

Linear Preservers of Copositive Matrices

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Abstract

An n -by- n real symmetric matrix is called copositive if its quadratic form is nonnegative on nonnegative vectors. Our interest is in identifying which linear transformations on symmetric matrices preserve copositivity either in the into or onto sense. We conjecture that in the onto case, the map must be congruence by a monomial matrix (a permutation times a positive diagonal matrix). This is proven under each of some additional natural hypotheses. Also, the into preservers of standard type are characterized. A general characterization in the into case seems difficult, and examples are given. One of them provides a counterexample to a conjecture about the into preservers.

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1 Introduction

Matrix $A \in M_n(\mathbb{R})$ is called *copositive* if $A^T = A$ and $x^T Ax \geq 0$ whenever $x \in \mathbb{R}^n$, and $x \geq 0$. If $x^T Ax > 0$ whenever $x \geq 0$ (entrywise), $x \neq 0$, then A is called *strictly copositive*. We write \mathcal{C} (\mathcal{C}_n) and \mathcal{SC} (\mathcal{SC}_n) to denote the two classes (with the subscript to indicate the size of the matrices, if useful). Of course \mathcal{C} generalizes the positive semidefinite (PSD) and \mathcal{SC} the positive definite (PD) matrices. Much is known about the copositive matrices [1, 2, 4, 6, 8] and there are tests for copositivity [4, 9, 15], but, in general, recognition is more difficult than for the PD/PSD cases.

Let $\mathcal{S}_n = \mathcal{S}_n(\mathbb{R})$ be the $\frac{n(n+1)}{2}$ -dimensional subspace of $M_n(\mathbb{R})$ consisting of symmetric matrices. In considering the action of linear transformations on copositive matrices, it suffices to consider linear transformations $L: \mathcal{S}_n \rightarrow \mathcal{S}_n$ (and it can be convenient because there are fewer variables to consider). However, it may also be convenient to consider L to be a linear map on $M_n(\mathbb{R})$. We shall do so interchangeably. We say that such a linear transformation *preserves copositivity* if $A \in \mathcal{C}$ implies $L(A) \in \mathcal{C}$, and similarly for strict copositivity. More precisely, such an L is an *into* copositivity preserver. If $L(\mathcal{C}) = \mathcal{C}$, we have an *onto* copositivity preserver. Our purpose here is to better understand both types of linear copositivity preservers. The into preservers of the PSD matrices are fully understood, and they are recognized to be a difficult problem. The onto linear preservers of PSD are straight forwardly known to be the congruences by a fixed invertible matrix [14].

Certain natural kinds of linear transformations are more amenable to preserver analysis. We say a linear transformation on $M_n(\mathbb{R})$ is of *standard form* if there are fixed matrices $R, S \in M_n(\mathbb{R})$ such that $L(A) = RAS$ (or $L(A) = RA^T S$). Such a linear transformation is invertible if and only if R and S are invertible matrices. More generally, L is a linear transformation on $M_n(\mathbb{R})$ if $L(A) = (l_{ij}(A))$, in which l_{ij} is a linear functional in the entries of A . It is known that an invertible linear transformation that preserves rank is of standard form [7, 12] and there are useful variations upon this sufficient

condition.

Both the “onto” and especially the “into” copositivity linear preserver problems appear subtle. For example, in [13] a conjecture of N. Johnston is relayed: Any (into) copositivity preserver is of the form

$$X \longrightarrow \sum_i A_i^T X A_i.$$

in which $A_i \geq 0$. Since any such map is (clearly) a copositivity preserver, this is a natural (though optimistic) conjecture. However, it is false even for 2-by-2 matrices. Suppose that a linear map on \mathcal{S}_2 is given by

$$\begin{bmatrix} a & c \\ c & b \end{bmatrix} \rightarrow \begin{bmatrix} a & a + b + 2c \\ a + b + 2c & b \end{bmatrix}.$$

If the argument is copositive, then $a + b + 2c \geq 0$ (since it is the value of the quadratic form of the argument at $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$), so that the image is nonnegative and, thus, copositive. But, as the conjectured form is a PSD preserver (by virtue of being a sum of congruences), the fact that $\begin{bmatrix} 10 & -1 \\ -1 & 10 \end{bmatrix}$ is PSD, while its image $\begin{bmatrix} 10 & 18 \\ 18 & 10 \end{bmatrix}$ is not, shows that our linear map is not of the conjectured form, though a copositivity preserver.

