USING YOUNG CHILDREN'S REAL WORLD TO SOLVE MULTIPLICATIVE REASONING PROBLEMS

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This study focuses on kindergarten children's multiplicative reasoning. The participants were 12 children (5-6-year-olds) from Viseu, Portugal. Pre- and post-tests were used to assess the effect of an intervention program focused on multiplicative reasoning. The intervention program comprised 12 multiplicative reasoning problems and was carried on in four sessions, during three weeks. Children's performance and arguments were analyzed when solving selective problems of multiplication, partitive and quotitive division. The results suggest that children can succeed in some multiplicative reasoning problems, presenting valid or partially valid arguments, and that their multiplicative reasoning can be improved relying on their informal knowledge.

INTRODUCTION

Children possess informal knowledge relevant for the learning of mathematical concepts. The mathematical ideas children acquire in kindergarten constitute the basis of future mathematical learning. Thus, the development of the mathematical skills in early age is crucial to the success for future learning (NCTM, 2008). In Portugal, the Curricular Guidelines for Pre-School Education (Silva, Marques, Mata & Rosa, 2016) emphasize the importance of mathematics, in everyday life as in the structuring of the child's thinking, with a special focus on problem solving. In practice, it can be said that solving problems enables the development of thinking skills and stimulates a creative search for solutions to everyday problems. Children involvement in resolution of tasks and problem solving that allow different strategies, improve their mathematical reasoning (NCTM, 2017).

Concerning quantitative reasoning, literature reveals children's difficulty establishing a multiplicative reasoning, and the long period of time that is necessary to develop the ideas involved on it (see Vergnaud, 1983; Clark & Kamii, 1996; Sullivan, Clarke, Cheeseman & Mulligan, 2001; Siemon, Breed & Virgona, 2005), contrasting with the relatively short time that is required to develop additive reasoning. However, there is evidence that many children have already an informal knowledge that allows them to solve some multiplicative reasoning problems (see Becker, 1993; Frydman & Bryant, 1994; Nunes et al., 2007). Children can use their informal knowledge to analyse and solve simple addition and subtraction problems before they receive any formal instruction on addition and subtraction operations (Nunes & Bryant, 1996). But they can also know quite a lot about multiplicative reasoning when they start school (Nunes &

Bryant, 2010). Here, some research results are presented from a study focused on kindergarten children's multiplicative reasoning, in Portugal.

THEORETICAL FRAMEWORK

Numbers are used to represent quantities and to represent relations. Nunes and Bryant (2010) argue that when numbers are used to represent quantities they are the result of a measurement operation from which a quantity can be represented by a number of conventional units (e.g., 3 children, 4 chairs). When a number is used to represent relations, the number does not refer to a quantity but to a relation between two quantities, expressing how many more or fewer (e.g., there is 1 more chair than children). In mathematics children are expected to be able to attribute a number to a quantity, which is measuring (Nunes & Bryant, 2010), but they also are expected to be able to quantify relations. When quantities are measured, they have a numerical value, but it is possible to reason about the quantities without measuring them. In agreement with Nunes, Bryant and Watson (2010), it is crucial for children to learn to make both connections and distinctions between number and quantity. Quantitative reasoning results from quantifying relations and manipulating them (Nunes & Bryant, 2010). Quoting Nunes and Bryant (2010), "[…] quantifying relations can be done by additive or multiplicative reasoning. Additive reasoning tells us about the difference between quantities; multiplicative reasoning tells us about the ratio between quantities." (p.8). In literature additive reasoning is associated to addition and subtraction and multiplicative reasoning is associated to multiplication and division problems (see Nunes & Bryant, 1996; Vergnaud, 1983).

The fact that children learn about addition and subtraction before multiplication and division maintains the idea that multiplicative reasoning is accessible to children only when they already master additive thinking. This idea supports the notion of an additive phase predictive of multiplicative reasoning (Piaget & Inhelder, 1975; Hart, 1981; Karplus, Pulos & Stage, 1983). Piaget and Inhelder (1975) argued that there should be any superior qualitative transformation in children's thinking to understand and perform such complex operations as multiplication and division. Moreover, because some multiplicative problems can be solved with additive strategies such as repeated addition, it has preserved the idea that multiplicative reasoning depends totally on the additive reasoning, so, this should be consolidated first. However, understanding multiplication as a complicated form of addition is a very reductive way of realizing multiplicative reasoning.

