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Double warped space–times

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An invariant characterization of double warped space–times is given in terms of Newman–Penrose formalism and a classification scheme is proposed. A detailed study of the conformal algebra of these space–times is also carried out and some remarks are made on certain classes of exact solutions. © *2003 American Institute of Physics.* [DOI: 10.1063/1.1605496]

I. INTRODUCTION

Given two metric manifolds (M_1,h_1) and (M_2,h_2) and given two smooth real functions $\theta_1:M_1\to\mathbb{R}$, $\theta_2:M_2\to\mathbb{R}$ (*warping functions*), one can build a new metric manifold (M,g) by setting $M=M_1\times M_2$ and

$$g=e^{2\theta_2}\pi_1^*h_1\otimes e^{2\theta_1}\pi_2^*h_2, \tag{1}$$

where π_1,π_2 above are the canonical projections onto M_1 and M_2, respectively, and will be omitted where there is no risk of confusion (thus writing, from now on: $g=e^{2\theta_2}h_1\otimes e^{2\theta_1}h_2$). One such structure will be called *double warped product manifold*, and gives rise to the so-called *warped product manifold* whenever one of the warping functions is constant, see Refs. 1 and 2.

If $\dim M_1+\dim M_2=4$ and g has Lorentz signature [i.e., one of the manifolds (M_i,h_i) is Lorentz and the other Riemann], then (M,g) will be referred to as a *double warped space–time*, and again, if one of the warping functions is constant, one recovers the definition of *warped space–time* (see Refs. 3 and 4).

In what is to follow and unless otherwise stated, we shall assume that we are dealing with "proper" double warped space–times (i.e., neither of the warping functions is constant); further, and without loss of generality (M_1,h_1) will be assumed Lorentzian and (M_2,h_2) Riemannian.

The considerations in this work will be mainly local, thus we shall assume that for each $p\in M$ there exists a neighborhood U of p such that there is a coordinate system x^a, $a=0,\ldots,3$ on U *adapted* to the product structure in the sense that the line element associated with g can be written as

$$ds^2=e^{2\theta_2(x^D)}h_{1\ \alpha\beta}(x^\gamma)dx^\alpha\, dx^\beta+e^{2\theta_1(x^\gamma)}h_{2\ AB}(x^D)dx^A\, dx^B; \tag{2}$$

where $x^{\alpha,\beta,\cdots}$ and $x^{A,B,\cdots}$ will designate the coordinates on the submanifolds M_1 and M_2 of M through p, respectively, while n_1 and n_2 denote their respective dimensions; thus, Greek indices will run from 0 to n_1-1 and capital Latin indices from n_1 to 3. Conversely, if a space–time contains an open neighborhood U on which there exists a coordinate system as the one described above, then it will be referred to as *locally double warped space–time*.

The aim of the present paper is to deal with double warped space–times in much the same way as warped space–times were dealt with previously (see Refs. 3 and 4 and references cited

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therein); thus their geometrical properties will be deducted and studied starting from those of their lower dimensional factors (M_i, h_i) which are, in general, much easier to deal with.

The paper is structured as follows: in Sec. II an invariant characterization of these space–times is given, including their characterization in terms of the Newman–Penrose formalism, and a classification scheme is put forward. Section III contains some results on the curvature structure of such space–times, whereas Secs. IV and V deal with their conformal algebra. Finally, in Sec. VI, a few remarks are made on double warped exact solutions.

II. INVARIANT CHARACTERIZATION AND CLASSIFICATION

Starting with the form (2) of the line element, let us re-write it as follows:

$$ds^2 = e^{2(\theta_1(x^\gamma)+\theta_2(x^D))}[e^{-2\theta_1(x^\gamma)} h_{1\ \alpha\beta}(x^\gamma) dx^\alpha dx^\beta + e^{-2\theta_2(x^D)} h_{2\ AB}(x^D) dx^A dx^B], \quad (3)$$

now, the two terms $e^{-2\theta_1(x^\gamma)} h_{1\ \alpha\beta}(x^\gamma)$ and $e^{-2\theta_2(x^D)} h_{2\ AB}(x^D)$ are metrics on the submanifolds M_1 and M_2, say $\hat{h}_1$ and $\hat{h}_2$. The sum of their associated line elements [that is, the expression within the square brackets in (3)], is the line element, say $d\hat{s}^2$ of a decomposable space–time $(M,\hat{g})$ with $M = M_1 \times M_2$ and $\hat{g} = \hat{h}_1 \otimes \hat{h}_2$ (Again, to be correct one should write $\hat{g} = \pi_1^* \hat{h}_1 \otimes \pi_2^* \hat{h}_2$, π_1, π_2 being the canonical projections onto M_1 and M_2, but since there is no risk of confusion, we omit them for the sake of simplicity), thus we have proven:

Lemma 1: A (locally) double warped space–time is always conformally related to a (locally) decomposable space–time, the conformal factor being separable in the coordinates associated with the two factor submanifolds.

In what follows, we shall refer to the factor submanifolds in the decomposable space–time $(M,\hat{g})$ as $(M_1,\hat{h}_1)$ and $(M_2,\hat{h}_2)$, respectively, assuming that $(M_1,\hat{h}_1)$ is Lorentz and $(M_2,\hat{h}_2)$ Riemann; and we shall write the metric of a double warped space–time as $g = \exp(2\theta)\hat{g}$ in the understanding that $\hat{g}$ is the metric of the underlying decomposable space–time and θ separates as the sum of two functions θ_1 and θ_2 on M_1 and M_2, respectively.

Now, the space–time $(M,\hat{g})$ is locally decomposable if its holonomy group is nondegenerately reducible (and globally decomposable if, on top of this, it is simply connected) (see for instance Ref. 5, and references therein), its holonomy type being then R_2, R_3, R_4, R_6, R_7, R_{10}, or R_{13} (see Ref. 6); one then has the following possibilities for $(M,\hat{g})$.[5]

(1) $(M,\hat{g})$ is $1+3$ decomposable if it admits a global, non-null, nowhere zero covariantly constant vector field $\vec{u}$. One then distinguishes between $1+3$ spacelike (holonomy type R_{13}) or $1+3$ timelike (holonomy types R_3, R_6 or R_{10}) depending on the nature of the three-dimensional submanifold orthogonal to the covariantly constant vector field. In a coordinate system adapted to the covariantly constant vector field, say $\vec{u} = \partial_u$, the line element $d\hat{s}^2$ then takes the following forms, respectively:

$$d\hat{s}^2 = -du^2 + \hat{h}_{AB}(x^D) dx^A dx^B$$

or

$$d\hat{s}^2 = \hat{h}_{\alpha\beta}(x^\gamma) dx^\alpha dx^\beta + du^2. \quad (4)$$

If another non-null covariantly constant vector field exists in the space–time, then $(M,\hat{g})$ decomposes further and can be referred to as being $1+1+2$ spacelike (type R_4) or $1+1+2$ timelike (type R_2) in an obvious notation.

(2) $(M,\hat{g})$ is $2+2$ decomposable and then two global, linearly independent recurrent null vector fields exist (holonomy type R_7). This is equivalent to saying that in $(M,\hat{g})$ there exist two linearly independent covariantly constant tensor fields of rank 2, say P and Q such that

$$\hat{g}_{ab} = P_{ab} + Q_{ab}$$

with

$$P_{ab/c}=Q_{ab/c}=0; \tag{5}$$

a stroke denoting covariant derivative in $(M,\hat{g})$; the line element reads in this case

$$d\hat{s}^2=\hat{h}_{1\ \alpha\beta}(x^\gamma)dx^\alpha\, dx^\beta+\hat{h}_{2\ AB}(x^D)dx^A\, dx^B, \tag{6}$$

where $\hat{h}_1$ and $\hat{h}_2$ are two two-dimensional metrics on M_1 and M_2, respectively, such that $\pi_1^*\hat{h}_1 = P$ and $\pi_2^*\hat{h}_2=Q$.

Going back to the double warped space–time (M,g) conformally related to $(M,\hat{g})$ via (3), it appears natural to consider the following two classes of double warped space–times.

Class A whenever the underlying space–time $(M,\hat{g})$ is $1+3$ decomposable. If necessary, and following the same notation as in the case of warped space–times, we shall distinguish between classes A_1 ($1+3$ spacelike) and A_2 ($1+3$ timelike). Taking into account (3) and (4), we shall write the canonical form of the line element of these space–times as

$$ds^2=e^{2(\theta_1(u)+\theta_2(x^D))}[-du^2+\hat{h}_{AB}(x^D)dx^A\, dx^B], \tag{7}$$

$$ds^2=e^{2(\theta_1(x^\gamma)+\theta_2(u))}[\hat{h}_{\alpha\beta}(x^\gamma)dx^\alpha\, dx^\beta+du^2], \tag{8}$$

respectively.

Class B whenever the underlying space–time $(M,\hat{g})$ is $2+2$ decomposable. The canonical form of the line element will be in this case

$$ds^2=e^{2(\theta_1(x^\gamma)+\theta_2(x^D))}[\hat{h}_{1\ \alpha\beta}(x^\gamma)dx^\alpha\, dx^\beta+\hat{h}_{2\ AB}(x^D)dx^A\, dx^B]. \tag{9}$$

In what is to follow and whenever no confusion may arise, we shall put $\theta\equiv\theta_1(x^\gamma)+\theta_2(x^D)$ and write accordingly

$$g_{ab}=e^{2\theta}\hat{g}_{ab},$$

and also

$$ds^2=e^{2\theta}\, d\hat{s}^2. \tag{10}$$

Also, we shall denote the covariant derivatives in (M,g) and $(M,\hat{g})$ by ∇ and $\hat{\nabla}$ or a semicolon (;) and a slash (/), respectively. Further, reference will be often made to conformal Killing vectors and their properties, hence it is in order at this point to recall their definition and basic properties; thus, given an n-dimensional manifold V endowed with a metric g of arbitrary signature, a vector field $\vec{X}$ on V is said to be a *conformal Killing vector* (CKV) iff $\mathcal{L}_{\vec{X}}g=2\phi g$ where ϕ is some function of the coordinates (*conformal factor*) and $\mathcal{L}_{\vec{X}}$ stands for the Lie derivative operator with respect to the vector field $\vec{X}$. The former equation can also be written in an arbitrary coordinate chart as

$$X_{a;b}=\phi g_{ab}+F_{ab} \tag{11}$$

and then, from the Bianchi identities, it follows

$$F_{ab;c}=R_{abcd}X^d-\phi_a g_{bc}+\phi_b g_{ac}, \tag{12}$$

$$\mathcal{L}_{\vec{X}}R_{ab}=-(n-2)\phi_{a;b}-\phi^c{}_{;c}g_{ab}, \tag{13}$$

$$\mathcal{L}_{\vec{X}}R=-2\phi R-2(n-1)\phi^c{}_{;c}, \tag{14}$$

$$\mathcal{L}_{\vec{X}}R_{abcd}=2\phi R_{abcd}-\phi_{a;c}g_{bd}+\phi_{a;d}g_{bc}-\phi_{b;d}g_{ac}+\phi_{b;c}g_{ad}, \qquad (15)$$

where $\phi_a \equiv \phi_{,a}$, a semicolon stands for the covariant derivative with respect to the connection associated with the metric, $F_{ab}=-F_{ba}$ is the so-called *conformal bivector*, and R_{abcd}, R_{ab}, and R stand, respectively, for the components in the chosen chart of the Riemann and Ricci tensors and the Ricci scalar. The special cases $\phi=$constant and $\phi=0$ correspond, respectively, to $\vec{X}$ being a *homothetic vector* (HV) and a *Killing vector* (KV), the associated bivector is then said to be the *homothetic bivector*, or *Killing bivector*, respectively. A CKV is said to be *proper* whenever it is nonhomothetic (i.e., $\phi \neq$const); likewise, we shall use ''proper homothetic'' to designate a HV which is not a KV (i.e., $\phi=$const$\neq 0$). A proper CKV is said to be a *special CKV* (SCKV) whenever its associated conformal factor ϕ satisfies $\phi_{a;b}=0$ in any coordinate chart. Further, it is easy to see that the CKV that (V,g) admits form, under the usual Lie bracket operation, a Lie algebra of vector fields which we shall designate as $\mathcal{C}_r(V,g)$, r being its dimension. Similar statements can be made regarding the SCKV, HV, and KV that (V,g) may admit [Lie algebras $\mathcal{S}_r(V,g)$, $\mathcal{H}_r(M,g)$ and $\mathcal{K}_r(V,g)$ respectively (note from the above considerations it follows that in any given space–time $\mathcal{C}_r \supseteq \mathcal{S}_m \supseteq \mathcal{H}_s \supseteq \mathcal{K}_n$, with $r \geqslant m \geqslant s \geqslant n$)]. We refer the reader to Ref. 7 for further details on CKV and their Lie algebra. Going back now to the problem of characterizing class A and B double warped space–times, we see that this can be carried out by ''translating'' into (M,g) the properties of the preferred vector fields (non-null covariantly constant or null recurrent) that characterize the underlying decomposable space–times $(M,\hat{g})$. Thus we get:

Theorem 1: *The necessary and sufficient condition for (M,g) to be a double warped class A space–time is that it admits a non-null, nowhere vanishing CKV $\vec{X}$ which is hypersurface orthogonal and such that the gradient of its associated conformal factor ψ is parallel to $\vec{X}$.*

Proof: Let (M,g) be a class A double warped space–time, its line element takes then the form (7) and (8) and it is easy to see that $\vec{X}=\partial_u$ is a CKV which satisfies the required properties, in particular, its associated conformal factor ψ is $\psi=\theta_{,u}$ which on account of the form that θ has (separable in u and the rest of the coordinates) is $\psi=\psi(u)$ and therefore $\psi_{,a} \propto X_a$.