2 Some Linear Copositivity Preservers

Here we indicate several types of linear copositivity preservers. Besides these, we do not know any others.

2.1 Linear Copositivity Preservers of Standard Form

Since our arguments are always in \mathcal{S}_n , a linear transformation of standard form is of the form

$$L(A) = RAR^T, \text{ or } L(A) = RA^T R^T$$

$R \in M_n(\mathbb{R})$. Here we characterize both the into and onto preservers of copositivity of standard form. Such a transformation is invertible on \mathcal{S}_n if and only if R is invertible. First, a useful lemma about copositive matrices. We say that a vector $v \in \mathbb{R}^n$, $n \geq 2$, is of *mixed sign* if v has both positive and negative entries. Nothing is assumed about 0 entries.

Lemma 1 *For each vector $v \in \mathbb{R}^n$ of mixed sign, there is a matrix $A \in \mathcal{C}_n$ such that $v^T Av < 0$.*

Proof. By the permutation similarity invariance of \mathcal{C}_n , we may assume that $v = [v_1, v_2, \dots, v_n]^T$ with $v_1 v_2 < 0$. Then, let $A \in \mathcal{C}_n$ be

$$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array} \right],$$

so that $v^T Av = 2v_1 v_2 < 0$, as claimed. ■

Theorem 2 *Suppose that L is an into linear preserver of \mathcal{C}_n of standard form. Then,*

$$L(A) = S^T AS,$$

with $S \in M_n(\mathbb{R})$ and $S \geq 0$.

Proof. Since L preserves symmetry and is of standard form, $L(A) = S^T AS$ with $S \in M_n(\mathbb{R})$. Suppose that there is an $x \geq 0$, $x \in \mathbb{R}^n$, such that Sx is of mixed sign. Then by Lemma 1, the argument A may be chosen so that $0 > (Sx)^T A(Sx) = x^T S^T ASx$ and $S^T AS = L(A)$ is not in \mathcal{C}_n . Hence, L is not a copositivity preserver. Thus, for any $x \geq 0$, Sx must be weakly uniformly signed. This means that $S \geq 0$ or ≤ 0 . Since S appears twice, we may take it to be the former. ■

Since \mathcal{C}_n contains a basis of \mathcal{S}_n , a linear map on \mathcal{S}_n that is an onto copositivity preserver must be an invertible linear map and the inverse map must also be an onto preserver. If the map is of standard form, the inverse map

just corresponds to inverting the S and S^T (which, of course, must be invertible). Thus, we have that $S^{-1} \geq 0$, as well as $S \geq 0$ (or $S^{-1} \leq 0$ and $S \leq 0$). It is well known that this happens if and only if S is a *monomial matrix*, the product of a permutation matrix and a positive diagonal matrix. Taken together, this gives the following characterization of linear transformations of standard form that maps \mathcal{C}_n onto itself.

Theorem 3 *Suppose that L is an onto linear preserver of \mathcal{C}_n of standard form. Then,*

$$L(A) = S^T A S,$$

in which $S \in M_n(\mathbb{R})$ and S is monomial.

Since \mathcal{C}_n forms a cone, we note that (i) any sum of into \mathcal{C}_n preservers is again an into \mathcal{C}_n preserver, though a sum of ones of standard form may no longer be of standard form, and (ii) the sum of onto \mathcal{C}_n preservers need no longer be onto. Also, it follows from the proven forms that both into and onto copositivity preservers of standard form are also (into and onto, respectively) PSD preservers.