In spite of his undoubted contribution to research, more recently research has been giving evidence of a different position. Thompson (1994), Vergnaud (1983) and Nunes and Bryant (2010) support the idea that additive and multiplicative reasoning have different origins. Vergnaud (1983), in his theory of conceptual fields, distinguishes the field of additive structures and the field of multiplicative structures, considering them as sets of problems involving operations of the additive or the multiplicative type. Vergnaud (1983) argues that “multiplicative structures rely partly on additive structures; but they also have their own intrinsic organization which is not reducible to additive aspects” (p.128). Nunes and Bryant (2010) also consider that additive and multiplicative reasoning have different origins, arguing that “Additive reasoning stems from the actions of joining, separating and placing sets in one-to-one correspondence. Multiplicative reasoning stems from the action of putting two variables in one-to-many correspondence (one-to-one is just a particular case), an action that keeps the ratio between the variables constant.” (p.11).

Multiplicative reasoning involves two (or more) variables in a fixed ratio. Thus, problems such as: “Joe bought 5 sweets. Each sweet costs 3p. How much did he spent?” Or “Joe bought some sweets; each sweet costs 3p. He spent 30p. How many sweets did he buy?” are examples of problems involving multiplicative reasoning. The former can be solved by a multiplication to determine the unknown total cost; the later would be solved by means of a division to determine an unknown quantity, the number of sweets (Nunes & Bryant, 2010).

Research has been giving evidence that children can solve multiplication and division problems of these kinds even before receiving formal instruction about multiplication and division in school. For that they use the schema of one-to-many correspondence. Carpenter, Ansell, Franke, Fennema and Weisbeck (1993), reported high percentages of success when observing kindergarten children solving multiplicative reasoning problems involving correspondence 2:1, 3:1 and 4:1. Nunes et al. (2005) analysed primary Brazilian school children performance when solving multiplicative reasoning problems. When children were shown a picture with 4 houses and then were asked to solve the problem: “In each house are living 3 puppies. How many puppies are living in the 4 houses altogether?”, 60% of the 1st-graders and above 80% of the children of the other grades succeeded. When children were asked to solve a division problem, such as: “There are 27 sweets to share among three children. The children want to get all the same amount of sweets. How many sweets will each one get?”, the levels of success for 1st-graders was 80% and above that for the other graders (2nd to 4th-graders).

In Portugal, there is still not much information about kindergarten children understanding of multiplicative reasoning, relying on their informal knowledge. This study focuses on children’s ideas when solving multiplicative reasoning problems. It tries to address three questions: 1) How do children perform when solving multiplication, partitive and quotitive division problems? 2) What arguments children present to justify their resolutions?

METHODS

An intervention program was conducted with 12 kindergarten children (5-years-old, n=6; 6-years-old, n=6), from a public supported kindergarten in Viseu, Portugal. These children belong to an economic middle class group. Pre- and Post-tests were used to identify changes on children's understanding during the intervention. The study integrates a wider research program conducted by Soutinho (2016).

Individual interviews were used in the Pre- and Post-tests, and were conducted in a separate room in the Kindergarten, prepared for it. In each interview children solved 28 problems (18 additive structure problems; 6 multiplicative structure problems; 4 control problems). The problems presented in the interview followed an established order, and was the same for all children. Due to the higher number of problems, each child was interviewed in two different moments, during two straight days. The same procedure was use with all the children.

The problems presented to the children were selected and adapted from Vergnaud's classification (see Vergnaud, 1982, 1983). The problems of both tests were similar. The additive structure problems presented to children in the tests comprised: i) composition of two measures; ii) transformation liking two measures, with the starting and element of transformation omitted, (2 for addition, 2 for subtraction); iii) static relation linking two measures, (2 involving "more than", 2 for "less than"). The multiplicative structure problems in the tests comprised: iv) Isomorphism of Measures, selecting the problems of Multiplication, Partitive Division, and Quotitive Division. The control problems included only geometry tasks (geometric regularities, shape with tangram). The problems presented to the children in each test comprised two problems of each type. Tables 1 and 2 give, respectively, some examples of problems of additive and multiplicative structures presented to the children in the Pre- and Post-tests.

Type of problem	Examples of problems of additive reasoning structures
Composition of two measures	Mary has 8 dolls but only 2 are in the box. How many dolls are outside the box?
Transformation liking two measures	Bill had 7 marbles. He gave some to Paul and now Bill has only 4. How many marbles did Bill give to Paul?
	There are 5 frogs in the lake. Some more join the group. Now there are 8 frogs. How many frogs came to join the group?
Static relation linking two measures	Anna has 4 puppies. John has 2 more than Anna. How many puppies does John have?
	Mary has 5 bananas and 2 strawberries. How many strawberries are there less than bananas?