The converse also holds for, assume that (M,g) admits a non-null, nowhere vanishing CKV $\vec{X}$ which is hypersurface orthogonal. Since $\vec{X}$ is nonvanishing and hypersurface orthogonal, a coordinate chart exists, say $\{u,x^k\}$, such that

$$\vec{X}=\partial_u, \quad ds^2=\epsilon e^{2U(u,x^k)}du^2+h_{ij}(u,x^k)dx^i\,dx^j,$$

where $\epsilon=\pm 1$ (see for example Ref. 8, p. 168). Further, the conformal equations for $\vec{X}$ above are simply $g_{ab,u}=2\psi g_{ab}$ (with $\psi=\psi(u,x^k)$) which in turn implies

$$\psi(u,x^k)=U_{,u}(u,x^k), \quad h_{ij}(u,x^k)=e^{2U(u,x^k)}\hat{h}_{ij}(x^k)$$

and the above line element can then be written as

$$ds^2=e^{2U(u,x^k)}[\epsilon du^2+\hat{h}_{ij}(x^k)dx^i\,dx^j].$$

Finally, imposing that $\psi_{,a} \propto X_a$ yields $\psi=\psi(u)$ and therefore $U(u,x^k)=\theta_1(u)+\theta_2(x^k)$ and the resulting space–time is then class A double warped. □

The characterization of warped space–times can now be easily recovered as the following corollary shows:

Corollary 2: If the CKV $\vec{X}$ in theorem 1 is a Killing vector (KV) then the space–time is warped of class A_2 in the classification given in Ref. 3. If $\vec{X}$ is a proper (non-KV) gradient CKV (i.e., if the associated conformal bivector vanishes $F_{ab}=X_{a;b}-X_{b;a}=0$) the space–time is class A_1 warped in that classification.

It is worthwhile noticing that Theorem 1 also provides an invariant characterization of space–times conformal to $1+3$ locally decomposable space–times:

Corollary 3: The necessary and sufficient condition for (M,g) *to be conformally related to a* $1+3$ *decomposable space–time* $(M,\hat{g})$ *is that it admits a non-null, nowhere vanishing conformal Killing vector (CKV)* $\vec{X}$ *which is hypersurface orthogonal.*

Theorem 1 can be conveniently rephrased in terms of Newman–Penrose (NP) formalism[9] through the two following theorems:

Theorem 4: (M,g) *is a class* A_1 *double warped space–time if and only if there exist a function* $U:M\rightarrow\mathbb{R}$ *and a canonical complex null tetrad* $\{k_a,l_a,m_a,\overline{m}_a\}$ $(k^a l_a=-m^a\overline{m}_a=-1)$ *in which:*

$$DU=\epsilon+\overline{\epsilon}, \tag{16}$$

$$\triangle U=-(\gamma+\overline{\gamma}), \tag{17}$$

$$\delta U=\kappa+\overline{\pi}=-(\tau+\overline{\nu}), \tag{18}$$

$$\sigma+\overline{\lambda}=0, \tag{19}$$

$$\alpha+\overline{\beta}=0, \tag{20}$$

$$\epsilon+\overline{\epsilon}+\gamma+\overline{\gamma}=\rho+\overline{\mu}, \tag{21}$$

$$D(\rho+\overline{\mu})=-\Phi, \tag{22}$$

$$\triangle(\rho+\overline{\mu})=\Phi, \tag{23}$$

$$\delta(\rho+\overline{\mu})=\overline{\delta}(\rho+\overline{\mu})=0, \tag{24}$$

where $\Phi=\Phi(u)$ *is a real function of the timelike coordinate* u.

Proof: With the notation of Theorem 1 we have that for a class A_1 double warped space–time a coordinate chart $\{u,x^k\}$ exists such that the line element takes the form (7), $\vec{X}=\partial_u$ is then a timelike hypersurface orthogonal CKV with associated conformal factor $\psi(u)=\theta_{1,u}(u)$, and $\vec{u}=\mathbf{e}^{-U}\partial_u$ is a unit timelike vector field parallel to $\vec{X}$ where we put $U(u,x^k)=\theta_1(u)+\theta_2(x^k)$ for convenience, it is then easy to see that, in the above coordinate chart, one has

$$u_{a;b}=(U_{,c}u^c)g_{ab}-U_{,a}u_b \tag{25}$$

and also

$$\psi_{,a}=\Phi\mathbf{e}^{-U}u_a, \tag{26}$$

where $\Phi=\Phi(u)$ is a real function of the timelike coordinate u [to be precise: $\Phi=-\theta_{1,uu}(u)$].

One can define a canonical null tetrad as follows:

$$k_a=\frac{1}{\sqrt{2}}(u_a+x_a^1),\quad l_a=\frac{1}{\sqrt{2}}(u_a-x_a^1),\quad m_a=\frac{1}{\sqrt{2}}(x_a^2+ix_a^3), \tag{27}$$

where x_a^1, x_a^2, x_a^3 are spacelike vectors orthogonal to u_a. Expressions (16)–(24) are then obtained by contracting (25) and (26) with the tetrad (27).

On the other hand, contracting (16)–(24) with the dual of (27) one recovers expressions (25) and (26), which, according to Theorem 1, imply that the space–time is class A_1 double warped. □

Theorem 5: (M,g) *is a class* A_2 *double warped space–time if and only if there exist a function* $U:M\rightarrow\mathbb{R}$ *and a canonical complex null tetrad* $\{k_a,l_a,m_a,\overline{m}_a\}$ $(k^a l_a=-m^a\overline{m}_a=-1)$ *in which one of the following sets of equations holds:*

$$DU=\epsilon+\overline{\epsilon}, \tag{28}$$

$$\Delta U=-(\gamma+\overline{\gamma}), \tag{29}$$

$$\delta U=-\kappa+\overline{\pi}=\tau+\overline{\nu}, \tag{30}$$

$$\sigma-\overline{\lambda}=0, \tag{31}$$

$$\alpha+\overline{\beta}=0, \tag{32}$$

$$\epsilon+\overline{\epsilon}-(\gamma+\overline{\gamma})=\rho-\overline{\mu}, \tag{33}$$

$$D(\rho-\overline{\mu})=\Phi, \tag{34}$$

$$\Delta(\rho-\overline{\mu})=\Phi, \tag{35}$$

$$\delta(\rho-\overline{\mu})=\overline{\delta}(\rho-\overline{\mu})=0, \tag{36}$$

$$DU=\sigma+\overline{\rho}, \tag{37}$$

$$\Delta U=-(\overline{\lambda}+\mu), \tag{38}$$

$$\delta U=\overline{\alpha}-\beta, \tag{39}$$

$$\delta U+\overline{\delta}U=\pi+\overline{\pi}=-(\tau+\overline{\tau}), \tag{40}$$

$$\kappa+\overline{\kappa}=0, \tag{41}$$

$$\nu+\overline{\nu}=0, \tag{42}$$

$$\epsilon-\overline{\epsilon}=0, \tag{43}$$

$$\gamma-\overline{\gamma}=0, \tag{44}$$

$$\delta(\pi+\overline{\pi})=\overline{\delta}(\pi+\overline{\pi})=\Phi', \tag{45}$$

$$\Delta(\pi+\overline{\pi})=D(\pi+\overline{\pi})=0, \tag{46}$$

$$DU=-\sigma+\overline{\rho}, \tag{47}$$

$$\Delta U=\overline{\lambda}-\mu, \tag{48}$$

$$\delta U=\overline{\alpha}-\beta, \tag{49}$$

$$\delta U-\overline{\delta}U=-\pi+\overline{\pi}=-\tau+\overline{\tau}, \tag{50}$$

$$\kappa-\overline{\kappa}=0, \tag{51}$$

$$\nu-\overline{\nu}=0, \tag{52}$$

$$\epsilon-\overline{\epsilon}=0, \tag{53}$$

$$\gamma-\overline{\gamma}=0, \tag{54}$$

$$-\delta(\pi-\overline{\pi})=\overline{\delta}(\pi-\overline{\pi})=-\Phi'', \tag{55}$$

$$\Delta(\pi-\overline{\pi})=D(\pi-\overline{\pi})=0, \tag{56}$$

where Φ,Φ' *and* Φ'' *are real functions of the spacelike coordinate* u.

Proof: The proof follows along the same lines as that of Theorem 4. If $\vec{X}=\partial_u$ is the hypersurface orthogonal spacelike CKV and $\vec{u}=\mathbf{e}^{-U}\partial_u$ is the unit spacelike vector field parallel to it whose existence are ensured by theorem 1, then a canonical tetrad can be constructed in one of the following ways:

$$k_a=\frac{1}{\sqrt{2}}(u_a+x_a^3),\quad l_a=\frac{1}{\sqrt{2}}(-u_a+x_a^3),\quad m_a=\frac{1}{\sqrt{2}}(x_a^2+\imath x_a^1), \tag{57}$$

$$k_a=\frac{1}{\sqrt{2}}(x_a^2+x_a^3),\quad l_a=\frac{1}{\sqrt{2}}(-x_a^2+x_a^3),\quad m_a=\frac{1}{\sqrt{2}}(u_a+\imath x_a^1), \tag{58}$$

$$k_a=\frac{1}{\sqrt{2}}(x_a^2+x_a^3),\quad l_a=\frac{1}{\sqrt{2}}(-x_a^2+x_a^3),\quad m_a=\frac{1}{\sqrt{2}}(x_a^1+\imath u_a), \tag{59}$$

where u_a,x_a^1,x_a^2 are spacelike vectors and x_a^3 is a timelike vector.

