if there exists some $x \in R$ satisfying

$$\text{(i) } axa = a, \quad \text{(ii) } xax = x, \quad \text{(iii) } (ax)^* = ax, \quad \text{(iv) } (xa)^* = xa.$$

An x satisfying (i)-(iv) is called a Moore-Penrose inverse of a. It is unique if it exists, and is denoted by $a^\dagger$. We call a to be $\{1,3\}$-invertible if it satisfies the conditions (i) and (iii). Such an x is a $\{1,3\}$-inverse of a and is denoted by $a^{(1,3)}$. Similarly, a is $\{1,4\}$-invertible if it satisfies the conditions (i) and (iv). Such an x is a $\{1,4\}$-inverse of a and is denoted by $a^{(1,4)}$. It is known that a is Moore-Penrose invertible if and only if it is both {1,3}-invertible and {1,4}-invertible. In this case, $a^\dagger = a^{(1,4)}aa^{(1,3)}$. By R^{-1}, R^D, $R^{\#}$ and $R^\dagger$ we denote the sets of all invertible, Drazin invertible, group invertible and Moore-Penrose invertible elements in R, respectively.

Over the last decade, many authors [2, 3, 5, 9, 10, 14, 15] studied the cleanness and $*$-cleanness of elements. As idempotents, projections and units can be constructed by generalized inverses of an element, then generalized inverses of an element can be related to its cleanness. For example, [5, Propositions 2.5 and 2.6] and [6, Theorem 5.5] relate generalized inverses of elements with the parts of the clean decomposition of these elements. Motivated by this, we aim to further study relations between generalized inverses of an element and its clean decomposition.

The paper is organized as follows. In Section 2, we prove that $a \in R^D$ if and only if there exist an idempotent e and a unit u such that $a^n = e + u$ is both a strongly clean decomposition and a special clean decomposition, for some positive integer n. Moreover, a^n is proved to be uniquely strongly clean. In Section 3, we show that $a \in R^{\dagger}$ if and only if aa^* has both a strongly $*$-clean decomposition and a special $*$-clean decomposition, under certain assumptions. Further, $a \in R^{\#} \cap R^{\dagger}$ if and only if a^2a^* has both a strongly $*$-clean decomposition and a special $*$-clean decomposition, under the assumption of the right strongly $*$-cancellability of a. As a special case, a characterization of EP elements is given.

2. The Drazin inverse of an element and its cleanness

We begin with the following known lemma, which plays an important role in the sequel.

Lemma 2.1. *Let $a, b \in R$. Then*

(i) *If $(ab-1)x = 1$ for some $x \in R$, then $(ba-1)(bxa-1) = 1$.*

(ii) *If $y(ab-1) = 1$ for some $y \in R$, then $(bya-1)(ba-1) = 1$.*

By Lemma 2.1, we know that $ab-1 \in R^{-1}$ if and only if $ba-1 \in R^{-1}$. In this case, $(ba-1)^{-1} = b(ab-1)^{-1}a-1$. This formula is known as Jacobson's Lemma.

The following characterization of the Drazin inverse of a ring element is slightly different from the classical existence of Drazin inverse of matrices over a ring [12, page 107]. Herein, $a^{2n}(a^n)^- + e \in R^{-1}$ is proved to be equivalent to $a^{n+1} - e \in R^{-1}$, where $e = 1 - a^n(a^n)^-$.

Lemma 2.2. *Let $a \in R$. Then the following conditions are equivalent*:

(i) *$a \in R^D$ with* $\operatorname{ind}(a) = n$.

(ii) *n is the smallest positive integer such that a^n is regular and $u = a^{n+1} - 1 + a^n(a^n)^- \in R^{-1}$.*

(iii) *n is the smallest positive integer such that a^n is regular and $v = a^{n+1} - 1 + (a^n)^- a^n \in R^{-1}$.*

In this case, $a^D = u^{-1}a^n = a^n v^{-1}$.