Table 1: Examples of problems presented to the children in Pre- and Post-tests.

Type of problem	Examples of problems of multiplicative reasoning structures
Partitive division	Sara has 10 candies to give to 5 children. She is doing it fairly. How many candies is each child receiving?
Multiplication	Bill has 3 boxes with pencils. Each box has 4 pencils. How many pencils does Bill have in total?
Quotitive division	The teacher Anna has 12 children in her group. She wants to seat the children in groups in the tables. Each group must have 4 children. How many tables does teacher Anna need?

Table 2: Examples of problems presented to the children in Pre- and Post-tests.

All the problems were presented to the children by the means of a story, and materials were available to represent the problems. After each resolution, each child was asked “Why do you think so?” in order to reach a better understanding of his/her reasoning. All the information was registered in video. A quantitative analysis of Pre- and Post-tests results was conducted using the Statistical Package for Social Science (SPSS 20.0).

In the intervention, the participants were divided into three groups of four children each, having each the same age and Pre-test results conditions. The intervention took place in the pre-test following week and lasted for 3 weeks. Four sessions were planned, organized by level of difficulty, equal to all the groups. In each session children solved 3 problems, and the same kind of problems was explored twice a week. Each group had the opportunity to discuss and solve the same type of problem 4 times, in a total of 12 problems. The tasks presented to the children, during the intervention comprised 4 partitive division problems, 4 multiplication problems, and 4 quotitive division problems. The problems presented to the children in the intervention program were similar to those of the multiplicative structure problems given in the tests (see Table 2).

The interviewer presented the problems to the children orally by the means of a story. In each session, the interviewer presented each problem to the group and the material related to the context of the problem was available for representation. The children were challenged to solve the problem individually and present his/her response to the group. After each resolution, the interviewer asked questions related to their resolutions in order to gain an insight of children’s reasoning and stimulate their discussion. All the information was video and audio recorded. Qualitative methods were used to analyse children’s interviews when solving the problems.

RESULTS

Children's performance in solving problems

One point was awarded to each child's correct response. Children's performance in solving Pre- and Post-tests problems was analysed to understand the effects of the intervention on the children's performance. Table 3 presents the mean of proportions (and standard deviation) of correct responses for Pre- and Post-tests, according to each type of problem.

	Mean (s.d.)	
Type of Problem	Pre-test	Post-test
Additive Structure	.45 (.22)	.55 (.21)
Multiplicative Structure	.40 (.31)	.61 (.31)
Control	.77 (.23)	.77 (.25)

Table 3: Mean of proportions (standard deviation) of correct responses, in Pre- and Post-tests.

The Wilcoxon's Test reveals that children's performances improved significantly from pre- to post-tests in both the additive structure problems (W = 50.000; $p<.05$), and in the multiplicative structure problems (W = 42.000; $p<.05$). No significant improvements on children's performances were observed regarding problems of control, despite the higher level of success in these kinds of problems. This indicates that the intervention on multiplicative structures problems was effective.

By focusing the attention on multiplicative reasoning problems, it becomes relevant to analyse children's performance when solving multiplication, partitive and quotitive division problems. Figure 1 presents the distribution of percentage of children's correct responses when solving these problems, in Pre- and Post-tests.

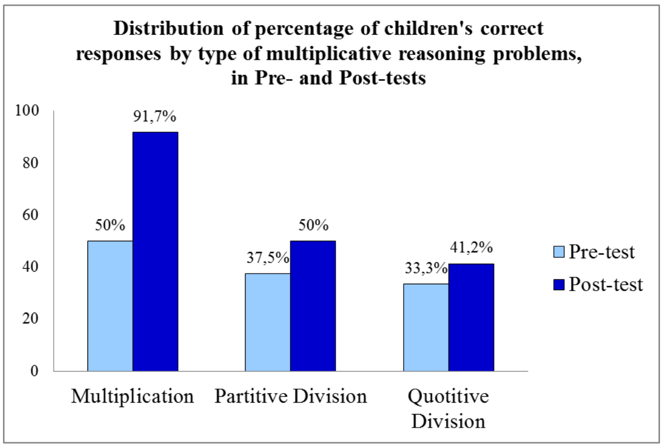

Figure 1: Distribution of percentage of correct answers when solving problems.