Equations (28)–(36) are obtained contracting (25) and (26) with tetrad (57), (37)–(46) arise from contracting (25) and (26) with tetrad (58), while contraction of (25) and (26) with tetrad (59) gives rise to (47)–(56). To recover expressions (25) and (26) one must in turn contract those sets of equations with the corresponding dual tetrad. □

Regarding the characterization of class B double warped space–times, we shall first recall the necessary and sufficient condition for a space–time to be conformally related to a $2+2$ decomposable one, as it was given in theorem 3 of Ref. 3, and next give the condition on the conformal factor that makes it separable in the two sets of coordinates adapted to the two two-dimensional factor submanifolds. We do this in the following theorem:

Theorem 6: *The necessary and sufficient condition for* (M,g) *to be conformally related to a* $2+2$ *decomposable space–time* $(M,\hat{g})$ *with* $g=\exp(2\theta)\hat{g}$ *(*θ *being a real function), is that there exist null vectors* $\vec{l}$ *and* $\vec{k}$ $(l^a k_a=-1)$ *satisfying*

$$l_{a;b}=Ae^{-\theta}l_a l_b-\theta_{,a}l_b+(\theta_{,c}l^c)g_{ab},\quad k_{a;b}=-Ae^{-\theta}k_a l_b-\theta_{,a}k_b+(\theta_{,c}k^c)g_{ab}; \tag{60}$$

for some function A. *Further,* (M,g) *is class B doublewarped if and only if*

$$H_a^c(h_b^d\theta_{,d})_{;c}+2(h_b^d\theta_{,d})(H_a^c\theta_{,c})=0,$$

where

$$h_{ab}\equiv-2k_{(a}l_{b)},\quad H_{ab}\equiv g_{ab}+h_{ab}. \tag{61}$$

Proof: The reader is referred to Theorem 3 of Ref. 3 for a proof of the first part of the theorem. As for the second part, namely that characterizing double warped space–times within the larger class of space–times which are conformal to $2+2$ decomposable ones, notice that Eq. (61) is nothing but the covariant expression of $\partial_\alpha(\partial_A\theta)=0$, where $\{x^\alpha\}$ and $\{x^A\}$ are coordinate charts on the two 2-dimensional submanifolds M_1 and M_2, respectively (see Sec. I). □

As in the former case, Theorem 6 can be expressed in terms of the NP formalism. To do so, a complex null tetrad $\{k_a,l_a,m_a,\overline{m}_a\}\equiv\{z_a^m\}$ is chosen such that $\vec{k}$ and $\vec{l}$ are the vectors in (60); i.e., $k^a l_a=-m^a\overline{m}_a=-1$ all other inner products vanishing.
One then has

Theorem 7: *The necessary and sufficient condition for* (M,g) *to be conformally related to a* $2+2$ *decomposable space–time* $(M,\hat{g})$, *with* $g=e^{2\theta}\hat{g}$, *is that there exist a function* $\theta:M\to\mathbb{R}$ *and a canonical complex null tetrad* $\{k_a,l_a,m_a,\overline{m}_a\}$ *as described above such that*

$$\kappa=\sigma=\lambda=\nu=\alpha+\overline{\beta}=\pi+\overline{\tau}=\rho+(\epsilon+\overline{\epsilon})=0,$$

$$Ae^{-\theta}=\mu+(\gamma+\overline{\gamma}),\tag{62}$$

$$\rho=-D\theta,\quad \mu=\Delta\theta,\quad \tau=-\delta\theta,$$

where A is the real function appearing in (60). Furthermore, (M,g) *is class B double warped if and only if*

$$\delta\rho=-2\rho\tau,\quad \delta\mu=-2\mu\tau,\quad \rho\mu=0.\tag{63}$$

Proof: Equation (60) for $\vec{l}$ and $\vec{n}=\vec{k}$ becomes in NP formalism

$$\gamma_{1mn}=b\,\eta_{1m}\eta_{1n}-\theta_{,m}\eta_{1n}+\eta_{mn}D\theta,\tag{64}$$

$$\gamma_{2mn}=-b\,\eta_{2m}\eta_{1n}-\theta_{,m}\eta_{2n}+\eta_{mn}\Delta\theta,\tag{65}$$

where $b\equiv Ae^{-\theta}$, $m,n,\ldots$ are tetrad indices and the notation is the same as in Ref. 9. Contracting (64) and (65) with the tetrad vectors, (62) are easily obtained.

On the other hand, the tetrad version of (61), together with the information contained in (62), yields (63).

Conversely, (62) and (63) contracted with the dual tetrad of $\{k_a,l_a,m_a,\overline{m}_a\}$ give Eqs. (60) and (61). □

The characterization of class A and class B double warped space–times given in Theorems 1 and 6, or alternatively 4, 5, and 7 should prove useful in formulating an algorithm for classifying such metrics. This is so because this characterization is coordinate independent although tetrad dependent. In what follows the tetrads described in Theorems 4, 5, and in Theorem 7 will be designated as *dw tetrads of class A and B, respectively*.

Thus, in order to determine whether a given metric g represents a double warped space–time, one can either use Theorems 1, and 6 (coordinate approach), or else their counterparts 4, 5, and 7 through the following scheme:

(1) Determine the Petrov type of the Weyl tensor associated with the metric g and choose a canonical tetrad $\{k_a,l_a,m_a,\overline{m}_a\}$ such that $g_{ab}=2[-l_{(a}k_{b)}+m_{(a}\overline{m}_{b)}]$.
(2) Determine the NP spin coefficients and their NP derivatives in the chosen tetrad (1).
(3) If the scalars determined in step (2) satisfy the relations of Theorem 4 or 5 (respectively, 7) for some function U (respectively, θ), then the space–time is double warped of class A (respectively, B) and the algorithm stops here, otherwise continue the algorithm.
(4) If possible, find the Lorentz transformation of the invariance group that transforms tetrad (1) into a dw tetrad; i.e., such that the corresponding NP spin coefficients and NP derivatives obey the conditions in Theorem 4 or 5 (respectively, 7). If such a transformation exists, the space–time is double warped of class A (respectively, B), otherwise it is not double warped.

The Lorentz transformations considered in step (4) must belong to the invariance group of the Petrov type of the metric since in step (1) one chooses a canonical tetrad. Thus, for instance, if the given metric is of the Petrov type D or N, then in step (4) one looks for spin and boost transformations or for null rotations respectively.

III. CURVATURE STRUCTURE

The purpose of this section is to study the Riemann tensor of double warped space–times in connection with that of the underlying, conformally related, decomposable space–time, with a special emphasis on the algebraic Petrov and Segre types of the associated Weyl and Ricci tensors, respectively.

First of all, notice that since the metric g of the double warped space–time is conformally related to that of the decomposable space–time $\hat{g}$, their respective Weyl conformal tensors and hence their Petrov types, will be equal. The Petrov types of decomposable space–times can be easily calculated and are in most cases related to the holonomy type of the space–time; thus Ref. 5 one has that if the space–time is $1+3$ spacelike (holonomy type R_{13}) the Petrov type can only be I, D, or O, whence it follows that class A_1 double warped space–times can only be of those Petrov types. In the case of $1+3$ timelike decomposable space–times the Petrov type of the Weyl tensor is unrestricted and the same will hold for A_2 double warped space–times. Finally, the Petrov type of $2+2$ decomposable space–times (holonomy type R_7), and hence that of class B double warped ones, can only be D or O. Further, if it is type D the null vectors $\vec{k}$ and $\vec{l}$ in Theorem 6 are principal null directions of the Weyl tensor $C^a{}_{bcd}$, since the corresponding null vectors in the underlying $2+2$ decomposable space–time (that is: the recurrent null vectors $\hat{l}_a = e^{-\theta} l_a$ and $\hat{k}_a = e^{-\theta} k_a$, see Ref. 3 for details) can be easily seen to be principal null directions of the Weyl tensor in $(M,\hat{g})$; i.e., $\hat{C}^a{}_{bcd}$ (which equals $C^a{}_{bcd}$); see Ref. 10.

Regarding the Segre classification of the Ricci tensor, similar comments to those in the case of warped space–times hold; that is: conformal scaling does change the Ricci tensor and therefore the Segre type of double warped space–times is unrestricted in principle. Further, in the case of class A warped space–times we have that the unit vector field $\vec{u}$ (see proof of Theorem 1) is always an eigenvector of the Ricci tensor[3] (and therefore the Segre type of class A_1 warped space–times is $\{1,111\}$ or one of its degeneracies), while in the case of double warped space–times this is no longer so for, from the Ricci identity specialized to $\vec{u}$ it follows (see Ref. 11):

$$R^a{}_b u_a = -\tfrac{2}{3}\Theta_{,b} + \tfrac{1}{3}(\dot{\Theta} + \Theta^2) u_b ,$$

now, in order for $\vec{u}$ to be an eigenvector of the Ricci tensor, it should be that $\Theta_{,b} \propto u_b$ and then a trivial calculation using the expressions for $\vec{u}$ and the metric that appear in the proof of Theorem 1 shows that either $\Theta=0$ and then the conformal factor associated with $\vec{X}$ is $\psi=0$; i.e., $\vec{X}$ is a KV and the space–time is A_2 warped (see Corollary 2) or else $U(u,x^k)=\theta_1(u)$ the space–time thus being type A_1 warped. Since the converse follows trivially, we have shown

Corollary 8: The necessary and sufficient condition for a class A double warped space–time to be a class A warped space–time is that the CKV $\vec{X}$ in Theorem 1 be a Ricci eigenvector (then it is of class A_1 if it is a proper CKV and of class A_2 if it is a KV).

In the case of type B space–times, all Segre types are possible in principle.

To close this section, we next give the expressions of the Ricci tensors and the Ricci scalar. They can be derived easily from Appendix D in Ref. 12. Notice that, in the notation established in the previous section, $\theta_{\alpha,A} = \theta_{\alpha/A} = 0$,

$$R_{\alpha\beta} = \hat{R}_{\alpha\beta} - 2[\theta_{1\ \alpha/\beta} - \theta_{1\ \alpha}\theta_{1\ \beta}] - S\hat{h}_{1\ \alpha\beta},$$

$$R_{\alpha B} = 2\theta_{1\ \alpha}\theta_{2\ B}, \tag{66}$$

$$R_{AB} = \hat{R}_{AB} - 2[\theta_{2\ A/B} - \theta_{2\ A}\theta_{2\ B}] - S\hat{h}_{2\ AB},$$

where

$$S \equiv \theta^d{}_{/d} + 2\theta^d\theta_d , \tag{67}$$

that is:

$$S=\frac{1}{\sqrt{-\hat{h}_1}}[\sqrt{-\hat{h}_1}\theta_1^{\mu}]_{,\mu}+2\hat{h}_1^{\mu\nu}\theta_{1\ \mu}\theta_{1\ \nu}+\frac{1}{\sqrt{\hat{h}_2}}[\sqrt{\hat{h}_2}\theta_2^{M}]_{,M}+2\hat{h}_2^{MN}\theta_{2\ M}\theta_{2\ N}, \tag{68}$$

where $\hat{h}_1\equiv\det(\hat{h}_1)_{\mu\nu}$, $\hat{h}_2\equiv\det(\hat{h}_2)_{AB}$, and $\theta_{1\ \alpha}\equiv\theta_{1\ ,\alpha}$, etc., and $\hat{R}_{ab}$ stands for the components of the Ricci tensor associated with the decomposable metric $\hat{g}$, which turn out to be $\hat{R}_{\alpha\beta}=\hat{R}_{1\ \alpha\beta}$ and $\hat{R}_{AB}=\hat{R}_{2\ AB}$ that is: the Ricci tensors of the metrics $\hat{h}_1$ and $\hat{h}_2$, respectively. Notice that S is separable as a sum in the coordinates x^{α} and x^{A}. For the Ricci scalar one easily gets

$$\begin{aligned}R=e^{-2(\theta_1+\theta_2)}\{\hat{R}-6[(-\hat{h}_1)^{-1/2}((-\hat{h}_1)^{1/2}\theta_1^{\mu})_{,\mu}+\hat{h}_1^{\mu\nu}\theta_{1\ \mu}\theta_{1\ \nu}\\+(\hat{h}_2)^{-1/2}((\hat{h}_2)^{1/2}\theta_2^{M})_{,M}+\hat{h}_2^{MN}\theta_{2\ M}\theta_{2\ N}]\},\end{aligned} \tag{69}$$

where $\hat{R}$ denotes the Ricci scalar of the metric $\hat{g}$, which is simply $\hat{R}_1+\hat{R}_2$, i.e., the sum of the Ricci scalars associated with the metrics $\hat{h}_1$ and $\hat{h}_2$.

IV. THE CONFORMAL LIE ALGEBRA OF CLASS B DOUBLE WARPED SPACE–TIMES

The purpose of this section is to make a few remarks on the Lie algebra of CKV, including Killing vectors (KV) and homothetic vectors (HV), of class B double warped space–times.

A double warped space–time (M,g) admits a CKV $\vec{X}$ iff $\mathcal{L}_{\vec{X}}g=2\psi g$ where ψ is some real function. If $\psi=$constant then $\vec{X}$ is a HV and if $\psi=0$ it is a KV.

Now, since a double warped space–time (M,g) is always conformally related to a decomposable one $(M,\hat{g})$, their respective conformal algebras will be equal; and as it turns out, it is relatively simple to deal with the conformal algebra of the decomposable space–time $(M,\hat{g})$. Conformal algebras in locally decomposable space–times have been studied by Coley and Tupper,[13] Capocci and Hall,[14] and (following a different approach) by Tsamparlis.[15] For the sake of completeness, we next summarize the basic results and refer the reader to the above papers for detailed proofs.

Theorem 9: *Let $(M,\hat{g})$ be a 2+2 decomposable space–time; the following results hold regarding its conformal Lie algebra:*

(1) If $(M,\hat{g})$ is conformally flat (CF) its conformal algebra is 15-dimensional, their generators being those of Minkowski's conformal algebra. In this case the two factor submanifolds must each be of constant curvature, say k_1 and k_2, respectively, with $k_1+k_2=0$.

(2) If it is not CF, the only CKV it may admit are KV or HV.