Proof. (i) $\Rightarrow$ (ii) As $a \in R^D$ and $\mathrm{ind}(a) = n$, then $a^n = a^n a a^D = a^n(aa^D)^{n+1} = a^{2n+1}(a^D)^{n+1}$. Also, $a \in R^D$ implies that a^n is regular with $(a^D)^n \in a^n\{1\}$. It follows from Lemma 2.1 that $a^{n+1} - 1 + a^n(a^n)^-$ is right invertible if and only if $a^{2n+1}(a^n)^- - 1 + a^n(a^n)^-$ is right invertible since $a^{n+1} = a^n(a^n)^- a^{n+1}$. Note that $(a^{2n+1}(a^n)^- - 1 + a^n(a^n)^-)(a^n(a^D)^{n+1}(a^n)^- - 1 + a^n(a^n)^-) = 1$. So, $a^{n+1} - 1 + a^n(a^n)^-$ is right invertible. Similarly, we can show that $u = a^{n+1} - 1 + a^n(a^n)^-$ is left invertible using $a^n = (a^D)^{n+1}a^{2n+1}$. So, $u = a^{n+1} - 1 + a^n(a^n)^- \in R^{-1}$.

(ii) $\Leftrightarrow$ (iii) follows from Lemma 2.1.

(iii) $\Rightarrow$ (i) If $v = a^{n+1} - 1 + (a^n)^- a^n \in R^{-1}$, then $a^n v = a^{2n+1}$ and hence $a^n = a^{2n+1}v^{-1} \in a^{n+1}R$. Since $u = a^{n+1} - 1 + a^n(a^n)^- \in R^{-1}$, we have $a^n = u^{-1}a^{2n+1} \in Ra^{n+1}$. Thus, $a^n \in a^{n+1}R \cap Ra^{n+1}$, that is $a \in R^D$.

As $ua^n = a^{2n+1} = a^n v$, then $u^{-1}a^n = a^n v^{-1}$. Next, we show that $z = u^{-1}a^n = a^n v^{-1}$ is the Drazin inverse of a with $\mathrm{ind}(a) = n$.

(1) Note that $ua^{n+1} = a^{2n+2} = a^{n+1}v$. Then $u^{-1}a^{n+1} = a^{n+1}v^{-1}$. Hence $za = u^{-1}a^{n+1} = a^{n+1}v^{-1} = aa^n v^{-1} = az$.

(2) $zaz = a^n v^{-1} a a^n v^{-1} = u^{-1}a^n a^{n+1} v^{-1} = u^{-1}a^{2n+1}v^{-1} = u^{-1}a^n = z$.

(3) $a^{n+1}z = a^{n+1}a^n v^{-1} = a^{2n+1}v^{-1} = a^n$.

Therefore, $a^D = u^{-1}a^n = a^n v^{-1}$. □

It follows from Lemma 2.2 that $a \in R^D$ implies $a^m - 1 + aa^D \in R^{-1}$ for some positive integer m. Applying this result, we present the relations between the Drazin inverse of an element and its clean decomposition in a ring.

Theorem 2.3. *Let $a \in R$. Then the following conditions are equivalent*:

(i) $a \in R^D$.

(ii) *There exist an idempotent e and a unit u such that $a^n = e + u$ is both a strongly clean decomposition and a special clean decomposition, for some positive integer n.*

In this case, $a^D = a^n(a^{n-1}u^{-1})^{n+1} = (u^{-1}a^{n-1})^{n+1}a^n$.

Proof. (i) $\Rightarrow$ (ii) Suppose $a \in R^D$. It follows from Lemma 2.2 that $u = a^n - 1 + aa^D \in R^{-1}$ for any integer $n > \mathrm{ind}(a)$. Let $e = 1 - aa^D$. Then

$a^n = e+u$ is a clean decomposition. By a direct check, we get $eu = -e = ue$ and hence $a^n = e+u$ is a strongly clean decomposition. Given $b \in eR \cap a^nR$, then $b = (1-aa^D)x = a^ny = (1-aa^D)a^ny = 0$ for some $x, y \in R$. So, $a^n = e+u$ is also a special clean decomposition.

