# YOUNG CHILDREN CAN LEARN TO REASON AND TO NAME FRACTIONS

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*This study investigates the effects of a teaching intervention on children's reasoning and labelling of fractions in Quotient, Part-whole and Operator situations. A Pre-test, Intervention and Post-test design was used with 37 six- to seven-year-olds from Primary schools in Braga, Portugal. The children had not been taught about fractions in school. Reasoning and labelling questions were presented in the three situations in the Pre- and Post-test. During teaching, each intervention group learned about fractions in only one of the three situation. Children who were taught in the Quotient situation made significant progress in the reasoning and naming fractions; Children taught in the Part-whole or in the Operator situations only learned how to label fractions.*

## INTRODUCTION

Fractions can be used to represent quantities in different types of situation. The aim of this study was to investigate the impact of the situation in which fractions are taught on children's learning. Three types of situation were included: Quotient, Part-whole and Operator. In quotient situations, a/b represents the relation between a number of items shared equally among b number of recipients (e.g., 2/3 represents 2 chocolate bars shared fairly by 3 children); a/b also represents the quantity received by each recipient (e.g., 2/3 represents the amount of chocolate received by each child). In part-whole situations, a/b represents the relation between *b*, the number of equal parts in which the whole is divided, and *a*, the number of these parts taken (e.g., 2/3 of a chocolate bar means that the bar was divided into 3 equal parts and 2 of these parts were taken). In operator situations, which involve a set of discrete items taken as a whole, *b* indicates the number of equal groups into which the set was divided and *a* is the number of groups taken (Nunes & Bryant, 2008).

Quotient situations involve sharing (Streefland, 1997; Mamede, Nunes & Bryant, 2005), where the denominator and the numerator of a fraction involves variables of distinct nature, recipients and items being shared, respectively (Nunes et al, 2007). Part-whole situations involve dividing continuous quantities into equal parts, and the denominator and the numerator involve variables of the same nature (Nunes et al, 2007), respectively the number of equal parts into which the whole was cut and the number of those parts taken. Fractions in operator situations also involve variables of same nature, the denominator and the numerator refer to the number of equal groups initially made and the number of groups taken, requiring the child to ignore the number of elements of each group. Although quotient, part-whole and operator situations may seem very similar to an adult, they may be perceived as quite different by children.

## FRAMEWORK

Previous work (Correa, Nunes & Bryant, 1998; Kornilaki & Nunes, 2005) on children's understanding of division has shown that children aged 6 and 7 understand that, the larger the number of recipients, the smaller the part that each one receives. So in sharing situations, they display some informal knowledge and are able to order the values of the quotient. It should be noted that these studies were carried out with divisions where the dividend was larger than the divisor. In the present study, all situations involve dividends that are smaller than the divisor. So it is necessary to see whether the children will still understand the inverse relation between the divisor and the quotient when the result of the division would be a fraction. The equivalent insight in part-whole situations - the larger the number of parts into which a whole was cut, the smaller the size of the parts (Behr et al., 1984), has not been documented in children of this age. Research is needed to know more about how do young children understand this inverse relation in situations where the divisor is larger than the dividend, when they do not have to deal with it numerically, but only make a judgement, similar to those required by Correa et al. and Kornilaki and Nunes in quotient situations.

There is little information regarding equivalence in quotient situations but Empson (1999) found some evidence for children's use of ratios with concrete materials when children aged 6 and 7 years solved equivalence problems. Concerning part-whole situations, Piaget, Inhelder and Szeminska (1960) found that children in this age level understand the equivalence between the sum of all the parts and the whole and some of the slightly older children could understand the equivalence between parts - 1/2 and 2/4 - if 2/4 was obtained by subdividing 1/2. Different informal strategies have been identified (drawing and shading, using knowledge from money situations) by other researchers but these were observed at later ages, after children had already received instruction on fractions.

Previous research on children's informal knowledge (Empson, 1999) shows that children aged 6 and 7 found it difficult to understand the operator concept, but this difficulty is reduced after receiving instruction. Research with older children, who received instruction on fractions (Behr et al., 1984; Post et al., 1985), shows that for some children the operator concept is still difficult. However, these studies were not focusing on children's informal knowledge and do not compare children's reactions across situations.