2.2 Hadamard Multiplier \mathcal{C}_n preservers

Recall that the *Hadamard*, or entry-wise, product of two matrices $A = (a_{ij})$ and $B = (b_{ij})$ of the same size is defined and denoted by $A \circ B = (a_{ij}b_{ij})$. If we consider a fixed n -by- n matrix H , then a natural type of linear transformation on n -by- n matrices A is given by

$$L(A) = H \circ A. \tag{1}$$

We may also ask for which H are such transformations (into) \mathcal{C}_n preservers.

An n -by- n matrix B is called *completely positive* (CP) if $B = FF^T$ with F n -by- k and $F \geq 0$. Thus, for $F = [f_1, f_2, \dots, f_k]$ partitioned by columns,

$$B = f_1 f_1^T + f_2 f_2^T + \dots + f_k f_k^T.$$

Then, the CP matrices, which also form a cone, are special PSD matrices. It is known [5] that the CP matrices are the cone theoretic dual of \mathcal{C}_n , as $Tr(B^T A) = \sum_{i=1}^k f_i^T A f_i$, which is ≥ 0 if $A \in \mathcal{C}_n$.

Now, consider a linear transformation of the form (1) with H a CP matrix of the form $H = \sum_{i=1}^k h_i h_i^T$, $h_i \in \mathbb{R}^n$, $h_i \geq 0$, $i = 1, \dots, k$. If $A \in \mathcal{C}_n$, then

$H \circ A = \sum_{i=1}^k h_i h_i^T \circ A$ and $x^T (H \circ A) x = \sum_{i=1}^k (x \circ h_i)^T A (x \circ h_i)$, which is ≥ 0 for $x \geq 0$.

Theorem 4 *A linear transformation of the form (1) is an into copositive preserver if and only if H is CP.*

Proof. Sufficiency follows from the calculation above. The quadratic form of $H \circ A$ on a nonnegative vector is a sum of quadratic forms of A on nonnegative vectors. On the other hand, if H is not CP, because of the known duality, $e^T (H \circ A) e = Tr H^T A < 0$ for some $A \in \mathcal{C}_n$, and $H \circ A \notin \mathcal{C}_n$. ■

If H is CP, $H = \sum_{i=1}^k h_i h_i^T$, $h_i \geq 0$, let $D_i = \text{diag}(h_i)$. Then $H \circ A = \sum_{i=1}^k D_i^T A D_i$, with $D_i \geq 0$, so that a linear transformation of the form (1) is also a sum of into transformations of standard form. With the exception of H being a *rank* 1, positive, symmetric matrix, such a transformation will not be onto.

2.3 General Linear Maps on \mathcal{S}_n

Let $L(A) = (l_{ij}(A))$ in which each entry l_{ij} is a linear functional in the entries of A . Symmetry requires that the functionals l_{ij} and l_{ji} be the same. It is possible to design such maps that are copositivity preservers (and in a similar way, PSD preservers).

Let $z_{ij}^{(k)} \in \mathbb{R}^n$, $z_{ij}^{(k)} \geq 0$ and $z_{ij}^{(k)} = z_{ji}^{(k)}$. If $A \in \mathcal{C}_n$, then $z_{ij}^{(k)T} A z_{ij}^{(k)} \geq 0$.

Define

$$l_{ij}(A) = \sum_k z_{ij}^{(k)T} A z_{ij}^{(k)}$$

and

$$L(A) = (l_{ij}(A)).$$

Then, for $A \in \mathcal{C}_n$, $L(A) \geq 0$ and $L(A)^T = L(A)$, so that $L(A) \in \mathcal{C}_n$ and L is an into \mathcal{C}_n preserver. PSD (into) preservers may be designed in a similar way. For example, let $L(A) = \text{diag}(l_1(A), \dots, l_n(A))$ with $l_i(A) = z_i^* A z_i$, $z_i \in \mathbb{C}^n$, so that if A is PSD, $L(A)$ is a nonnegative diagonal matrix and, thus, PSD.