(ii) $\Rightarrow$ (i) As $a^n = e+u$ is a strongly clean decomposition, then $ea^n = a^ne$. By the special cleanness of $a^n = e+u$, we know $eR \cap a^nR = 0$. Hence, $ea^n \in eR \cap a^nR = (0)$. Multiplying $a^n = e+u$ by a^n on the right yields $a^{2n} = ua^n$ and hence $a^n = u^{-1}a^{n-1}a^{n+1} \in Ra^{n+1}$.

Also, $a^n = a^{2n}u^{-1} = a^{n+1}a^{n-1}u^{-1} \in a^{n+1}R$ by the commutativity of u and a^n. So, $a \in R^D$ and $a^D = a^n(a^{n-1}u^{-1})^{n+1} = (u^{-1}a^{n-1})^{n+1}a^n$. $\square$

Corollary 2.4. *Let $a \in R$. Then the following conditions are equivalent*:

(i) $a \in R^D$.

(ii) *There exist $e^2 = e \in R$ and some positive integer n such that $a^ne = ea^n = 0$ and $a^n - e \in R^{-1}$.*

(iii) *There exist $e^2 = e \in R$ and some positive integer n such that $a^ne = ea^n = 0$ and $a^n + e \in R^{-1}$.*

PROOF. (i) $\Rightarrow$ (ii) Take $e = 1 - aa^D$, and $n > \mathrm{ind}(a)$, then the result follows from Lemma 2.2.

(ii) $\Rightarrow$ (i) Let $u := a^n - e \in R^{-1}$. When multiplying the relation $a^n = u + e$ by a^n on the left and right, we obtain that $a^{2n} = ua^n = a^nu$. So, $a^n = u^{-1}a^{2n} = a^{2n}u^{-1} \in a^{n+1}R \cap Ra^{n+1}$, which means $a \in R^D$.

(i) $\Leftrightarrow$ (iii) is similar to the proof of (i) $\Leftrightarrow$ (ii). $\square$

Recall that an element $a \in R$ is called uniquely strongly clean if it has a uniquely strongly clean decomposition (see [4]).

It follows from Theorem 2.3 that $a \in R^D$ implies that a^n has a strongly clean decomposition, for some positive integer n. The following result shows that such a strongly clean decomposition is unique, under certain conditions.

Theorem 2.5. *Let $a \in R^D$. Then a^n is uniquely strongly special clean, for some positive integer n.*

PROOF. Suppose $a \in R^D$. Then, by Theorem 2.3, there exist an idempotent e and a unit u such that $a^n = e + u$ is a strongly clean decomposition. Let $a^n = f+v$ be another strongly special clean decomposition, where $f = f^2$ and $v \in R^{-1}$. To show that a^n is uniquely strongly special clean, it is sufficient to prove $e = f$.

From Corollary 2.4, there exists some positive integer n such that $ea^n = 0 = fa^n$, and consequently $a^n = (1-e)a^n = (1-e)(e+u) = (1-e)u$. Then $1-e = a^n u^{-1}$. Multiplying $1-e = a^n u^{-1}$ by f on the left we obtain $f(1-e) = fa^n u^{-1} = 0$, i.e., $f = fe$.

Similarly, we have $1-f = v^{-1}a^n$. Multiplying $1-f = v^{-1}a^n$ by e on the right yields $(1-f)e = v^{-1}a^n e = 0$, i.e., $e = fe = f$, as required. $\square$

3. The Moore-Penrose inverse of an element and its $*$-cleanness

Throughout this section, we assume R to be a $*$-ring. It is known from [8, Theorem 5.4] that $a \in R^\dagger$ if and only if $a \in aa^*R \cap Ra^*a$. It was proved [17] that $a \in R^\dagger$ if and only if $a \in aa^*aR$ if and only if $a \in Raa^*a$. In particular, if $a = aa^*ax$ or $a = yaa^*a$ for some $x, y \in R$, then $a^\dagger = (ax)^* = (ya)^*$ (see [18, Theorem 3.12]).