Thus, one still needs to investigate children's understanding of equivalence and ordering of quantities represented by fractions in distinct situations, before being taught about it in school. Although there are some studies on informal knowledge, systematic and controlled comparisons between the quotient, part-whole and operator situations have not been carried out. These situations may seem very similar to an adult, but it is hypothesized that they are perceived as quite different by children as the meaning of numerator and denominator varies across situations. Thus it is predicted that, if children learn about fractions in one type of situation, they will not transfer easily what they have learned to the other two types of situation.

Literature presents some studies on the effects of situations in which fractions are used on children's understanding. Previous research shows that children perform differently in parallel items presented in the context of quotient and part-whole situations. For example, 8- and 9-year-old British children answered items about fraction equivalence in quotient and part-whole situations; when comparing 1/2 and 2/4, the rate of correct responses was 35% in part-whole and 66% in quotient situations (Nunes et al., 2007). Similar results were found amongst Portuguese children aged 6-7 years: when ordering fractional quantities, 42% of the answers were correct in part-whole and 61% in quotient situation; in equivalence items, 14% correct answers were observed in part-whole and 22% correct answers in quotient situations (Mamede, Nunes & Bryant, 2005). In another survey Nunes and Bryant (2008) asked to 318 Year 4 and 5 pupils to judge whether the fractions 1/3 and 2/6 were equivalent, or not. The items were presented simply as numbers, without a context, in the context of part-whole situations, and in the context of quotient situations. Pupils were most successful in quotient situations (68% correct), followed by part-whole situations (41% correct) followed by numerical problems without context (39% correct). Similar results were obtained in a study with 8- and 9-year-olds in England, who had been taught about fractions in part-whole situations and attained 40% (8-year-olds) and 74% (9-year-olds) correct responses in part-whole problems; their rates of correct responses to the quotient questions were 71% and 83% (Nunes & Bryant, 2011).

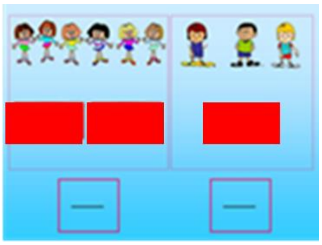
In Brazil, Campos, Magina, Canova and Silva (2012) compared the impact of intervention sessions focused on fractions in quotient, part-whole, operator and intensive quantities on 138 Brazilian 3rd and 4th-graders. The authors refer that students of the quotient situation intervention group registered the higher improvement. More recently, Canova (2013) analysed the effect of a teaching experiment, comprising reasoning and naming fractions tasks with part-whole and quotient intervention groups, involving 378 fourth- to sixth-graders from Brazilian primary schools. The quotient intervention group performed better on the reasoning fractions problems, and the part-whole intervention group performed better in the naming of fractions.

These results strongly support the significance of the distinction between quotient and part-whole situations for educational practices. However, previous studies did not investigate the consequences of teaching and learning about fractions in these different situations; teaching had been done in schools without the researchers' interference. The present study analyses the effects of teaching children about fractions in each of these types of situation in comparison to the others. It is hypothesized that what children learn about fractions is at first connected to the situation in which they were taught. If the situations are truly distinct from the children's perspective, their newly acquired knowledge will be situated rather than generalized. Thus further teaching and experiences with fractions would be required to allow for a more general understanding of fractions that can be used in a variety of situations.

## METHODS

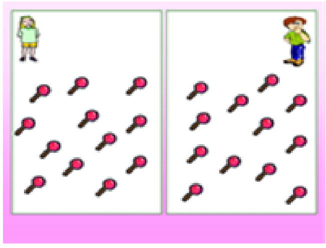
Participants were 37 six and seven-year-olds (mean age 6.6 years) from two state supported primary schools, in Braga, Portugal. According to the information given by the teachers, the children had not received formal instruction on fractions at school. This study was carried out with un-instructed children, otherwise the results would be influenced by the type of instructions that they had received. In Portugal, the children contact with equal sharing activities in the 2$^{nd}$ grade (7- to 8-years-old) and were formally introduced to fractions in the 3$^{rd}$ grade, and part-whole and operator situations were the most common ones to explore fractions in the 3$^{rd}$ and 4$^{th}$ grades.