We note that, with this machinery, it is possible to design into, not onto, but invertible \mathcal{C}_n preservers. Here is an example. For

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix},$$

let

$$l_{11}(A) = \begin{bmatrix} 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \end{bmatrix} A \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$l_{12}(A) = l_{21}(A) = \begin{bmatrix} 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and

$$l_{22}(A) = \begin{bmatrix} 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \end{bmatrix} A \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then,

$$L \left(\begin{bmatrix} a & b \\ b & c \end{bmatrix} \right) = \begin{bmatrix} 2a + 2b + c & a + 2b + c \\ a + 2b + c & a + 2b + 2c \end{bmatrix}.$$

and, for $A \in \mathcal{C}_2$, $L(A) \geq 0$ and $L(A)$ is PSD. So L is a \mathcal{C}_2 preserver. Since $l_{12} \geq 0$, it is only into. However, L is invertible, as

$$L^{-1} \begin{bmatrix} x & z \\ z & y \end{bmatrix} = \begin{bmatrix} x - z & \frac{3z - x - y}{2} \\ \frac{3z - x - y}{2} & y - z \end{bmatrix}.$$

We note that more elaborate maps may be designed, including the possibility of negative off-diagonal entries.

2.4 PSD Preservers that are not Copositivity Preservers

Of course, a PSD preserver need not be a copositivity preserver, and we note here a famous example of a PSD preserver that is not a copositivity preserver.

The Choi map [3] is a linear transformation from $M_3(\mathbb{R})$ to $M_3(\mathbb{R})$ that preserves PSD, but is not of any typical type. It is defined by

$$L \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 2a_{11} + 2a_{22} & -a_{12} & -a_{13} \\ -a_{21} & 2a_{22} + 2a_{33} & -a_{23} \\ -a_{31} & -a_{32} & 2a_{33} + 2a_{11} \end{bmatrix}.$$

It is known, and easily checked, that any 3-by-3 (symmetric) PSD matrix is transformed to another PSD matrix. Of course, L is not a fixed congruence, nor onto. However, note that a copositive matrix with 0 diagonal and positive off diagonal entries is transformed to a matrix that is not copositive.

In general, copositivity preservers need not be PSD preservers and PSD preservers need not be copositivity preservers. However, we conjecture that onto copositivity preservers are also onto PSD preservers.

3 Onto Linear Copositivity Preservers

We first make a fundamental observation about onto preservers of \mathcal{C}_n .

Theorem 5 *Let $L : \mathcal{S}_n \rightarrow \mathcal{S}_n$ be a linear transformation that maps \mathcal{C}_n onto \mathcal{C}_n . Then L is invertible and L^{-1} maps \mathcal{C}_n onto \mathcal{C}_n .*

Proof. Since the copositive matrices include the standard basis for the symmetric matrices, their span is all of \mathcal{S}_n , which means that the map is invertible. Since $L(\mathcal{C}_n) = \mathcal{C}_n$, application of L^{-1} to both sides of the equality yields the desired statement. ■

Now we may see that linear onto preservers of \mathcal{C}_n also preserve several related sets. For $B \in \mathcal{C}_n$, let

$$\mathcal{C}(B) = \{A \in \mathcal{C}_n : \exists \alpha > 0 : B - \alpha A \in \mathcal{C}_n\}$$

and

$$\mathcal{R} = \{B \in \mathcal{C}_n : \mathcal{C}(B) = \mathcal{C}_n\}.$$

Corollary 6 *Let $L : \mathcal{S}_n \rightarrow \mathcal{S}_n$ be a linear transformation that maps \mathcal{C}_n onto \mathcal{C}_n . Then L also preserves (in the onto sense)*