An element $a \in R$ is called left $*$-cancellable if $a^*ax = a^*ay$ implies $ax = ay$, and $a \in R$ is called right $*$-cancellable if $baa^* = caa^*$ implies $ba = ca$. Moreover, a is left $*$-cancellable if and only if a^* is right $*$-cancellable. An element $a \in R$ is $*$-cancellable if it is both left and right $*$-cancellable, which is equivalent to the implication $a^*a = 0 \Rightarrow a = 0$. A ring R is called $*$-cancellable if every element of R is $*$-cancellable. Recall that a ring is $*$-cancellable is also said to have a proper involution (see e.g. [1]).

The following result presents the relation between $*$-clean elements and Moore-Penrose invertible elements, under one-sided $*$-cancellability.

Theorem 3.1. *Let $a \in R$ be right $*$-cancellable. Then the following conditions are equivalent*:

(i) $a \in R^\dagger$.

(ii) *There exist a projection p and a unit u such that $aa^* = p + u$ is both a strongly $*$-clean decomposition and a special $*$-clean decomposition.*

In this case, $a^\dagger = a^*u^{-1}$.

Proof. (i) $\Rightarrow$ (ii) As $a \in R^\dagger$, then $p = 1 - aa^\dagger$ is a projection, and $u = aa^* - 1 + aa^\dagger \in R^{-1}$ from [16, Theorem 2.3]. Hence, $aa^* = p + u$ is a $*$-clean decomposition. We have $pu = -p = up$ by $aa^*aa^\dagger = a(aa^\dagger a)^* = aa^*$. So, $aa^* = p + u$ is a strongly $*$-clean decomposition. Let $c \in (1-aa^\dagger)R \cap aa^*R$. Then there exist $g, h \in R$ such that $c = (1-aa^\dagger)g = aa^*h = (1-aa^\dagger)aa^*h = 0$, that is, $aa^* = p + u$ is a special $*$-clean decomposition.

(ii) $\Rightarrow$ (i) Suppose that $aa^* = p + u$ is a strongly $*$-clean decomposition. Then $paa^* = aa^*p$. Also, as $aa^* = p + u$ is a special $*$-clean decomposition, then $paa^* = 0$ since $paa^* \in pR \cap aa^*R$. Multiplying $aa^* = p + u$ by aa^* on the left yields $aa^*aa^* = aa^*u$, which implies $a^*aa^* = a^*u$ since a^* is left $*$-cancellable. Hence, $a = (u^{-1})^*aa^*a$, which guarantees that $a \in R^\dagger$ and $a^\dagger = ((u^{-1})^*a)^* = a^*u^{-1}$. $\square$

By Theorem 2.3, we get that $a \in R^\#$ if and only if it has both a strongly clean decomposition and a special clean decomposition. Theorem 3.1 ensures that $a \in R^\dagger$ if and only if aa^* has both a strongly $*$-clean decomposition and a special $*$-clean decomposition, under the $*$-cancellability of a. It is natural to consider whether $a \in R^\# \cap R^\dagger$ is equivalent to the statement that a^2a^* has both a strongly $*$-clean decomposition and a special $*$-clean decomposition, under certain conditions. The following theorem addresses this problem. We will need the following lemma.

Lemma 3.2. *Let $a \in R$. Then the following conditions are equivalent*:

(i) $a \in R^\# \cap R^\dagger$.