Pre- and Post-tests, administered individually, were used to assess whether there was progress after the intervention. These tests comprised 12 reasoning items, involving equivalence or ordering fractions, presented in each type of situation – quotient (Qt), part-whole (Pw) and operator (Op) - without the use of fraction labels. Figure 1 gives an example of an equivalence problem presented in the Pre- and Post-tests.

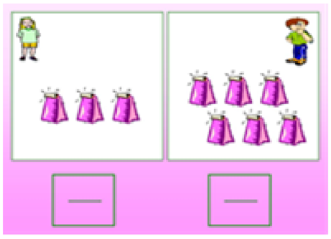
| Type of situation | Example |
|---|---|
| Quotient  | Three boys are going to share 1 chocolate bar fairly. Six girls are going to share 2 chocolate bars fairly. Does each boy eat more chocolate than each girl? Does each girl eat more chocolate than each boy? Or do the boys and girls eat the same amount of chocolate? Circle the one that you think that ate more or both if they ate the same. Explain your answer. Write in the box a number to show how much chocolate each girl (each boy) is going to eat. |
| Part-whole  | Betty and Ruth have each a chocolate bar. But as they are not very hungry, they decide not to eat all the chocolate bar at once. Betty divides hers into 3 equal parts and eats 1 part; Ruth divides hers into 6 equal parts and eats 2 parts. Does Betty eat more chocolate than Ruth? Does Ruth eat more chocolate than Betty, or are they eating the same amount of chocolate? Circle |

the one that you think that ate more or both if they ate the same. Explain your answer. Write in the box a number to show how much chocolate each girl is going to eat.

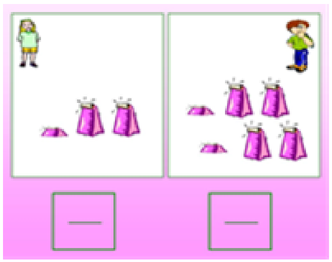
Operator



1. Anna and Phil have each 12 sweets (first slide).



2. Anna decided to share hers into 3 equal bags, with the same number of sweets in each; Phil shares his into 6 equal bags, all with the same number of sweets (second slide).



3. Anna eats 1 bag of sweets and Phil eats 2 bags (third slide). Does Anna eat more sweets than Phil, does Phil eat more sweets than Anna, or do they eat the same number of sweets? Circle the one that you think that ate more or both if they ate the same. Explain your answer. Write in the box a number to show how much chocolate each one is going to eat.

Figure 1: Examples of an equivalence problem of the Pre- and Post-tests

After solving the reasoning questions, the children were also asked to name the 12 pairs of fractions in each of these situations. Fractional language is relatively rare in Portuguese culture in everyday life. The most common fraction in everyday language is “metade” (half), but is often used to refer to a division in two parts without rigor in the equality of parts. So in order to examine whether children can adopt fractions signs in writing and oral language more easily in one type of situation than another, the children received a brief explanation of how to use fractional representation and then were assessed on their ability to use these representations.

The same fractions were used across the different situations making it possible to compare the children’s performance on reasoning and naming problems in each situation. Table 1 presents the pairs of fractions used in the problems of equivalence and ordering of quantities represented by fractions in the Pre- and Post-tests.

| Pre-Test | | Post-Test | |
|---|---|---|---|
| Equivalence | Ordering | Equivalence | Ordering |
| 1/3 ; 2/6 | 1/3 ; 1/4 | 1/3 ; 2/6 | 1/3 ; 1/4 |
| 1/2 ; 2/4 | 1/3 ; 1/6 | 1/2 ; 2/4 | 1/3 ; 1/6 |
| 3/5 ; 6/10 | 2/3 ; 2/9 | 3/5 ; 6/10 | 2/3 ; 2/9 |
| 2/3 ; 4/6 | 2/5 ; 2/10 | 2/3 ; 4/6 | 2/5 ; 2/3 |
| 1/2 ; 3/6 | 3/4 ; 3/6 | 1/4 ; 2/8 | 3/4 ; 3/8 |
| 3/6 ; 6/12 | 4/5 ; 4/10 | 3/4 , 6/8 | 4/5 ; 4/10 |

Table 1: Fractions used in the problems of equivalence and ordering of fractions in each condition of study for Pre- and Post-tests.