The intervention focused on multiplicative reasoning problems seemed to improve children's understanding of multiplication, but also partitive and quotitive division. Regarding the multiplicative structure problems, the Multiplication problems seemed to be easier for children to understand than division problems. Quotitive division problems revealed to be the most difficult ones for children. Nevertheless, some improvements were observed with the intervention. According to Friedman's test, in post-tests there are significant differences between Multiplication and Quotitive division problems ($\chi^2_F(2) = 7.786$; $p<.05$). Friedman's test also revealed that differences between children's performances in Pre- and Post-test are only significant in Multiplication problems, ($W=28.000$; $p<.05$). Thus, this intervention program seemed to be effective for children understanding of multiplicative reasoning problems.

In order to clarify that children's performance was not reached by chance when solving the multiplicative reasoning problems, their arguments were analyzed as they were always challenged to explain their answers.

Children's arguments after solving the problems

After solving each problem in Pre- and Post-tests, children's verbal explanations were required when asked "Why do you think so?". An analysis of children's arguments was conducted among those who solved the problems correctly, in order to have an insight of their reasoning when solving the tasks. Four categories of arguments were distinguished when solving the multiplicative structure problems: valid argument (V), comprising an explanation that articulates correctly all the quantities involved in the problem; partially valid argument (PV), comprising an explanation in which a child attends only to part of the quantities of the problem, producing an incomplete argument; no argument (NA), comprising expressions such "I don't know", and the absence of an argument; and invalid argument (I), comprising an explanation that could not be understood or is decontextualized from the problem. Table 4 summarizes the frequency of type of argument given by the children when solving multiplicative reasoning tasks correctly.

Type of argument	Pre-test (%)	Post-test (%)
Valid	48.3	50
Partially Valid	20.7	13.6
Invalid	27.6	25
No argument	3.4	11.4

Table 4: Type of arguments presented for correct responses, in Pre- and Post-test.

Many children presented valid arguments in the explanations of their correct resolutions, revealing an understanding of the problem. In many cases, the partially valid arguments were presented using material, representing the situation correctly, in spite of the difficulty in the verbal communication. To present an explanation is not a simple task for young children; some children were able to solve the problems correctly providing no explanation at all. This can be explained by the difficulty that some children have in putting into words their reasoning. This difficulty was expected among children of ages 5 and 6 as, according to Piaget (1977), they have to reflect upon their action. With the intervention, children seemed to become more aware of process and explanation, being more prone to be quieter, giving no explanation, than to give an answer that was not compatible with their procedure. The decrease of invalid arguments and the increase of valid arguments after the intervention discards the possibility of the success in problem solving be achieved randomly.

Also children's explanations in the intervention sessions when solving the multiplicative structure problems revealed some improvement on their way of thinking. The following Transcript gives evidence of children arguing solving the multiplication problem "Bill has 3 bicycles without wheels in his garage. Each bicycle must have 2 wheels. How many wheels does Bill need to fix all the bicycles?". Figure 2 shows children presenting their arguments solving the multiplication problem.

Child 1, 3: Two.

Child 2: Six.

Child 4: [Remain in silence.]

Child 1: It's 2! [Argues while getting 2 wheels to represent it.]

Researcher: There are 2 wheels in each bicycle... Show me why do you think so?

Child 2: I got 2 [shows it with paper material] and it is only for 1, got more and it is for 2 [takes more 2 paper wheels], got 2 more and is for 3 [takes 2 more paper wheels].

Researcher: So, how many wheels do you need?

Child 1,2,3: Six!

Child 4: It's 4… No… Two are for 1, and more wheels are for another [put it below the previous ones], and these are for bicycle 3 [put them below the last ones].

Figure 2: Children presenting their arguments when solving a multiplication problem.

In many problems, the material was mostly used by children not to solve the situations but to explain their resolutions.

FINAL REMARKS

This study explores the effects of a short intervention program focused on multiplicative reasoning on young children solving problems of additive and multiplicative structure. The intervention was effective as children improved their understanding of multiplicative reasoning problems. Multiplication problems revealed to be easier for children than division ones. Also children's arguments revealed improvements. Young children provided arguments and explanations that sustain the idea that their successful resolutions were not obtained randomly.

Previous research carried out with kindergarten children solving multiplicative reasoning problems (see Carpenter et al., 1993) reports levels of success, but does not refer to children's explanations or arguments to give a better insight of children's way of thinking. Also Nunes et al. (2005) report remarkable success levels when 1st-graders solve multiplication and division problems, but give no reference to their explanations. The study reported here gives evidence that young children can reach success levels when solving multiplication and division problems, relying on their informal knowledge, presenting arguments that show that they are able to establish the correct reasoning when solving the tasks, articulating properly all the quantities involved in the given problems.