(ii) *a is regular and $u = aa^*a - 1 + aa^- \in R^{-1}$.*

(iii) *a is regular and $v = a^2a^* - 1 + aa^- \in R^{-1}$.*

Proof. (i) $\Rightarrow$ (ii) It follows from [16, Corollary 2.7] that $a \in R^\dagger$ implies $aa^* - 1 + aa^- \in R^{-1}$ and hence $aa^*aa^- - 1 + aa^- \in R^{-1}$ by Jacobson's Lemma. Also, $a \in R^\#$ implies $a + 1 - aa^- \in R^{-1}$. Hence, we have $(aa^*aa^- - 1 + aa^-)(a + 1 - aa^-) = aa^*a - 1 + aa^- \in R^{-1}$.

(ii) $\Rightarrow$ (i) As $u = aa^*a - 1 + aa^- \in R^{-1}$, then $t = a^*a^2 - 1 + a^-a \in R^{-1}$ by Jacobson's Lemma. Since $at = aa^*a^2$, it follows $a = aa^*a^2t^{-1} \in aa^*aR$, which means $a \in R^\dagger$. Again, by [16, Corollary 2.7], we get $aa^* - 1 + aa^- \in R^{-1}$ and hence $aa^*aa^- - 1 + aa^- \in R^{-1}$. As $a + 1 - aa^- = (aa^*aa^- - 1 + aa^-)^{-1}(aa^*a - 1 + aa^-) \in R^{-1}$, it follows $a \in R^\#$, and consequently $a \in R^\# \cap R^\dagger$.

(i) $\Leftrightarrow$ (iii) can be proved similarly. $\square$

Let $a \in R$. We call a right square $*$-cancellable if $ba^2a^* = ca^2a^*$ implies $ba = ca$ for any $b, c \in R$. A ring R is said to be right square $*$-cancellable if all its elements are right square $*$-cancellable.

Theorem 3.3. *Let $a \in R$ be right square $*$-cancellable. Then the following conditions are equivalent*:

(i) $a \in R^\# \cap R^\dagger$.

(ii) *There exist a projection p and a unit u such that $a^2a^* = p + u$ is both a strongly $*$-clean decomposition and a special $*$-clean decomposition.*

In this case, $a^\dagger = (u^{-1}a^2)^*$ and $a^\# = (aa^*u^{-1})^2a$.

Proof. (i) $\Rightarrow$ (ii) It follows from Lemma 3.2 that $a \in R^{\#} \cap R^{\dagger}$ implies $u = a^2a^* - 1 + aa^{\dagger} \in R^{-1}$. Take $p = 1 - aa^{\dagger}$. Then $a^2a^* = p + u$ is a $*$-clean decomposition. As $pu = -p = up$, then $a^2a^* = p + u$ is a strongly $*$-clean decomposition. Moreover, $pR \cap a^2a^*R = (0)$. Indeed, for any $n \in pR \cap a^2a^*R$, we have $n = px = a^2a^*y = pa^2a^*y = 0$ for some $x, y \in R$. Hence, $a^2a^* = p+u$ is both a strongly $*$-clean decomposition and a special $*$-clean decomposition.

(ii) $\Rightarrow$ (i) As $a^2a^* = p + u$ is a strongly $*$-clean decomposition, then $pa^2a^* = a^2a^*p$. From the special $*$-cleanness, we get $pa^2a^* \in pR \cap a^2a^*R = (0)$. Multiplying $a^2a^* = p + u$ by a^2a^* on the right yields $a^2a^*a^2a^* = ua^2a^*$. From the right square $*$-cancellability of a, it follows that $a^2a^*a = ua$ and hence $a = u^{-1}a^2a^*a \in Raa^*a$. So, $a \in R^{\dagger}$ and $a^{\dagger} = (u^{-1}a^2)^*$ by [18, Theorem 3.12].