(3) If $(M,\hat{g})$ is not CF its KV are the KV of the submanifolds $(M_i,\hat{h}_i)$, for $i=1,2$; that is: if $\zeta^a=(\zeta^0,\zeta^1)$ is a KV of $(M_1,\hat{h}_1)$, then $\xi^a=(\zeta^0,\zeta^1,0,0)$ is a KV of $(M,\hat{g})$, etc. Also, $(M,\hat{g})$ will admit a HV if and only if each of $(M_i,\hat{h}_i)$ for $i=1,2$ admit a HV, i.e., if $\kappa^a=(\kappa^0,\kappa^1)$ and $\lambda^a=(\lambda^2,\lambda^3)$ are HV of the 2-spaces (adjusted to the same numerical values of the respective homothetic scalars), then $\eta^a=(\kappa^0,\kappa^1,\lambda^2,\lambda^3)$ is an HV of $(M,\hat{g})$ with the same value for its homothetic scalar.

For the case referred to in the above theorem, the reader is also referred to Ref. 16 where a thorough discussion of conformally decomposable $2+2$ space–times is given, along with a classification in terms of their conformal algebra.

V. THE CONFORMAL LIE ALGEBRA OF CLASS A DOUBLE WARPED SPACE–TIMES

We shall dedicate this section to the study of the conformal algebra of class A double warped space–times, which by our previous remarks, will be the same as that of the underlying $1+3$ decomposable space–time in each case. In so doing, we shall give some interesting results on particular types of CKV (namely: gradient CKV or GCKV for short) in three-dimensional manifolds which, to the best of our knowledge, are new. Most of the results on proper CKV in $1+3$ decomposable space–times can be found in Ref. 13 and also (although no explicit expressions are given) in Ref. 14, we re-derive them here following a different approach which provides interesting information on the geometry of three-dimensional manifolds, and renders along the way useful and interesting results on particular types of CKV (namely: gradient CKV, GCKV for short) also in three-dimensional manifolds which, to the best of our knowledge, do not exist in the literature.

For this section alone, we shall change our notation slightly so as to avoid unnecessary complications, thus, the line element of the decomposable space–time $(M,\hat{g})$ will be written as

$$d\hat{s}^2=[\epsilon du^2+h_{AB}(x^D)dx^A\,dx^B], \quad \epsilon=\pm 1. \tag{70}$$

The three-dimensional submanifold coordinated by x^A, $A=1,2,3$ will be noted as V and its metric (of either signature) as h (instead of $\hat{h}$). We shall represent the covariant derivative with respect to the three-dimensional metric h by a slash ("/"), whereas a semicolon " ; " will be used to note that with respect to the four-dimensional metric $\hat{g}$. (The reader is reminded that this notation holds only in the present section: notice that, in the rest of the paper, a semicolon stands for the covariant derivative associated with g, the metric of the double warped space–time, whereas a slash stands for that associated with $\hat{g}$, the metric of the decomposable space–time.) The covariantly constant vector is then $\vec{u}=\partial_u$ (i.e., $u_{a;b}=0$ and therefore it is a non-null gradient KV). Finally note that, in the above coordinate system, the covariant derivatives satisfy

$$X_{ab\cdots\ ;u}=X_{ab\cdots\ ,u}, \quad X_{ab\cdots\ ;A}=X_{ab\cdots\ /A}$$

for any tensor $X_{ab\cdots}$, and also that

$$R_{ua}=0, \quad R_{AB}=\overset{(3)}{R}_{AB},$$

where $\overset{(3)}{R}_{AB}$ stands for the Ricci tensor associated to the 3-metric h on V.

In order to investigate its conformal algebra, we first make a few trivial remarks in the paragraphs that follow.

First of all, and making an obvious abuse in the notation, we shall represent points in M by their coordinates in the above chart [that is: $p\in M$ with coordinates $x^a(p)=(u,x^A)$ will be represented simply as (u,x^A)]; next we consider the three-dimensional submanifold (hypersurface) consisting of all the points with the same value of the x^0 coordinate, say $x^0=u$, and note it as $V(u)$; i.e., $V(u)=\{(u,x^A)\ :\ u\ \text{fixed}\}$; the induced metric on $V(u)$ is h and, clearly, any two such submanifolds are diffeomorphic amongst themselves [and diffeomorphic to (V,h)] by the one-parameter group of isometries $\{\tau_t\}$ generated by $\vec{u}$ [that is: $\tau_t:V(u)\to V(u+t)$ where $\tau_t(u,x^A)=(u+t,x^A)$ wherever this makes sense].

Note that $\tau_t^*h=h$; that is, the three-dimensional metric h is invariant under the isometries generated by $\vec{u}$. Further, a vector field $\vec{X}$ in M will be invariant under these isometries ($\tau_{t*}\vec{X}=\vec{X}$) iff $[\vec{u},\vec{X}]=0$. In particular, if $\vec{X}$ is tangent to the submanifolds $V(u)$ it follows that it will be invariant under $\{\tau_t\}$ iff its components with respect to the above coordinate basis do not depend on u, i.e., $\vec{X}=X^A(x^D)\partial_A$.

Finally, we shall use the notation $\mathcal{C}_n(V,h)$ ($\mathcal{S}_n(V,h)$, $\mathcal{H}_n(V,h)$ or $\mathcal{K}_n(V,h)$) to designate the n-dimensional conformal (respectively: special conformal, homothetic or Killing) algebra of (V,h). Such an algebra (and therefore all of its subalgebras) is finite dimensional, its dimension being 10 at most (and (V,h) is then conformally flat). If (V,h) is nonconformally flat, then, a remarkable theorem by Hall and Capocci (see Ref. 17) shows that its dimension can be at most 4. In our subsequent developments we will often have to refer to some basis of $\mathcal{C}_n(V,h)$, which we will generically represent by $\{\vec{\zeta}_k\}$, $k=1,\ldots,n$ with associated conformal factors ψ_k; that is

$$\mathcal{L}_{\vec{\zeta}_k} h = 2\psi_k h. \tag{71}$$

Notice that any such basis is invariant under the isometries generated by $\vec{u}$; that is: $[\vec{u},\vec{\zeta}_k]=0$, $k=1,\ldots,n$, hence $\vec{\zeta}_k=\zeta_k^B(x^D)\partial_B$ and therefore also $\psi_k=\psi_k(x^D)$. (Some of the conformal factors ψ_k may be constant if they correspond to homotheties, or zero in the case of Killing vectors.)

We can now consider the problem of finding the CKV of a $1+3$ reducible space–time $(M,\hat{g})$. Let $\vec{Y}\in\mathcal{C}_r(M,\hat{g})$, one then has

$$Y_{a;b}+Y_{b;a}=2\phi\hat{g}_{ab}.$$

In the chosen coordinate chart, $\vec{Y}=Y^u(u,x^B)\partial_u+Y^A(u,x^B)\partial_A$ and the above equation then reads (on account of our previous remarks):

$$Y_{u,u}=\epsilon_1\phi, \tag{72}$$

$$Y_{u,A}+Y_{A,u}=0, \tag{73}$$

$$Y_{A/B}+Y_{B/A}=2\phi h_{AB}. \tag{74}$$

Now, (74) effectively says that for u fixed the vector field $Y^B(u,x^D)\partial_B$ is a CKV in $V(u)$ [equivalently: if a proper CKV is admitted in $(M,\hat{g})$ then, its projection on the submanifolds $V(u)$ is a CKV there], therefore, given $\{\vec{\zeta}_k\}$ a basis for $\mathcal{C}(V,h)$, it follows that it will also be a basis of $\mathcal{C}(V(u),h)$ (u fixed but otherwise arbitrary) whence, on $V(u)$ we shall necessarily have $Y^B(u,x^D)\partial_B=\lambda^k\vec{\zeta}_k$ with $\lambda^k=$constant and summation over $k=1,\ldots,n$ is to be understood; again this will be so for any $V(u)$ (i.e., u fixed but otherwise arbitrary). Finally, since $[\vec{u},\vec{\zeta}_k]=0$, we will have

$$\vec{Y}=Y^u(u,x^B)\partial_u+\lambda^k(u)\vec{\zeta}_k(x^B), \tag{75}$$

where $\lambda^k(u)$, $k=1,\ldots,n$ are n functions of the coordinate u. Substituting this back into (72)–(74) and putting $\phi\equiv\dot{\Sigma}$, where a dot indicates differentiation with respect to u, yields

$$Y_u=\epsilon_1\Sigma, \quad \phi=\dot{\Sigma}=\lambda_k(u)\psi_k(x^B), \tag{76}$$

$$\epsilon_1\Sigma_{,A}+\dot{\lambda}^k\zeta_{k\ \ A}=0. \tag{77}$$

Further, $\dot{\lambda}^k(u)\vec{\zeta}_k$ is also a CKV in each $V(u)$ [since for u fixed it is a linear combination of the CKV in the basis of $\mathcal{C}(V,h)$] which, on account of (77), is locally a gradient, i.e., $\dot{\lambda}^k\zeta_{k\ \ A}=-\epsilon_1\Sigma_{,A}$. The question arises as to how many independent GCKV may (V,h) admit, what are they; namely proper CKV, proper HV, or KV, and what does their existence imply on the 3-metric h.

Before proceeding, the following remarks, which follow trivially from the above equations, are in order:

R0: If no GCKV exist in (V,h), then $\dot{\lambda}^k=0$ [i.e., $\lambda^k(u)=$constant] in the above equations and $\phi=\lambda^k\psi_k(x^B)=$constant (since then $\phi=\lambda^k\psi_k(x^D)\equiv\phi(x^D)$, which yields $Y_u=u\phi(x^D)+B(x^D)$, but then $Y_{u,A}+Y_{A,u}=0$ implies $\phi_{,A}=B_{,A}=0$.), that is: $\vec{Y}$ is homothetic in $(M,\hat{g})$ [and $\lambda^k(u)\vec{\zeta}_k$ is also homothetic in (V,h)]. If (V,h) is such that no HV are admitted, then the only CKV that $(M,\hat{g})$ admits are KV.

R1: Let $\vec{\xi}$ be a KV in (V,h), then $\vec{\xi}$ is also a KV of $(M,\hat{g})$.

R2: Let $\vec{Y}$ be a KV in $(M,\hat{g})$. The following situations may then arise:

(a) (V,h) admits no KV, then $\vec{Y}=a\partial_u$ necessarily.

(b) (V,h) admits KV none of which is locally a gradient. Then, if $\{\vec{\xi}_k\}$ is a basis of $\mathcal{K}(V,h)$, one has

$$\vec{Y}=a\partial_u+b^k\vec{\xi}_k \tag{78}$$

with a,b^k arbitrary constants.

(c) (V,h) admits KV some of which are locally gradients (and therefore, by the Killing equation, covariantly constant vectors). Then one can choose a basis for $\mathcal{K}(V,h)$, say $\{\vec{\xi}_1,\ldots,\vec{\xi}_p,\vec{\xi}_{p+1},\ldots,\vec{\xi}_n\}$ (with $p\leqslant 3$) in a way such that $\vec{\xi}_1,\ldots,\vec{\xi}_p$ are covariantly constant. Then

$$\vec{Y}=A(x^B)\partial_u+a^s u\ \ \vec{\xi}_s+b^k\vec{\xi}_k, \tag{79}$$

where a^s,b^k are arbitrary constants, $(s=1,\ldots,p,\ \ k=1,\ldots,n)$ and $A(x^B)$ satisfies $A_{,B}=a^s\xi_{s\,B}$. Notice that if one of the gradient KV in (V,h) is non-null, the space–time $(M,\hat{g})$ decomposes still further, becoming a $1+1+2$ decomposable space–time. If (V,h) admits two, then a third one is automatically admitted and the space–time $(M,\hat{g})$ is locally flat. We shall return to this later on in the paper.

R3: Let $\vec{\eta}$ be a proper HV in (V,h) with homothetic constant $k(\neq 0)$, then $\vec{Y}=ku\partial_u+\vec{\eta}$ is also a HV of $(M,\hat{g})$ with homothetic constant k. Further, if $\vec{Y}$ is a proper HV of $(M,\hat{g})$ with homothetic constant $k(\neq 0)$, then it is of the form

$$\vec{Y}=ku\partial_u+\vec{\eta}, \tag{80}$$

$\vec{\eta}$ being a (proper) HV in (V,h) scaled so as to have the same value k for its homothetic constant; the above HV is unique up to the addition of KV such as those given by (78) and/or (79) (if GKV exist).

The various possibilities regarding the existence of GCKV in (V,h) can be summarized as follows:

(1) (V,h) admits no GCKV (either proper or homothetic, including Killing). In that case (77) implies $\dot{\lambda}^k=0$ and the rest of the equations imply then that $\vec{Y}$ is a HV, see Eqs. (78) or (80) above. Thus, in this case $(M,\hat{g})$ admits no proper CKV.