- a) *the boundary copositive matrices;*
- b) *the interior of \mathcal{C}_n ;*
- c) *\mathcal{SC}_n ; and*
- d) *\mathcal{R} .*

Proof. a) and b) follow since the copositive matrices are the closure of the strictly copositive matrices and an invertible linear transformation is continuous. c) follows since \mathcal{SC}_n is the interior of \mathcal{C}_n . Now we show d). Let $R \in \mathcal{R}$ and $A \in \mathcal{C}$. Then $R - \alpha A \in \mathcal{C}$ for some $\alpha > 0$. So, $L(R - \alpha A) \in \mathcal{C}$ and $L(R) - \alpha L(A) \in \mathcal{C}$. Since L is onto, $L(A)$ runs over \mathcal{C} when A runs over \mathcal{C} . So $L(R) \in \mathcal{R}$ and $L(\mathcal{R}) \subseteq \mathcal{R}$. Similarly, $L^{-1}(\mathcal{R}) \subseteq \mathcal{R}$. Thus, $\mathcal{R} \subseteq L(\mathcal{R}) \subseteq \mathcal{R}$ implying that $L(\mathcal{R}) = \mathcal{R}$. ■

Since both a fixed permutation similarity (equivalently, congruence) or a fixed positive diagonal congruence is an onto copositivity preserver, we have that monomial congruence is an onto copositivity preserver. We

Conjecture. The onto copositivity preservers are exactly the fixed monomial congruences.

In a moment, we prove this in the 2-by-2 case. However, a proof, without additional hypothesis, appears subtle. We have already shown that the conjecture holds if

- 1) the map is of standard form (Section 2.1). However, each of the following alternative additional hypotheses is sufficient:
- 2) the map is an (onto) PSD preserver;

3) the map is rank nonincreasing (in which case it is of standard form [10, 11]);

4) in the language of the Introduction, each of the component maps l_{ij} is a function of only one entry of A (i.e., it is a monomial map on $\text{vec } \mathcal{S}_n$);

5) each of the component maps l_{ij} must have nonnegative coefficients.

We now study the onto 2-by-2 copositivity preservers. A symmetric matrix $A \in \mathcal{S}_2$ is copositive if and only if

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

with $a \geq 0$, $c \geq 0$ and $b \geq -\sqrt{ac}$. The matrix A is strictly copositive if all the inequalities are strict.

From now on, we assume that $L : \mathcal{S}_2 \rightarrow \mathcal{S}_2$ is a linear transformation that maps \mathcal{C}_2 onto \mathcal{C}_2 and we use the following notation

$$\begin{aligned} L \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) &= \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{12} & \alpha_{22} \end{bmatrix} =: \Pi_\alpha \\ L \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) &= \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{12} & \gamma_{22} \end{bmatrix} =: \Pi_\gamma \\ L \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) &= \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{12} & \beta_{22} \end{bmatrix} =: \Pi_\beta, \end{aligned}$$

so that

$$L \left(\begin{bmatrix} a & b \\ b & c \end{bmatrix} \right) = a\Pi_\alpha + b\Pi_\beta + c\Pi_\gamma.$$

Lemma 7 *We have $\alpha_{11}\alpha_{22} = 0$ and $\gamma_{11}\gamma_{22} = 0$.*

Proof. We show the first claim. The proof of the second claim is analogous. According to Corollary 6, either $\alpha_{11}\alpha_{22} = 0$ or

$$\Pi_\alpha = \begin{bmatrix} \alpha_{11} & -\sqrt{\alpha_{11}\alpha_{22}} \\ -\sqrt{\alpha_{11}\alpha_{22}} & \alpha_{22} \end{bmatrix},$$

with $\alpha_{11} > 0$ and $\alpha_{22} > 0$. In order to get a contradiction, suppose that Π_α has this latter form.