Children were randomly assigned to learning in one of the three situations – Quotient (Qt), Part-whole (Pw), or Operator (Op) intervention – or to a Comparison group, who solved multiplication and division problems with whole numbers.

Eight groups of 5 children (one of them with 3 children) participated in two teaching sessions of about 35 minutes each. These teaching sessions took place outside the classroom, in a small room of their school. Thus, no changes on the curriculum were provided due to this intervention.

In the first session, the children received instruction on how to label fraction in their working situation, and then they had to solve 2 problems of labelling and 2 of ordering of fractions. In the instruction sessions on how to label fractions, the unitary fractions up to 1/5 and the non-unitary fractions 2/3 were used. After being taught to label the fractions, the children were asked to name the fractions in the subsequent labelling and ordering problems, and their answers were discussed in the group by the researcher. In the second session, the children had to solve 2 problems of equivalence of fractions. Table 2 summarizes the fractions involved in the intervention sessions when solving reasoning and naming problems.

| Naming | Ordering | Equivalence |
|---|---|---|
| 3/7 | 1/2; 1/3 | 2/3; 4/6 |
| 5/8 | 2/3; 2/4 | 3/4; 6/8 |

Table 2: The fractions involved in the problems used in the intervention sessions.

All problems were presented using an approach similar to the test items exemplified in Figure 1, in which the researcher showed the children an illustration while presenting the problem orally, and the children had a booklet with the same illustration in which they could write or draw as they wish.

The researcher presented the problem and then explained the question; each child answered in their own booklet. For the problems of labelling, each child had to write down the answer; for the problems of reasoning, they had to judge about the equivalence and ordering of fractions individually, drawing a circle around those that they considered to be having/eating more. When all the children had finished and all the answers were written dawn, each child had to say why they answered so. Finally, the researcher discussed their answers with the children of the group.

No judgements were made by the researcher whose role was to pose the questions, create opportunities for the children to present their individual responses to the group, and manage the group discussion.

## RESULTS

One point was awarded for each child’s correct response, the maximum score on reasoning problems of fractions is 12. Table 3 presents the means and standard deviations for accuracy on reasoning items in each situation by testing occasion. The means are separated by intervention group. At Pre-test (Table 3), all children performed better on reasoning problems presented in quotient situations, irrespective of the group to which they were later assigned. There were almost no correct responses to reasoning problems presented in part-whole or operator situations. At Post-test, children in the Quotient Intervention Group improved in accuracy in the quotient reasoning items, but no other improvement in reasoning is noticeable.

| | Reasoning problems (Maximum score = 12) | | | | | |
|---|---|---|---|---|---|---|
| | Pre-test | | | Post-test | | |
| | Qt | Pw | Op | Qt | Pw | Op |
| Qt (n=10) | 5,6 (3,3) | 0 | 0 | 8,6 (3,1) | 0 | 0 |
| Pw (n=10) | 2,7 (3,4) | 0,1 (0,3) | 0 | 3,0 (3,7) | 0,6 (1,9) | 0 |
| Op (n=10) | 2,5 (2,6) | 0 | 0 | 3,8 (3,7) | 0 | 0 |
| Control (n=7) | 3,0 (3,9) | 0,29 (0,8) | 0,43 (1,1) | 3,0 (4,5) | 1,57(4,2) | 1,71 (4,5) |

Table 3: Mean accuracy (standard deviations in brackets) by Testing Occasion on Reasoning Items in Each Situation by Intervention Group.

In the naming problems, one point was awarded to each fraction correctly named. The total score of naming problems ranged from 0 (minimum) to 24 (maximum). At Pre-test, no child was able to label a fraction correctly but there are improvements in the children's accuracy in labelling items in the post-test (Table 4).

| Labelling problems (Maximum score = 24) | | | |
|---|---|---|---|
| | Post-test | | |
| | Qt | Pw | Op |
| Qt (n=10) | 8,6 (3,1) | 0 | 0 |
| Pw (n=10) | 3,0 (3,7) | 0,6 (1,9) | 0 |
| Op (n=10) | 3,8 (3,7) | 0 | 0 |
| Control (n=7) | 3,0 (4,5) | 1,57(4,2) | 1,71 (4,5) |

Table 4: Mean accuracy (standard deviations in brackets) by Testing Occasion on Labelling Items in Each Situation by Intervention Group.