(2) The only GCKV that (V,h) admits are gradient KV (GKV). In this case $(M,\hat{g})$ admits a proper CKV (which turns out to be a SCKV) if and only if the GKV is null and (V,h) admits a proper SCKV (i.e., nonhomothetic) such that the gradient of its conformal factor is parallel to the null GKV. Otherwise the only CKV that $(M,\hat{g})$ admits are HV.

(3) (V,h) admits proper gradient HV (GHV); it may also admit GKV, but no proper GCKV exist in (V,h). In this case, $(M,\hat{g})$ does admit a proper CKV which turns out to be special (i.e., SCKV); that is: its associated conformal factor ϕ satisfies $\phi_{a;b}=0$. This SCKV is unique up to the addition of KV and HV which must then take the forms discussed above.

(4) (V,h) admits proper GCKV (GHV and/or GKV can also be admitted in principle). In this case, the space–time admits proper CKV.

Regarding the maximum number of GCKV that a three-dimensional space may admit, one can easily prove the following results:

Proposition 1: Let (V,h) be a three-dimensional Lorentz or Riemann space admitting two independent proper GCKV, say $\vec{\zeta}$ and $\vec{\chi}$, with associated conformal factors ψ and ϕ, respectively, then:

(1) The Lie bracket $[\vec{\zeta},\vec{\chi}]\equiv\vec{\xi}$ is a KV.

(2) The conformal factors are $\psi=k\zeta$ and $\phi=k\chi$, where k is a constant and ζ and χ are the

functions whose gradients are the GCKV $\vec{\eta}$ *and* $\vec{\chi}$, *respectively*.

(3) (V,h) *is of constant curvature and therefore the Cotton–York tensor vanishes, thus being conformally flat*.

Proof: Suppose $\vec{\eta}$ and $\vec{\chi}$ are linearly independent GCKV satisfying

$$\zeta_{A/B}=\psi h_{AB}, \quad \chi_{A/B}=\phi h_{AB},$$

where $\zeta_A=\zeta_{,A}$, $\chi_A=\chi_{,A}$ and also [see comments following Eq. (111)] $\psi=\psi(\zeta)$ and $\phi=\phi(\chi)$. Now, a direct calculation shows that

$$[\vec{\zeta},\vec{\chi}]=\phi\vec{\zeta}-\psi\vec{\chi}\equiv\vec{\xi}$$

that is, $\vec{\eta}$ and $\vec{\chi}$ are surface-forming.

Compute next

$$\mathcal{L}_{[\vec{\zeta},\vec{\chi}]}h_{AB}=\mathcal{L}_{\vec{\zeta}}(\mathcal{L}_{\vec{\chi}}h_{AB})-\mathcal{L}_{\vec{\chi}}(\mathcal{L}_{\vec{\zeta}}h_{AB})=2(\zeta^D\chi_D)(\tilde{\phi}-\tilde{\psi})h_{AB},$$

where $\tilde{\phi}\equiv \mathrm{d}\phi/\mathrm{d}\chi$ and $\tilde{\psi}\equiv \mathrm{d}\psi/\mathrm{d}\zeta$, and also

$$\mathcal{L}_{\vec{\xi}}h_{AB}=\mathcal{L}_{\phi\vec{\zeta}-\psi\vec{\chi}}h_{AB}=(\tilde{\phi}-\tilde{\psi})(\zeta_A\chi_B+\chi_A\zeta_B)$$

therefore

$$(\tilde{\phi}-\tilde{\psi})(\zeta_A\chi_B+\chi_A\zeta_B)=2(\zeta^D\chi_D)(\tilde{\phi}-\tilde{\psi})h_{AB}.$$

An elementary consideration on the ranks of the tensors at both sides of the equation readily shows that

$$\tilde{\phi}-\tilde{\psi}=0;$$

therefore

$$\psi=k\eta, \quad \phi=k\chi$$

and $\vec{\xi}$ is then a KV given by

$$\vec{\xi}=k(\chi\vec{\zeta}-\zeta\vec{\chi}),$$

which is not a gradient: $\xi_{A/B}=k(\zeta_A\chi_B-\chi_A\zeta_B)$.

Now, since both $\vec{\zeta}$ and $\vec{\chi}$ are GCKV their respective conformal bivectors are zero and (12) applied to them yields

$$R_{ABCD}\zeta^D=k(\zeta_A h_{BC}-\zeta_B h_{AC}),$$

$$R_{ABCD}\chi^D=k(\chi_A h_{BC}-\chi_B h_{AC}),$$

which in turn implies, upon contraction with h^{AC},

$$R_{BD}\zeta^D=-2k\zeta_B, \quad R_{BD}\chi^D=-2k\chi_B.$$

Now, in three dimensions one has

$$R_{ABCD}=R_{AC}h_{BD}-R_{AD}h_{BC}+h_{AC}R_{BD}-h_{AD}R_{BC}+(R/2)(h_{AD}h_{BC}-h_{AC}h_{BD})$$

hence, the above equations imply

$$R_{AC}\zeta_B - \zeta_A R_{BC} + (k + R/2)(\zeta_A h_{BC} - \zeta_B h_{AC} = 0,$$

$$R_{AC}\chi_B - \chi_A R_{BC} + (k + R/2)(\chi_A h_{BC} - \chi_B h_{AC} = 0,$$

contracting the two equations above with ζ^A and χ^A, respectively, we get, since $\zeta^A\zeta_A, \chi^A\chi_A \neq 0$:

$$R_{AC} = (k + R/2)h_{AC} - (3k + R/2)\zeta_A\zeta_C,$$

$$R_{AC} = (k + R/2)h_{AC} - (3k + R/2)\chi_A\chi_C$$

contracting again both equations with, say ζ^C, equating and rearranging terms we get

$$(3k + R/2)\zeta_A = (3k + R/2)(\zeta_C\chi^C)\chi_A$$

but this contradicts our hypothesis of linear independence unless $3k + R/2 = 0$, i.e., $R = -6k$ ($=$constant). Then $R_{AB} = -2kh_{AB}$ and $R_{ABCD} = k(h_{AD}h_{BC} - h_{AC}h_{BD})$; that is: (V,h) is of constant curvature and therefore the associated Cotton–York tensor[11] is zero, i.e., h is conformally flat. □

The converse theorem also holds; namely: if (V,h) is a three-dimensional space or space–time of constant curvature (and therefore conformally flat), it admits two linearly independent GCKV whose associated conformal factors are multiples (with the same multiplicative constant) of the functions whose gradients they are.

Furthermore, with the same notation and hypotheses as in the preceding theorem and following a similar procedure to that outlined in its proof, it is easy to prove the following three results:

Lemma 2: Let $\vec{\zeta}$ be a GCKV and $\vec{\xi}$ a GKV (i.e., $\vec{\xi}$ is covariantly constant). Then $\vec{\zeta}$ is necessarily homothetic, that is, it is a GHV.

Proof: Since $\zeta_{A/B} = \psi h_{AB}$ and $\xi_{A/B} = 0$ it follows $[\vec{\zeta},\vec{\xi}] = -\psi\vec{\xi}$. Computing next the Lie derivative of h in two different ways, as in the proof of Theorem 1, and then equating yields

$$2(\xi^D\psi_D)h_{AB} = \tilde{\psi}(\zeta_A\xi_B + \xi_A\zeta_B).$$

Again, considerations on the rank of the tensors that appear on both sides of the equation, imply $\psi_A = 0$; that is: $\vec{\zeta}$ is a GHV. □

Lemma 3: Let $\vec{\zeta}$ be a GCKV and $\vec{\eta}$ a GHV. Then $\vec{\zeta}$ is necessarily homothetic and therefore it is the linear combination of $\vec{\eta}$ with some GKV.

Proof: Now $\zeta_{A/B} = \psi h_{AB}$ and $\eta_{A/B} = kh_{AB}$, and their Lie bracket is $[\vec{\zeta},\vec{\xi}] = k\vec{\zeta} - \psi\vec{\eta}$. Computing as above the Lie derivative of h in two different ways and then equating implies

$$2(\eta^D\psi_D)h_{AB} = (\eta_A\psi_B + \psi_A\eta_B),$$

which again implies $\psi_A = 0$ and the result follows. □

Lemma 4: Let $\vec{\eta}$ and $\vec{\xi}$ be a proper GHV and a GKV, respectively; (V,h) *is then flat*.

Proof: In this case we have $R_{ABCD}\eta^D = R_{ABCD}\xi^D = 0$, hence $R_{AB}\eta^D = R_{AB}\xi^B = 0$, and taking into account the expression of the Riemann tensor in terms of the metric and the Ricci tensor (see the proof of Theorem 1 and recall that $\vec{\eta}$ cannot be null), one gets $R_{AB} = (R/2)(h_{AB} - (\eta^D\eta_D)^{-1}\eta_A\eta_B)$; contracting with ξ^B both sides and equating to zero yields immediately $\eta_D\xi^D = R = 0$ (since $\vec{\eta}$ and $\vec{\xi}$ are linearly independent), and this in turn implies $R = R_{AB} = 0$ and then $R_{ABCD} = 0$. □

The same result holds trivially if two linearly independent GKV exist; since in this case two linearly independent constant vector fields in a manifold of dimension three readily imply (constancy of the metric) that a third one must also exist. Thus, we have proven:

Proposition 2: A three-dimensional space or space–time admitting two linearly independent GHV (proper or Killing) is necessarily flat.

Note that from the above Propositions 1 and 2, it follows that if two (or more) independent GCKV exist in the three-space (V,h), then it is of constant curvature (and therefore conformally flat, being flat in several cases) and then it admits 10 CKV (those of flat three-dimensional space). If this is not the case [i.e., (V,h) is not of constant curvature] then it can admit, at most, one GCKV which will then give rise to a proper CKV in $(M,\hat{g})$. If (V,h) admits no GCKV, then no proper CKV exist in $(M,\hat{g})$, just HV (case 1 above).

In the following sections we shall deal with cases 2, 3, and 4 separately, assuming that the GCKV admitted in each case is unique.

A. (V,h) admits a GKV and no proper GHV or GCKV

From the preceding results it follows that unless (V,h) is conformally flat (in which case its conformal algebra is completely known), the GKV, say $\vec{\xi}$, is the unique GCKV it admits. Taking now a basis of $\mathcal{C}_n(V,h)$ as $\{\vec{\xi},\vec{\eta},\vec{\zeta}_k\}$ where $\vec{\zeta}_k$ and $\vec{\eta}$ denote CKV (including KV) and a HV, respectively [in case one exists in (V,h), if not, just set $\vec{\eta}=0$], we can write, from (74)

$$Y_A=\lambda(u)\xi_A+\mu(u)\eta_A+\lambda^k(u)\zeta_A,$$

which substituted into (73) yields

$$-Y_{u,A}=\dot{\lambda}(u)\xi_A+\dot{\mu}(u)\eta_A+\dot{\lambda}^k(u)\zeta_A.$$

Since by hypothesis, $\vec{\xi}$ is the only GCKV in (V,h) and $\vec{\xi}$, $\vec{\eta}$ (if nonzero) and $\vec{\zeta}_k$ are linearly independent vector fields, it follows that $\dot{\mu}(u)=\dot{\lambda}_k(u)=0$ [otherwise the above equation would imply that, for u fixed, $\dot{\mu}(u)\eta_A+\dot{\lambda}^k(u)\zeta_A$ is a GCKV independent of $\vec{\xi}$]; hence $\mu=a_0$ $(=\text{constant})$ [and $a_0=0$ if (V,h) admits no proper HV], and $\lambda^k=a^k$ $(=\text{constant})$. Therefore

$$Y_A=\lambda(u)\xi_A+a_0\eta_A+a^k\zeta_A$$

and substituting this back into (74), (72), and (73) one has

$$\phi=\dot{\Sigma}=a_0k+a^k\psi_k(x^D), \tag{81}$$

$$Y_u=\epsilon_1(a_0k+a^k\psi_k(x^D))u+B(x^D),\quad \epsilon_1(a_0k+a^k\psi_k(x^D))_{,A}u+B(x^D)_{,A}+\dot{\lambda}(u)\xi_A=0 \tag{82}$$

hence

$$\dot{\lambda}(u)=au+b,$$

i.e.,

$$\lambda(u)=\frac{a}{2}u^2+bu+c$$

and also

$$a^k\psi_k(x^D)=\epsilon_1(-a\xi+m),\quad B(x^D)=-b\xi+n$$

and substituting this into the expressions for the covariant components of $\vec{Y}$, we would get $Y_u=(\epsilon_1a_0k+m)u-(au+b)\xi+n$ and $Y_A=[(a/2)u^2+bu+c]\xi_A+X_A$ where $X_A\equiv a_0\eta_A+a^k\zeta_{k\,A}$. Notice that the constants n and c can be set equal to zero without loss of generality, as they amount to adding multiples of $\vec{u}=\partial_u$ and $\vec{\xi}$, respectively. On the other hand, X_A are the covariant components of a CKV whose associated conformal factor is $\epsilon_1(-a\xi+m)+a_0k$ [if no HV exists in (V,h) then the CKV has components $X_A=a^k\zeta_{k\,A}$ and conformal factor $-\epsilon_1(-a\xi+m)$], that is

$$Y_a \mathrm{d}x^a = [(\epsilon_1 k a_0 + m)u - (au+b)\xi]du + \left[\left(\frac{a}{2}u^2 + bu\right)\xi_A + X_A\right]\mathrm{d}x^A, \tag{83}$$

where

$$X_A = a_0 \eta_A + a^k \zeta_{k\,A}$$

is such that

$$2X_{(A/B)} = 2[\epsilon_1(-a\xi + m) + a_0 k]h_{AB} \tag{84}$$

and the conformal factor associated to $\vec{Y}$ is

$$\phi = \epsilon_1(-a\xi + m) + a_0 k \tag{85}$$

and satisfies $\phi_{a;b}=0$, that is: $\vec{Y}$ is a SCKV in $(M,\hat{g})$, whereas $\vec{X}$ is also a SCKV in (V,h).