- Suppose that $\beta_{11}\beta_{22} \neq 0$. Then, by Corollary 6,

$$\Pi_\beta = \begin{bmatrix} \beta_{11} & -\sqrt{\beta_{11}\beta_{22}} \\ -\sqrt{\beta_{11}\beta_{22}} & \beta_{22} \end{bmatrix}.$$

Since Π_α and Π_β are linearly independent, we have $\alpha_{11}\beta_{22} - \alpha_{22}\beta_{11} \neq 0$. Thus,

$$\det(a\Pi_\alpha + b\Pi_\beta) = ab \left(\alpha_{11}\beta_{22} + \beta_{11}\alpha_{22} - 2\sqrt{\alpha_{11}\alpha_{22}}\sqrt{\beta_{11}\beta_{22}} \right)$$

is positive if $ab > 0$. Hence, for $a > 0$ and $b > 0$, $a\Pi_\alpha + b\Pi_\beta$ has positive diagonal entries and positive determinant, and, thus, is strictly copositive. Therefore, for $a > 0$, $b > 0$, and $c < 0$ sufficiently close to 0, A is not copositive and $L(A)$ is copositive.

- Suppose that $\beta_{11} = 0$. Then, $\beta_{12} \geq 0$.
 - If $\beta_{12} > 0$, let $a > 0$ and $c < 0$ be so that $a\alpha_{11} + c\gamma_{11} \geq 0$ and $a\alpha_{22} + c\gamma_{22} \geq 0$. Then, for $b > 0$ large enough, A is not copositive and $L(A) \geq 0$ is copositive.
 - If $\beta_{12} = 0$, then $\beta_{22} > 0$. For $a > 0$, $c < 0$ sufficiently close to 0, and $b > 0$ large enough, A is not copositive and $L(A)$ is copositive.
- The proof is similar if $\beta_{22} = 0$.

■

Lemma 8 *We have $\alpha_{11} = \alpha_{12} = \beta_{11} = \beta_{22} = \gamma_{22} = \gamma_{12} = 0$ or $\alpha_{22} = \alpha_{12} = \beta_{11} = \beta_{22} = \gamma_{11} = \gamma_{12} = 0$.*

Proof. By Lemma 7, $\alpha_{11}\alpha_{22} = 0$ and $\gamma_{11}\gamma_{22} = 0$.

- Suppose that $\alpha_{11} = 0$. We will show that $\alpha_{12} = \beta_{11} = \beta_{22} = \gamma_{22} = \gamma_{12} = 0$, which implies $\alpha_{22}, \beta_{12}, \gamma_{11} > 0$.

- If $\beta_{11} \neq 0$ then for $b < 0$ and $a, c > 0$ with $b < -c\frac{\gamma_{11}}{\beta_{11}}$ and $a > \frac{b^2}{c}$, A is copositive and $L(A)$ is not copositive as its 1, 1 entry is negative. So $\beta_{11} = 0$ and, then, $\beta_{12} \geq 0$.
- Suppose that $\alpha_{12} = 0$ (and $\beta_{11} = 0$).
 - * If $\beta_{22} \neq 0$, for $b > 0$, $c = 0$ and $a < 0$ sufficiently close to 0, A is not copositive and $L(A)$ is copositive. If $\gamma_{22} \neq 0$, then $\gamma_{11} = 0$ and $\gamma_{12} \geq 0$. Thus, for $c > 0$, $b = 0$ and $a < 0$ sufficiently small, A is not copositive and $L(A)$ is copositive. Thus, $\beta_{22} = \gamma_{22} = 0$.
 - * If $\alpha_{11} = \alpha_{12} = \beta_{22} = \gamma_{22} = \beta_{11} = 0$ and $\gamma_{12} \neq 0$, then $\gamma_{12} > 0$. For $a = 0$, $c > 0$ large and $b < 0$ sufficiently small, A is not copositive and $L(A)$ is copositive. Thus, $\gamma_{12} = 0$.
- Suppose that $\alpha_{12} \neq 0$ (and $\beta_{11} = 0$). Then $\alpha_{12} > 0$.
 - * If $\alpha_{22} \neq 0$ or $\beta_{22} = 0$, for $a > 0$, $c = 0$, and $b < 0$ sufficiently close to 0, A is not copositive and $L(A)$ is copositive.
 - * If $\alpha_{22} = 0$ and $\beta_{22} \neq 0$, for $b < 0$ and $a, c > 0$ with $b < -c\frac{\gamma_{22}}{\beta_{22}}$ and $a > \frac{b^2}{c}$, A is copositive and $L(A)$ is not copositive as its 2, 2 entry is negative.
- With similar arguments we show that if $\alpha_{22} = 0$ then $\alpha_{12} = \beta_{11} = \beta_{22} = \gamma_{11} = \gamma_{12} = 0$.