This study suggests that children's multiplicative reasoning can be enhanced when they can experience problem solving being able to interact with peers and discuss their ideas, after receiving some prompts from teacher, and the problems are presented by means of stories. It also suggests that both additive and multiplicative reasoning, in their simplistic forms, seem to be simultaneously reachable to kindergarten children, making sense to them. Thus, perhaps kindergarten mathematics should include more of these experiences in order to develop children's mathematical reasoning. When problems are presented to young children through a story connecting them to the children's real world, the mathematics make sense for children.

References

Carpenter, T., Ansell, E., Franke, M., Fennema, E., & Weisbeck, L. (1993). Models of problem solving: A study of kindergarten children's problem-solving processes. *Journal for Research in Mathematical Education, 24*, 428-441.

Clark B. & Kamii, C. (1996). Identification of Multiplicative Thinking in Children in Grades 1–5. *Journal for research in Mathematics Education* 27, 41–51.

Frydman, O. & Bryant, P. (1994). Children's understanding of multiplicative relationships in the construction of quantitative equivalence. *Journal of Experimental Child Psychology, 58,* 489-509.

Hart, K. (1981). *Children's understanding of mathematics:*11-16. London: John Murray.

Karplus R., Pulos S., & Stage E. (1983). Early adolescents proportional reasoning on "rate" problems. *Educational Studies in Mathematics 14*, 219–233.

NCTM. (2008). *Princípios e normas para a matemática escolar (Principles and Standards for School Mathematics)* (2.ª ed.). Lisboa: APM.

NCTM. (2017). *Princípios para a ação: assegurar a todos o sucesso em matemática (Principles for Action: Ensuring Success in Mathematics for All).* Lisboa: APM.

Nunes, T. & Bryant, P. (2010). Understanding relations and their graphical representation. In T. Nunes, P. Bryant & A. Watson (Eds), *Key understanding in mathematics learning*. (Accessed on 20 of April, 2011, http://www.nuffieldfoundation.org/sites/defaukt/files/P4.pdf.

Nunes, T., Bryant, P., Evans, D., Bell, D., Gardner, S., Gardner, A., & Carraher, J et al. (2007). The contribution of logical reasoning to the learning of mathematics in primary school. *British Journal of Developmental Psychology, 25*, 147-166.

Nunes, T.; Campos, T; Magina, S. & Bryant, P. (2005). *Educação matemática – Números e operações numéricas (Mathematical Education - Numerals and Numerical Operations)*. São Paulo: Cortez Editora.

Piaget, J. & Inhelder, B. (1975). *The Origin of the Idea of Chance in Children*. London: Routledge and Kegan Paul.

Siemon, D., Breed, M., & Virgona, J. (2005). *From additive to multiplicative thinking – the big challenge of the middle years*. Retrieved from www.education.vic.gov.au/studentlearning/teachingresources/maths/

Silva, I., Marques, L., Mata, L. & Rosa, M. (2016). *Orientações Curriculares para a Educação Pré-Escolar (Curricular Guidelines for Pre-School Education).* Lisboa: Ministério da Educação/DGE.

Soutinho, F. (2016). *A compreensão dos problemas de estrutura aditiva e estrutura multiplicativa por crianças do pré-escolar (The understanding of additive and multiplicative structure problems of preschool children).* (*PhD Thesis*, University of Minho). Braga: UM. http://hdl.handle.net/1822/41590.

Stein, M., Smith, M., Henningsen, M., & Silver, E. (2009). *Implementing Standards-Based Mathematics Instruction: A Casebook for Professional Development, 2nd ed.* New York: Teachers College Press.

Sullivan, P., Clarke, D. M., Cheeseman, J., & Mulligan, J. (2001). Moving beyond physical models in learning multiplicative reasoning. In M. Van Den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 233–240). Utrecht, The Netherlands: PME.

Vergnaud, G. (1983). Multiplicative structures. In R. Resh, & M. L. (Orgs.), *Acquisition of mathematics concepts and processes* (pp. 127-174). New York: Academic Press.

Vergnaud, G. (1982). A classification of cognitive tasks and operations of thought involved in addition and subtraction problems. In T. P. Carpenter, J. M. Moser, & T. A. (Eds.), *Addition and subtraction: A cognitive perspective* (pp. 60-67). Hillsdale, NJ: Lawrence Erlbaum Ass.