Notice that some of the constants appearing in the above expressions could have been removed by means appropriate redefinitions of the objects (functions and coordinates) in them. However, as it turns out, it is useful to keep them as they appear because this makes the subsequent analysis much more clear. The following possibilities now arise regarding the nature of $\vec{X}$; namely

Case 1: (V,h) admits no proper SCKV nor proper HV, then $a_0=0$ and $X_A = a^k \zeta_{k\,A}$ is a KV, that is $a^k \psi_k = \epsilon_1(-a\xi + m) = 0$; i.e., $a=m=0$ and the conformal factor ϕ above becomes zero, hence, $\vec{Y}$ is a KV which can be seen to be given by

$$Y_a \mathrm{d}x^a = -b\xi\, \mathrm{d}u + b\xi_A\, \mathrm{d}x^A + X_A\, \mathrm{d}x^A, \quad \vec{X} \in \mathcal{K}(V,h). \tag{86}$$

Case 2: (V,h) admits no proper SCKV but it admits a proper HV (that is: $a_0 \neq 0$). It then follows that $a^k \zeta_{k\,A}$ must be a KV, hence $a^k \psi_k = \epsilon_1(-a\xi + m) = 0$; and then $a=m=0$ as before. The conformal factor is then constant $\phi = k a_0$, $\vec{Y}$ then being a HV which can be written as

$$Y_a\, \mathrm{d}x^a = a_0[\epsilon_1 k u\, \mathrm{d}u + \eta_A\, \mathrm{d}x^A] - b\xi \mathrm{d}u + b\xi_A\, \mathrm{d}x^A + X_A\, \mathrm{d}x^A, \tag{87}$$

where the first term within square brackets is a proper HV and the remaining terms are easily recognizable as a KV [see Eq. (86)].

Case 3: (V,h) admits a proper SCKV, $\vec{X}$ such that $2X_{(A/B)} = 2[\epsilon_1(-a\xi)]h_{AB}$; i.e., $a \neq 0$ and the constants ka_0, m (if nonzero) have been absorbed by suitably redefining the function ξ. We then have

$$X_{A/B} = -\epsilon_1 a \xi h_{AB} + F_{AB},$$

where F_{AB} is the conformal bivector. Computing now the Lie bracket of $\vec{\xi}$ and $\vec{X}$ and making use of the above expression together with the fact that $\xi_{A/B}=0$ we get

$$[\vec{\xi},\vec{X}] = \vec{\eta}, \quad \eta_A = -\epsilon_1 a \xi \xi_A + F_{AB}\xi^B \tag{88}$$

computing now $\eta_{A/C}$ and making use of Eq. (12) it follows

$$\eta_{A/C} = -\epsilon_1 a(\xi^D \xi_D) h_{AC} \tag{89}$$

that is: $\vec{\eta}$ is either a GHV (whenever $\vec{\xi}$ is non-null, for in that case it can be scaled so that $\xi^D \xi_D = \epsilon_2$, where $\epsilon_2 = \pm 1$), or else $\vec{\eta}$ is a GKV (including $\vec{\eta}=0$ as a special case). In the former case ($\vec{\eta}$ is a GHV and therefore $\xi^D \xi_D = \epsilon_2$), Proposition 2 above implies that (V,h) is flat. In the latter case ($\vec{\eta}$ is a GKV and $\xi^D \xi_D = 0$), one has from (88) that $\xi^D \eta_D = 0$ and therefore either

$\eta^D\eta_D\neq 0$, in which case again, proposition 2 implies that (V,h) is flat, or else $\vec{\eta}=k\vec{\xi}$ where k is a constant which may be zero (if it is not zero, it can always be chosen equal to 1 by re-scaling $\vec{X}$ appropriately). This is the only nontrivial case [in the sense that (V,h) is not necessarily flat], and it is easy to see that coordinates v,w,y can be chosen so that the three-dimensional line element takes the form:

$$d\sigma^2=-2dv\,dw+p_\pm(w)M^2(y)[q_\pm(w)^{-2}dw^2+dy^2], \tag{90}$$

where

$$q_\pm(w)=n^2w^2\pm 1,\quad p_+(w)\equiv q_+(w)\exp[\epsilon_1 kn/a\cot^{-1}nw]$$
$$p_-(w)\equiv q_-(w)\exp[\epsilon_1 kn/a\coth^{-1}nw],\quad n=\text{constant}, \tag{91}$$

the KV $\vec{\xi}$ and SCKV $\vec{X}$ being, respectively,

$$\vec{\xi}=\partial_v,\quad \vec{X}=kv\,\partial_v+\frac{1}{n^2}q_\pm(w)\partial_w. \tag{92}$$

Alternatively, new coordinates can be chosen, which we still call v,w,y, so that Ref. 11 the line element takes the more familiar form

$$d\sigma^2=-2dv\,dw-2H(w,y)dw^2+dy^2 \tag{93}$$

and still $\vec{\xi}=\partial_v$ but the function $H(w,y)$ satisfies then a partial differential equation and $\vec{X}$ then takes a form which depends on H. In this case, the Ricci tensor is

$$R_{AB}=H_{,yy}l_Al_B, \tag{94}$$

where $l_A=\xi_A$.

B. (V,h) admits a proper GHV and no proper GCKV

Since a proper HV is unique up to the addition of KV, we can assume that there is just one GHV (in the sense that, if another exists, then their difference must be a gradient KV—in that respect, if any GKV exists in (V,h) we shall consider that has been added to the GHV, therefore, any remaining proper CKV or KV in $\mathcal{C}(V,h)$ will be nongradient), say $\vec{\eta}$ with homothetic constant $k(\neq 0)$; i.e., $\eta_A\equiv\eta_{,A}$ for some function $\eta(x^B)$.

At this point, it is easy to find an expression for the line element associated with h in coordinates adapted to the GHV $\vec{\eta}$. First of all notice that from $\eta_{A/B}=kh_{AB}$ it readily follows that $\vec{\eta}$ cannot be null; next and provided we are not in the vicinity of a fixed point of the HV, we can always choose a coordinate, say $x^1\equiv v$ adapted to $\vec{\eta}$, i.e., $\vec{\eta}=\partial_v$, now the fact that $\vec{\eta}$ is locally a gradient and a HV with homothetic constant k readily implies (by a similar argument to that used previously) that coordinates $x^i\equiv x^2,x^3$ can be chosen so that the line element associated with h reads

$$d\sigma^2=e^{2kv}(\dot{\epsilon}_2 dv^2+\overline{h_{ij}}(x^k)dx^i\,dx^j),\quad \epsilon_2=\pm 1 \tag{95}$$

and then $\eta_A\,dx^A=\epsilon_2\exp(2kv)dv$, hence $\eta=(\epsilon_2/2k)\exp(2kv)$. Also, since $\overline{h_{ij}}(x^k)$ is a two-dimensional metric, the coordinates x^i can be chosen so that it takes an explicit conformally flat form, i.e.,

$$d\sigma^2=e^{2kv}(\epsilon_2\,dv^2+\Omega^2(x^k)(\epsilon_3(dx^2)^2+(dx^3)^2),\quad \epsilon_2,\epsilon_3=\pm 1, \tag{96}$$

where $\Omega(x^k)$ is some function of its arguments. The line element of $(M,\hat{g})$ then reads

$$d\Sigma^2=\epsilon_1\,\mathrm{d}u^2+e^{2kv}[\epsilon_2\,\mathrm{d}v^2+\Omega^2(x^k)(\epsilon_3(\mathrm{d}x^2)^2+(\mathrm{d}x^3)^2)],\tag{97}$$

where

$$\epsilon_\alpha=\pm1\quad(\alpha=1,2,3),\quad \epsilon_1\epsilon_2\epsilon_3=-1,\quad \epsilon_1+\epsilon_2+\epsilon_3=+1.$$

Alternatively, the following change of coordinates can be carried out:

$$w\equiv k^{-1}\exp(kv),\quad v=k^{-1}\ln kw,\tag{98}$$

which renders $\vec{\eta}$, η and the line element associated with h in the form

$$\vec{\eta}=kw\,\partial_w,\quad \eta=\frac{k\epsilon_2}{2}w^2,\quad \mathrm{d}\sigma^2=\epsilon_2\,\mathrm{d}w^2+w^2\Omega^2(x^k)(\epsilon_3(\mathrm{d}x^2)^2+(\mathrm{d}x^3)^2)\tag{99}$$

hence

$$\mathrm{d}\hat{s}^2=\epsilon_1\,\mathrm{d}u^2+\epsilon_2\,\mathrm{d}w^2+w^2\Omega^2(x^k)(\epsilon_3(\mathrm{d}x^2)^2+(\mathrm{d}x^3)^2).\tag{100}$$

Now going back to the problem of finding the CKV that $(M,\hat{g})$ admits in this case, let $\{\vec{\eta},\vec{\zeta}_k\}$ be a basis for $\mathcal{C}(V,h)$ with $\vec{\eta}$ satisfying

$$\eta_{A/B}=kh_{AB}\tag{101}$$

and $\vec{\zeta}_k$ being CKV (including KV) such that no proper CKV (nor any linear combination of them) is a gradient, we then have

$$\zeta_{k\;A/B}+\zeta_{k\;B/A}=2\psi_k h_{AB},$$

where ψ_k is the associated conformal factor.

Equation (74) states that $Y^B(u,x^D)\partial_B$ is a CKV in every $V(u)$ for u fixed, and therefore according to our previous developments, we may write

$$Y^B(u,x^D)\partial_B=\lambda(u)\,\vec{\eta}+\lambda^k(u)\vec{\zeta}_k,\tag{102}$$

which when substituted back again into (72)–(74) yields (recall, the conformal factor ϕ has been renamed as $\dot{\Sigma}$):

$$\dot{\Sigma}(u,x^B)=k\lambda(u)+\lambda^k(u)\psi_k(x^B),\quad Y_u=\epsilon_1[k\mu(u)+\mu^k(u)\psi_k(x^B)+B(x^A)],\tag{103}$$

$$\epsilon_1\mu^k\psi_{k,\,A}+B_{,A}+\dot{\lambda}\,\eta_A+\dot{\lambda}^k\zeta_{k\;\;A}=0,\tag{104}$$

where $\mu(u)$ and $\mu^k(u)$ are such that $\dot{\mu}(u)=\lambda(u)$ and $\dot{\mu}^k(u)=\lambda^k(u)$, respectively, and $B(x^A)$ is a function of integration which does not depend on u. Now, (104) above implies that, for u fixed, $\dot{\lambda}^k\zeta_{k\;\;A}$ must be a GCKV (since η_A is a gradient by assumption), but since, by hypothesis there is none and $\vec{\eta}$, $\vec{\zeta}_k$ are independent, it must be $\dot{\lambda}^k=0$, that is $\lambda^k=a^k(=\text{const})$. Plugging this back again into (72) and (73) we get

$$Y_u=\epsilon_1[k\mu(u)+ua^k\psi_k(x^B)+B(x^A)],\quad \epsilon_1 ua^k\psi_{k,\,A}+B_{,A}+\dot{\lambda}\,\eta_A=0,\tag{105}$$

which, when differentiated with respect to u yields

$$\epsilon_1 a^k\psi_{k,\,A}+\ddot{\lambda}\,\eta_A=0\tag{106}$$

and two possibilities arise:

case 1: $\ddot{\lambda}=0$, i.e.: $\lambda=au+b$, with a,b constants. Then $a^k\psi_{k,A}=0$, that is: $a^k\psi_k=C$ $(=\text{constant})$.

case 2: $\ddot{\lambda}=a$ (constant), hence $\lambda=a/2u^2+bu+c$ and then $B=-b\eta+m$ and $\epsilon_1 a^k\psi_k=a\eta+C$ where m,C are constants and η is the function such that $\eta_{,A}=\eta_A$.