■

Theorem 9 $L : \mathcal{S}_2 \rightarrow \mathcal{S}_2$ is a linear transformation that maps \mathcal{C}_2 onto \mathcal{C}_2 if and only if

$$L \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \alpha_{11} & 0 \\ 0 & 0 \end{bmatrix}, \quad L \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \gamma_{22} \end{bmatrix}, \quad L \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \beta_{12} \\ \beta_{12} & 0 \end{bmatrix}, \quad (2)$$

or

$$L \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \alpha_{11} \end{bmatrix}, \quad L \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \gamma_{22} & 0 \\ 0 & 0 \end{bmatrix}, \quad L \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \beta_{12} \\ \beta_{12} & 0 \end{bmatrix}, \quad (3)$$

for some $\alpha_{11} > 0$, $\gamma_{22} > 0$ and $\beta_{12} = \sqrt{\alpha_{11}\gamma_{22}}$.

Proof. The sufficiency is obvious. Now we show the necessity. From Lemma 8 it follows that either (2) or (3) holds for some $\alpha_{11} \geq 0$, $\gamma_{22} \geq 0$ and $\beta_{12} \geq 0$. Suppose that (2) holds. The proof is similar if (3) holds. Since L is a linear transformation that maps \mathcal{C}_2 onto \mathcal{C}_2 it follows that $\alpha_{11}\gamma_{22}\beta_{12} \neq 0$. Thus, we just need to see that $\beta_{12} = \sqrt{\alpha_{11}\gamma_{22}}$. Let $a, c > 0$ and $b = -\sqrt{ac}$. Then

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

is copositive and

$$L \begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} a\alpha_{11} & -\sqrt{ac}\beta_{12} \\ -\sqrt{ac}\beta_{12} & c\gamma_{22} \end{bmatrix}$$

is copositive if and only if $-\sqrt{ac}\beta_{12} \geq -\sqrt{ac\alpha_{11}\gamma_{22}}$, which implies $\beta_{12} \leq \sqrt{\alpha_{11}\gamma_{22}}$.

Suppose that $0 < \beta_{12} < \sqrt{\alpha_{11}\gamma_{22}}$. Let $b = \frac{-\sqrt{ac}\sqrt{\alpha_{11}\gamma_{22}}}{\beta_{12}} < -\sqrt{ac}$. Then,

$$\begin{bmatrix} a\alpha_{11} & b\beta_{12} \\ b\beta_{12} & c\gamma_{22} \end{bmatrix}$$

is copositive and

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

is not copositive. Thus, we conclude that $\beta_{12} = \sqrt{\alpha_{11}\gamma_{22}}$. ■

Corollary 10 *If the linear transformation $L : \mathcal{S}_2 \rightarrow \mathcal{S}_2$ maps \mathcal{C}_2 onto \mathcal{C}_2 then there is a monomial matrix $E \in M_n(\mathbb{R})$ such that $L(A) = EAE^T$ for any $A \in \mathcal{S}_2$.*

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