Since $ua^2a^* = a^2a^*a^2a^* = a^2a^*u$, we get $a^2a^*u^{-1} = u^{-1}a^2a^*$. Hence, $a = u^{-1}a^2a^*a = a^2a^*u^{-1}a \in a^2R$. As $a^{\dagger} = (u^{-1}a^2)^*$, then $a = aa^*(a^{\dagger})^* = aa^*u^{-1}a^2 \in Ra^2$. So, $a \in a^2R \cap Ra^2$, i.e., $a \in R^{\#}$ and $a^{\#} = aa^*u^{-1}aa^*u^{-1}a = (aa^*u^{-1})^2a$.

Therefore, $a \in R^{\#} \cap R^{\dagger}$. □

Remark 3.4. The right square $*$-cancellability of a in Theorem 3.3 above cannot be dropped. In fact, let $R = \mathbb{Z}_4$ and let $*\colon x \mapsto x$ be an involution of R. Then 2 is not right square $*$-cancellable, because $1 \cdot 2^2 \cdot 2^* = 0 = 0 \cdot 2^2 \cdot 2^*$ but $2 \neq 0$. By a direct check, we get $2 \notin R^{\#} \cap R^{\dagger}$. However, there exist a projection 1 and a unit 3 such that $2^2 \cdot 2^* = 0 = 3 + 1$ is both a strongly $*$-clean decomposition and a special $*$-clean decomposition.

Recall that an element $a \in R$ is EP if $a \in R^{\#} \cap R^{\dagger}$ and $a^{\#} = a^{\dagger}$. A characterization of EP elements is that a is EP if and only if $aa^{\dagger} = a^{\dagger}a$. Finally, we give a characterization of an EP element in a $*$-ring.

Theorem 3.5. *Let $a \in R$. Then the following conditions are equivalent*:

(i) *a is* EP.

(ii) *There exist a projection p and a unit u such that $a = p + u$ is both a strongly $*$-clean decomposition and a special $*$-clean decomposition.*

In this case, $a^{\#} = a^{\dagger} = u^{-2}a = au^{-2}$.

Proof. (i) $\Rightarrow$ (ii) As a is EP, then $u = a - 1 + aa^{\dagger} \in R^{-1}$ by [13, Corollary 1]. Since $p = 1 - aa^{\dagger}$ is a projection, we have $up = pu$ by direct calculations. Let $b \in aR \cap pR$. Then $b = (1 - aa^{\dagger})x = ay = (1 - aa^{\dagger})ay = 0$ for some

$x, y \in R$. Hence, $aR \cap pR = (0)$. So, $a = p + u$ is both a strongly $*$-clean decomposition and a special $*$-clean decomposition.

(ii) $\Rightarrow$ (i) Note that $pa = ap$ and $aR \cap pR = (0)$. Multiplying $a = p + u$ by a on the left yields $a^2 = au$ and hence $a = a^2u^{-1} \in a^2R$. Multiplying $a = p + u$ by a on the right yields $a^2 = ua$ and hence $a = u^{-1}a^2 \in Ra^2$. So, $a \in a^2R \cap Ra^2$, i.e., $a \in R^{\#}$ and $a^{\#} = u^{-1}au^{-1}$.

As $pa = ap$, then $pa^* = a^*p$. Since $aR \cap pR = (0)$, it follows that $ap = pa = 0$ and hence $a^*p = pa^* = 0$. Multiplying $a = p + u$ by a^* on the left yields $a^*a = a^*u$ and hence $a = (u^{-1})^*a^*a$. By [19, Lemma 2.2], we know that a is $\{1,3\}$-invertible and that u^{-1} is a $\{1,3\}$-inverse of a. Multiplying $a = p+u$ by a^* on the right gives $aa^* = ua^*$ and hence $a = aa^*(u^{-1})^*$. Again, from [19, Lemma 2.2], it follows that a is $\{1,4\}$-invertible and that u^{-1} is a $\{1,4\}$-inverse of a. Thus, $a \in R^\dagger$ and $a^\dagger = a^{(1,4)}aa^{(1,3)} = u^{-1}au^{-1} = a^{\#}$.

Therefore, a is EP. $\square$

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