Case 1: In this case it is straightforward to get from the equations above that $B=-\epsilon a\eta$ and also that $a^k\vec{\zeta}_k$ must satisfy:

$$\mathcal{L}_{a^k\vec{\zeta}_k}h=2Ch$$

that is: $a^k\vec{\zeta}_k$ is a HV, which, on account of the assumed independence of $\vec{\eta}$, $\vec{\zeta}_k$ can be set directly equal to zero. [Alternatively, since homotheties are essentially unique, it would follow $a^k\vec{\zeta}_k=(C/k)\vec{\eta}+\vec{\xi}$, where $\vec{\xi}$ is a KV, which can be absorbed by a suitable redefinition of the constant b in $\lambda=au+b$.] Taking all this into account, redefining nonessential combinations of constants and subtracting any proper KV [i.e., linear combinations of ∂_u and KV in (V,h) such as $\vec{\xi}$ above], we get

$$\vec{Y}=\left[k\left(\frac{a}{2}u^2+pu\right)-\epsilon_1 a\eta\right]\partial_u+(au+p)\vec{\eta},$$

where p is a constant.

It is immediate to check that the above CKV $\vec{Y}$, whose associated conformal factor is $\phi=k(au+p)$, is in fact a SCKV, that is $\phi_{a;b}=0$. Also note that the HV $pu\partial_u+p\vec{\eta}$ can be subtracted from $\vec{Y}$, the resulting vector

$$\vec{Y}'=[k(a/2)u^2-\epsilon_1 a\eta]\partial_u+au\vec{\eta} \tag{107}$$

being, indeed, a SCKV.

Case 2: We now have $\lambda=(a/2)u^2+bu+c$, $B=-b\eta+m$, and also $\epsilon_1 a^k\psi_k=a\eta+C$, where a,b,m and C are constants. This implies that $\vec{X}\equiv a^k\vec{\zeta}_k$ is a CKV in (V,h) whose associated conformal factor is precisely $\epsilon_1(a\eta+C)$. A direct calculation using the forms (96) or (99) readily shows that no such CKV $\vec{X}$ can exist, and therefore this case turns out to be impossible.

C. (V,h) admits a proper GCKV

Let us turn our attention now to the case in which (V,h) admits a proper GCKV. Before analyzing the consequences this has on the conformal algebra of the $1+3$ reducible space–time $(M,\hat{g})$, we shall first explore the situation in a three-space (V,h). To this end, let $\vec{\zeta}$ be a GCKV in (V,h) with associated conformal factor ψ, we then have

$$\zeta_{A/B}=\psi h_{AB}, \quad \zeta_A=\zeta_{,A}, \tag{108}$$

where $\zeta=\zeta(x^D)$ is some function. The first equation above readily implies that $\vec{\zeta}$ cannot be null unless it is a KV. Taking a further covariant derivative we have

$$\zeta_{A/BC}=\psi_C h_{AB}, \quad \psi_C=\psi_{,C} \tag{109}$$

and the Bianchi identities imply, since $\zeta_{A/B}=\zeta_{B/A}$,

$$R_{ABCD}\zeta^D=\psi_A h_{BC}-\psi_B h_{AC}. \tag{110}$$

Contracting both sides of the above equation with ζ^C yields

$$0=\psi_A\zeta_B-\psi_B\zeta_A \tag{111}$$

and then, unless $\vec{\zeta}$ is a HV (in which case the equation above is satisfied identically) which we are assuming is not, it follows that $\psi=\psi(\zeta)$, hence, from now on we shall write $\psi_A=\tilde{\psi}\zeta_A$, where the tilde stands for the derivative with respect to the function ζ, i.e., $\tilde{\psi}=\mathrm{d}\psi/\mathrm{d}\zeta$. Also, differentiating (111) and using (108) it follows that $\vec{\zeta}$ cannot be a SCKV (unless it is a KV).

Following a procedure similar to the one in Sec. V B, we choose a coordinate v adapted to $\vec{\zeta}$ and two other coordinates x^2,x^3 so that

$$\vec{\zeta}=\partial_v, \quad \zeta_A\,\mathrm{d}x^A=\epsilon_2\exp(2V(v))\mathrm{d}v, \quad \zeta=\epsilon_2\int \mathrm{d}v\,\exp(2V(v)), \tag{112}$$

$$\mathrm{d}\sigma^2=e^{2V(v)}[\epsilon_2+\Omega^2(x^k)(\epsilon_3(\mathrm{d}x^2)^2+(\mathrm{d}x^3)^2)] \tag{113}$$

the conformal factor then being $\psi=V'(v)$ where the prime indicates derivative with respect to v. Note that

$$\tilde{\psi}=\frac{\mathrm{d}\psi}{\mathrm{d}\zeta}=\frac{\mathrm{d}\psi}{\mathrm{d}v}\left(\frac{\mathrm{d}\zeta}{\mathrm{d}v}\right)^{-1}=\epsilon_2V''e^{-2V(v)}. \tag{114}$$

Alternatively, a new coordinate w can be defined such that

$$w\equiv\int dv\,\exp(2V(v)) \tag{115}$$

and then

$$\vec{\zeta}=M(w)\partial_w, \quad M(w)=\exp(V(v(w))), \quad \zeta_A\mathrm{d}x^A=\epsilon_2M(w)\mathrm{d}w, \quad \text{and} \quad \zeta=\epsilon_2\int \mathrm{d}w\,M(w), \tag{116}$$

$$\mathrm{d}\sigma^2=\epsilon_2\mathrm{d}w^2+M^2(w)\Omega^2(x^k)(\epsilon_3(\mathrm{d}x^2)^2+(\mathrm{d}x^3)^2) \tag{117}$$

the conformal factor is $\psi=M'(w)$ (the prime now meaning derivative with respect to w) and, as before,

$$\tilde{\psi}=\frac{\mathrm{d}\psi}{\mathrm{d}\zeta}=\frac{\mathrm{d}\psi}{\mathrm{d}w}\left(\frac{\mathrm{d}\zeta}{\mathrm{d}w}\right)^{-1}=\epsilon_2\frac{M''}{M}. \tag{118}$$

The above metric describes the situation in which one proper (non-HV) GCKV exists in (V,h), with h being of arbitrary signature.

Let us now go back to the original problem of finding CKV in the $1+3$ reducible space–time whose three-dimensional factor (V,h) we are assuming to admit a GCKV. We next reproduce, for the sake of convenience, the original equations (72)–(74) with the conformal factor ϕ renamed as $\dot{\Sigma}$:

$$Y_{u,u}=\epsilon_1\dot{\Sigma}, \tag{119}$$

$$Y_{u,A}+Y_{A,u}=0, \tag{120}$$

$$Y_{A/B}+Y_{B/A}=2\dot{\Sigma}\,h_{AB}. \tag{121}$$

Again, (121) implies that $Y^B(u,x^D)\partial_B$ is a CKV in every $V(u)$ for u fixed.

Now, assume first that only one proper GCKV is admitted in (V,h), say $\vec{\zeta}$ with conformal factor ψ. From our previous developments it follows that no proper GHV or GKV can exist in (V,h); therefore we may consider a basis of $\mathcal{C}(V,h)$ given by $\{\vec{\zeta},\vec{\chi}_k\}$ where, again, $\vec{\chi}_k$ are nongradient CKV (possibly HV and KV) with conformal factors ϕ_k.

From the remark above it follows that

$$Y^B(u,x^D)\partial_B=\lambda(u)\vec{\zeta}+\lambda^k(u)\vec{\chi}_k$$

which, upon substitution into (121), yields

$$\dot{\Sigma}=\lambda(u)\psi(x^D)+\lambda^k(u)\phi_k(x^D)$$

that can be formally integrated to give

$$\Sigma=\psi(x^D)\int \mathrm{d}u\,\lambda(u)+\phi_k(x^D)\int \mathrm{d}u\,\lambda^k(u)+B(x^D),$$

where the terms resulting from the constants of integration arising from $(\int du\lambda(u))$, etc., have been absorbed into the function of integration $B(x^D)$.

Substituting this into (120) and taking into account that $\psi_{,A}=\tilde{\psi}\zeta_A$ we get

$$\left(\int \mathrm{d}u\,\lambda(u)\right)\tilde{\psi}\zeta_A+\left(\int \mathrm{d}u\,\lambda^k(u)\right)\phi_{k,\,A}+B_{,A}+\dot{\lambda}(u)\zeta_A+\dot{\lambda}^k(u)\chi_{k\,A}=0 \tag{122}$$

and this implies that, for u fixed, $\dot{\lambda}^k\chi_{k\,A}$ must be a GCKV independent of ζ_A. Since this is not possible from our assumptions, it follows that $\dot{\lambda}^k=0$, that is $\lambda^k=a^k(=\text{const})$. Therefore the above-given equation reads now

$$\left[\left(\int \mathrm{d}u\,\lambda(u)\right)\tilde{\psi}+\dot{\lambda}(u)\right]\zeta_A+ua^k\phi_{k,\,A}+B_{,A}=0 \tag{123}$$

and differentiating with respect to u,

$$[\lambda(u)\tilde{\psi}+\ddot{\lambda}(u)]\zeta_A+a^k\phi_{k,\,A}=0, \tag{124}$$

which readily implies:

$$\tilde{\psi}=k\quad(\text{constant}),\quad k\lambda(u)+\ddot{\lambda}(u)=a\quad(\text{constant}),\quad a^k\phi_k=-a\zeta+c, \tag{125}$$

where c is a constant. Substituting this information back into (123) and taking into account that from $\tilde{\psi}=k$ and $k\lambda(u)+\ddot{\lambda}(u)=a$ it follows $\tilde{\psi}\int\mathrm{d}u\,\lambda(u)+\dot{\lambda}(u)=a\,u$, one easily gets

$$B_{,A}=0,$$

i.e.,

$$B=b\quad(\text{constant}) \tag{126}$$

and then, using $k\int\mathrm{d}u\,\lambda(u)+\dot{\lambda}(u)=a\,u$, Eq. (119) implies

$$Y_u=\epsilon_1(-\zeta\dot{\lambda}+cu+b).$$

Note that b can be set equal to zero without loss of generality, since it simply amounts to adding a constant multiple of the KV $\vec{u}=\partial_u$, and we shall do that in what follows, thus writing

$$Y_u=\epsilon_1(-\zeta\dot{\lambda}+cu). \tag{127}$$

Also, from the above-mentioned developments we get

$$Y_A=\lambda(u)\zeta_A+X_A, \quad X_A\equiv a^k\chi_{k\,A}, \tag{128}$$

where $\lambda(u)$ satisfies $\ddot{\lambda}+k\lambda=a$, the conformal factor associated with the GCKV $\vec{\zeta}$ is $\psi=k\zeta$, and $\vec{X}$ is a CKV (nongradient by assumption) whose associated conformal $\phi'\equiv a^k\phi_k$ factor is, from Eq. (125) $\phi'=-a\zeta+c$.