The improvements are selective: children in the Quotient Intervention Group improve their performance in naming fractions in Quotient situations being able to name more than half of the fractions presented in Part-whole and Operator situations. The children in the Part-whole and Operator intervention groups improve their accuracy in naming fractions in both types of situation and are able to transfer these learning to name fractions among Part-whole and Operator situations. Nevertheless, the Part-whole and Operator intervention groups find more difficult to name fractions presented in Quotient situations.

In view of the floor effects in pre- and post-test accuracy scores in reasoning items in Part-whole and Operator situations, it was only possible to analyse the effect of the intervention on reasoning items in Quotient situations. In order to analyse whether one type of intervention led to greater improvement than the other on Quotient reasoning items, an ANCOVA was carried out, controlling for the Pre-test. The score for the pre-test Quotient reasoning problems was a factor and Type of Intervention session (Quotient, Part-whole, Operator, Control) was a between–participants factor. The dependent variable was the score for post-test Quotient reasoning problems. The results showed that the covariate predicts significantly the children's performance in solving the Quotient reasoning items ($F(1,32)=86.74$, $p<.001$). There was also an interaction of Quotient reasoning items by Session Intervention Group ($F(3,32)=4.48$, $p<.05$). Post-hoc (Bonferroni) tests revealed that the Intervention Sessions on Quotient Situations significantly increased children's performance compared to both the Part-whole Intervention Session Group, $t(32)=-3.15$, $p< .05$), and the Control Intervention Sessions Group ($t(32)=-319$, $p<.05$), but not with the Operator Intervention Sessions Group ($t(32)=-2.07$, n.s).

As there was no variation in the children's accuracy in naming fractions in the pre-test, only post-test performance can be analysed. A repeated Measures ANOVA was carried out, with naming problems as a repeated measure in Post-test and the Type of Intervention Group as a between participants factor. There is an interaction between the type of Group Intervention and the situation to name fractions, (F(6, 66) = 36,37, p<.001); Children in the Quotient Intervention Group performed better on naming problems presented in Quotient situations than those of the Part-whole or Operator Intervention Groups (p<.001), but weaker on problems presented in Part-whole or in Operator situations; on naming problems in Part-whole situations, the children of both Part-whole Intervention Group (p<.001) and Operator Intervention Group (p<.001) performed better than the Control and Quotient Intervention Groups; on naming problems on Operator situations, children of both Part-whole (p<.001) and Operator Intervention Groups (p<.001) also performed similarly and better than Control and Quotient Intervention Groups.

Thus, the type of situation in which fractions are used to present the problems to children affects differently children's reasoning and naming of fractions.

## FINAL REMARKS

The findings of this study show that some changes occurred with the teaching experiment in which the children were introduced to fractions, in each type of situation analysed. The children who were introduced to fractions in Quotient situations improved their performance on reasoning problems, involving equivalence and ordering, revealing some understanding of the inverse divisor-quotient relation. This understanding was also found previously in the literature (see Mamede, Nunes & Bryant, 2005), when fractions were introduced to young children, but also when comparing fractions problems were solved by older children in Quotient situations (see Nunes & Bryant, 2008; Canova, 2013). Contrasting with these findings, the children who were introduced to fractions either in Part-whole or Operator situations did not show improvement with the instruction sessions when solving reasoning problems. These findings suggest that Part-whole and Operator situations are very difficult situations for the children to attend to all the dimensions involved in the problem.

It is concluded that learning in Quotient situations was more effective, as the children progressed both in reasoning and naming items, but it was situated: there was no transfer. In contrast, learning in Part-whole and Operator situations was limited, as there was no progress in reasoning, but the use of fraction labels was generalized between the two situations.

Teaching about fractions in many countries is often done in part-whole and operator situations, with emphasis on learning to name fractions. Children easily learn to name fractions in specific situations, so it is easy to believe that they understand the reasoning underlying this new numerical form. This study underscores the limitations of teaching in these situations and the need to combine different situations in teaching fractions, as each of them has strengths and weaknesses.

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