Consider next the CKV $\vec{Z}$ defined by $\vec{Z}\equiv(a/k)\vec{\zeta}+\vec{X}$. From the previous paragraph it follows that

$$\mathcal{L}_{\vec{Z}}h_{AB}=2ch_{AB},$$

that is: $\vec{Z}$ is a HV with homothetic constant c, therefore, put $\vec{X}$ above as $\vec{X}=\vec{Z}-(a/k)\vec{\zeta}$ and then

$$Y_A=\left(\lambda(u)-\frac{a}{k}\right)\zeta_A+Z_A, \tag{129}$$

where $\vec{Z}$ is a HV in (V,h) with homothetic constant c. If (V,h) admits no HV, then $c=\vec{Z}=0$ above; thus we finally have

$$\vec{Y}=(-\zeta\dot{\lambda}+cu)\partial_u+\left(\lambda(u)-\frac{a}{k}\right)\vec{\zeta}+\vec{Z}.$$

Since $cu\partial_u+\vec{Z}$ is a HV with homothetic constant c we can subtract it from the above to get

$$\vec{Y}=(-\zeta\dot{\lambda})\partial_u+\left(\lambda(u)-\frac{a}{k}\right)\vec{\zeta} \tag{130}$$

and $\vec{\zeta}$ is now that given by (112) or (99), the corresponding three-dimensional line elements then being (113) or (117), thus we can finally write for the line element of $(M,\hat{g})$ a $1+3$ reducible space–time admitting a CKV under these hypotheses:

$$d\hat{s}^2=\epsilon_1\,du^2+\epsilon_2\,dw^2+M^2(w)\Omega^2(x^k)(\epsilon_3(dx^2)^2+(dx^3)^2), \tag{131}$$

where

$$\epsilon_\alpha=\pm 1 \quad (\alpha=1,2,3) \quad \epsilon_1\epsilon_2\epsilon_3=-1, \quad \epsilon_1+\epsilon_2+\epsilon_3=+1.$$

The CKV is then

$$\vec{Y}=(-\zeta\dot{\lambda})\partial_u+\left(\lambda(u)-\frac{a}{k}\right)M(w)\partial_w, \tag{132}$$

where $\lambda(u)$ and $M(w)$ must satisfy [see (118)]

$$\ddot{\lambda}(u)+k\lambda(u)=a, \quad M''(w)=\epsilon_2 kM. \tag{133}$$

These equations can be easily integrated for $\epsilon_2 k>0, \epsilon_2 k<0$ and $\epsilon_2 k=0$ obtaining then explicit expressions for both the line element $d\hat{s}^2$ and proper CKV $\vec{Y}$.

In the next theorem, we make an attempt at summarizing the results thus far obtained.

Theorem 10: *Let $(M,\hat{g})$ a $1+3$ decomposable space–time; the following results hold regarding its conformal Lie algebra:*

(1) If (V,h) admits no GCKV then the only CKV that $(M,\hat{g})$ admits are HV and KV—their forms are those discussed in remarks $R0$-$R3$ at the beginning of this section.

(2) $(M,\hat{g})$ can admit a proper CKV $\vec{Y}$ if and only if (V,h) admits a GCKV, which can be either a GKV $\vec{\xi}$, or a GHV $\vec{\eta}$, or a (proper) GCKV $\vec{\zeta}$.

(3) If two or more GCKV are admitted by (V,h), then (V,h) is of constant curvature and conformally flat (and it is flat if one of the GCKV admitted is a GHV).

(4) If only one GCKV is admitted by (V,h), then:

(a) If it is a GKV $\vec{\xi}$, then $(M,\hat{g})$ admits a proper CKV $\vec{Y}$ if and only if $\vec{\xi}$ is null and a proper SCKV exists also in (V,h), $\vec{X}$, such that $[\vec{\xi},\vec{X}]=k\vec{\xi}$; the CKV $\vec{Y}$ is also a SCKV and the metric is a special type of generalized pp-wave. Otherwise [i.e., if no proper SCKV $\vec{X}$ exists in (V,h) or it exists but it does not satisfy $[\vec{\xi},\vec{X}]=k\vec{\xi}$], then $\vec{Y}$ is a HV, possibly KV.

(b) If it is a GHV $\vec{\eta}$ then $(M,\hat{g})$ admits a proper SCKV $\vec{Y}$ (unique up to the addition of HV). The line element and $\vec{Y}$ are:

$$d\hat{s}^2=\epsilon_1\, du^2+\epsilon_2\, dw^2+w^2\Omega^2(x^i)[\epsilon_3(dx^2)^2+(dx^3)^2], \tag{134}$$

$$\epsilon_\alpha=\pm 1 \quad (\alpha=1,2,3), \quad \epsilon_1\epsilon_2\epsilon_3=-1, \quad \epsilon_1+\epsilon_2+\epsilon_3=+1$$

$$\vec{Y}=\left(\frac{a}{2}ku^2-\epsilon_1\epsilon_2\frac{a}{2}kw^2\right)\partial_u+au\,\vec{\eta}, \quad \vec{\eta}=kw\,\partial_w. \tag{135}$$

(c) If it is a GCKV $\vec{\zeta}$ then $(M,\hat{g})$ admits a proper CKV $\vec{Y}$ (unique up to the addition of HV). The line element and $\vec{Y}$ are:

$$d\hat{s}^2=\epsilon_1\, du^2+\epsilon_2\, dw^2+M^2(w)\Omega^2(x^i)[\epsilon_3(dx^2)^2+(dx^3)^2], \tag{136}$$

$$\epsilon_\alpha=\pm 1 \quad (\alpha=1,2,3), \quad \epsilon_1\epsilon_2\epsilon_3=-1, \quad \epsilon_1+\epsilon_2+\epsilon_3=+1,$$

$$\vec{Y}=-\zeta\dot{\lambda}(u)\,\partial_u+\left(\lambda-\frac{a}{k}\right)\vec{\zeta}, \quad \zeta=\epsilon_2\int dw\, M(w), \quad \vec{\zeta}=M(w)\partial_w \tag{137}$$

and the functions $\lambda(u)$ and $M(w)$ satisfy

$$\frac{d^2\lambda}{du^2}+k\lambda=a, \quad \frac{d^2M}{dw^2}=\epsilon_2 kM. \tag{138}$$

VI. EXAMPLES

In this section we shall briefly discuss instances of double warped space–times. Notice that warped space–times are special cases of double warped ones (i.e., whenever one of the warping functions is constant), and they include relevant classes of space–times such as all the spherically, plane and hyperbolic symmetric space–times, the whole class of Friedmann–Robertson–Walker solutions, the Bertotti–Robertson space–time and many others, see Refs. 3 and 4 for further information.

Fluid space–times. We have not been able to find a proper double warped (i.e., nonwarped) perfect fluid solution due to the complicated form that the field equations take on account of the Ricci tensor (see Sec. III). However, it is indeed possible to find anisotropic fluid solutions satisfying the dominant energy condition (see Ref. 11), as the following two examples show. The energy–momentum tensor is in both cases of the Segre type $\{1,1(11)\}$, and therefore can be written as

$$T_{ab}=\mu u_a u_b+p_1 z_a z_b+p_2[x_a x_b+y_a y_b], \tag{139}$$

where $\{u_a,z_a,x_a,y_a\}$ form an orthogonal tetrad ($-u_a u^a=x^a x_a=y^a y_a=z^a z_a=1$, the rest of the products being zero), u^a is aligned with the velocity of the fluid, and μ, p_1, p_2 are, respectively, the energy density and two (different) pressures as measured by an observer co-moving with the fluid. The dominant energy condition implies then

$$\mu \geqslant 0, \quad \mu \pm p_1 \geqslant 0, \quad \mu \pm p_2 \geqslant 0. \tag{140}$$

Case 1: Consider the line element given by

$$ds^2 = z(-dt^2 + dx^2) + t(dy^2 + dz^2), \tag{141}$$

where t and z are both non-negative. This space–time can be seen to represent an anisotropic fluid with an energy–momentum tensor given by (139) with

$$u_a = (-z^{1/2}\cosh\Phi, 0, 0, t^{1/2}\sinh\Phi), \qquad z_a = (-z^{1/2}\sinh\Phi, 0, 0, t^{1/2}\cosh\Phi),$$

$$x_a = (0, z^{1/2}, 0, 0), \quad y_a = (0, 0, t^{1/2}, 0),$$

$$\cosh\Phi = \sqrt{\frac{t}{t-z}}, \qquad \sinh\Phi = -\sqrt{\frac{z}{t-z}} \qquad \text{for} \quad t - z > 0$$

$$\cosh\Phi = \sqrt{\frac{z}{z-t}}, \qquad \sinh\Phi = -\sqrt{\frac{t}{z-t}} \qquad \text{for} \quad z - t > 0$$

and also

$$\mu = \frac{|z-t|}{4t^2z^2}, \quad p_1 = \mu, \quad p_2 = \frac{z-t}{4t^2z^2}.$$

This has to be understood as two different open submanifolds; namely, the one defined by $t - z > 0$ and that defined by $z - t > 0$. In both cases the dominant energy condition (140) is satisfied. As a final comment to this example, it can noted that the above metric admits the Killing vectors $\vec{\xi}_1 = \partial_x$, $\vec{\xi}_2 = \partial_y$, and the homothetic vector $\vec{\eta} = t\partial_t + x\partial_x + y\partial_y + z\partial_z$ with homothetic constant $\psi = 3/2$.

Case 2: Consider next the following line element:

$$ds^2 = (m - kz^2)(-dt^2 + dx^2) + (q + kt^2)(dy^2 + dz^2), \tag{142}$$

where m, k, q are constants, $m, q > 0$ and $k \geqslant 0$ in order for the energy conditions and other positivity requirements to be satisfied, and the range of the z coordinate is restricted to $m - kz^2 > 0$. Again, this represents an anisotropic fluid with energy–momentum tensor given by (139) where

$$u_a = (-(m-kz^2)^{1/2}\cosh\Phi, 0, 0, (q+kt^2)^{1/2}\sinh\Phi),$$

$$z_a = (-(m-kz^2)^{1/2}\sinh\Phi, 0, 0, (q+kt^2)^{1/2}\cosh\Phi),$$

$$x_a = (0, (m-kz^2)^{1/2}, 0, 0), \quad y_a = (0, 0, (q+kt^2)^{1/2}, 0),$$

$$\cosh 2\Phi = \frac{qz^2 + mt^2}{|qz^2 - mt^2 + 2kt^2z^2|}.$$

The density μ and the pressures p_1, p_2 are given in this case by

$$\mu = k(m-kz^2)^{-2}(q+kt^2)^{-2}\{k|qz^2 - mt^2 + 2kt^2z^2| + (m-kz^2)(q+kt^2)\},$$

$$p_1 = k(m-kz^2)^{-2}(q+kt^2)^{-2}\{k|qz^2 - mt^2 + 2kt^2z^2| - (m-kz^2)(q+kt^2)\},$$

$$p_2 = \frac{(-3\,qm - 2\,kt^2m + 2\,qkz^2 + k^2t^2z^2)k}{(-m+kz^2)^2(q+kt^2)^2}$$

and it can be seen that the dominant energy condition (140) is satisfied.

Vacuum space–times. Notice from the second equation in (66) that any double warped space–time representing a vacuum solution of Einstein's field equations, must in fact be warped (i.e., either $\theta_{1,\alpha}=0$ or $\theta_{2,A}=0$). Examples in this class include Schwarzschild solution and its plane and hyperbolic symmetric equivalents.

The characterization of class A and class B double warped space–times given in Theorems 4, 5, and 7 should prove useful in formulating an algorithm for classifying such metrics. this is so because this characterization is coordinate independent although tetrad dependent. In what follows the tetrads described in Theorems 4, 5, and 7 will be designated as dw tetrads of class A and B, respectively. In order to determine weather g represents a double warped metric we suggest the following classification scheme:

(1) Choose a coordinate system.
(2) Choose a canonical complex null tetrad $\{k_a, l_a, m_a, \overline{m_a}\}$ and write their components in the coordinate system chosen in (1).
(3) Determine the NP spin coefficients and their NP derivatives in tetrad (2).
(4) If the scalars determined in (3) satisfy the relations of Theorems 4 or 5 (Theorem 7) then the metric is a double warped space–time of class A (class B) and the algorithm stops here, otherwise go to step (5).
(5) If possible, find the Lorentz transformations that transform tetrad (2) into a dw tetrad, i.e., such that the corresponding NP spin coefficients and their NP derivatives obey the conditions of Theorems 4 or 5 (Theorem 7). If such transformations exist then the space–time is a double warped space–time of class A (class B), otherwise it is not double warped.

Unfortunately, step (5) of this procedure is not straightforward, since finding the Lorentz transformation which maps tetrad (2) into a dw tetrad can be difficult and such a transformation might not exist in which case the metric is not double warped.

The algorithm described here not only describes a way of determining whether a particular metric is double warped or not but also suggests a method for obtaining such space–times. For example, we suspect that one can obtain type D vacuum warped metrics of class B, for which $\rho=\mu=0$, by following a similar integration procedure to the one performed by Kinnersley.[18] In this paper, Kinnersley chooses coordinates such that $l^\alpha=\delta_2^\alpha$, making $x^2=r$ an affine parameter along l^α. These are the special coordinates given in Ref. 9. The idea would then be to express a dw tetrad in these special coordinates, write the NP equations taking into account (63). In order to determine explicitly the tetrad components in these special coordinates one must integrate the corresponding equations. By following this procedure we hope to obtain such metrics in future work.

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