The one-sided inverse along an element in semigroups and rings

Jianlong Chen*, Honglin Zou†, Huihui Zhu‡, Pedro Patrício§
*Department of Mathematics, Southeast University, Nanjing 210096, China.
†School of Mathematics and Statistics, Hubei Normal University, Huangshi 435002, China.
‡School of Mathematics, Hefei University of Technology, Hefei 230009, China.
§CMAT-Centro de Matemática and Departamento de Matemática e Aplicacões, Universidade do Minho, Braga 4710-057, Portugal.

Abstract: The concept of the inverse along an element was introduced by X. Mary in 2011. Later, H. H. Zhu etc. introduced the one-sided inverse along an element. In this paper, we first give a new existence criterion for the one-sided inverse along a product and characterize the existence of Moore-Penrose inverse by means of one-sided invertibility of certain element in a ring. In addition, we show that $a \in S^{\dagger} \bigcap S^{\#}$ if and only if $(a^*a)^k$ is invertible along a if and only if $(aa^*)^k$ is invertible along a in a $*$-monoid S, where k is an arbitrary given positive integer. Finally, we prove that the inverse of a along aa^* coincides with core inverse of a under the condition $a \in S^{\{1,4\}}$ in a $*$-monoid S.

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1 Introduction

Throughout this paper, S is a monoid (semigroup with identity) and R is a ring with identity. We say a is (von Neumann) regular in S if there exists $x \in S$ such that $axa = a$. Such x is called an inner inverse of a and denoted by a^-. An involution $*$: S $\to$ S is an anti-isomorphism which satisfies $(ab)^* = b^*a^*$ and $(a^*)^* = a$, where $a, b \in S$. $*$-monoid denotes the monoid with an involution.

Let us recall some definitions of generalized inverses. Let S be a $*$-monoid, an element $a \in S$ is said to Moore-Penrose invertible if the following equations:

$$(1)\ axa = a, \quad (2)\ xax = x, \quad (3)\ (ax)^* = ax, \quad (4)\ (xa)^* = xa$$

*Corresponding author. E-mail: jlchen@seu.edu.cn
†E-mail: honglinzou@163.com
‡E-mail: ahzhh08@sina.com
§E-mail:pedro@math.uminho.pt

has a common solution [12]. Such solution is unique if it exists, and is usually denoted by $a^\dagger$. The set of all Moore-Penrose invertible elements of S will be denoted by $S^\dagger$. If $x \in S$ satisfies both (1) and (3), then x is called an $\{1,3\}$-inverse of a and denoted by $a^{(1,3)}$. The set of all $\{1,3\}$-invertible elements of S will be denoted by $S^{\{1,3\}}$. Similarly, if $x \in S$ satisfies both (1) and (4), then x is called an $\{1,4\}$-inverse of a and denoted by $a^{(1,4)}$. The set of all $\{1,4\}$-invertible elements of S will be denoted by $S^{\{1,4\}}$.

The Drazin inverse [2] of $a \in S$ is the element $x \in S$ which satisfies

$$a^k = a^{k+1}x, \quad xax = x, \quad ax = xa, \quad \text{for some} \quad k \geq 1.$$

The element x above is unique if it exists and is denoted by a^D. The least such k is called the index of a, and denoted by $\operatorname{ind}(a)$. In particular, when $\operatorname{ind}(a)$=1, the Drazin inverse a^D is called the group inverse of a and it is denoted by $a^\#$. The set of all Drazin (resp. group) invertible elements of S will be denoted by S^D (resp. $S^\#$).

The core (resp. dual core) inverse [13] of $a \in R$ is the element $x \in R$ which satisfies

$$axa = a, \quad xR = aR \text{ (resp. } Rx = Ra), \quad Rx = Ra^* \text{ (resp. } xR = a^*R).$$

The element x above is unique if it exists and is denoted by $a^{\oplus}$ (resp. $a_{\oplus}$). The set of all core (resp. dual core) invertible elements of R will be denoted by $R^{\oplus}$ (resp. $R_{\oplus}$).

In [9], X. Mary introduced a new generalized inverse using Green's preorders and relations [3], named the inverse along an element. The element $a \in S$ will be said to be invertible along $d \in S$ if there exists $b \in S$ such that

$$bad = d = dab, \quad bS \subset dS, \quad Sb \subset Sd.$$

If such b exists, then it is unique and will be denoted by $a^{\|d}$. This inverse unify some well-known generalized inverse such as group inverse, Drazin inverse and Moore-Penrose inverse, that is $a^\# = a^{\|a}$, $a^D = a^{\|a^k}$ for some integer m and $a^\dagger = a^{\|a^*}$.

In [10], X. Mary and P. Patrício gave a very useful existence criterion of $a^{\|d}$ by means of a unit in the ring, that is a is invertible along d if and only if $ad + 1 - d^- d$ is invertible if and only if $da + 1 - dd^-$ is invertible, when d is regular.

In [14], H. H. Zhu etc. introduced left (right) invertible along an element. An element $a \in S$ is left (resp. right) invertible along $d \in S$ if there exists $b \in S$ such that

$$bad = d \text{ (resp. } dab = d), \quad Sb \subset Sd \text{ (resp. } bS \subset dS).$$

They proved a surprising conclusion in a $*$-monoid S, $a \in S$ is left invertible along a^* if and only if a is right invertible a^* if and only if a is Moore-Penrose invertible.

In this paper, our motivation is that if a is left (right) invertible along d, then we will consider that when d is left (right) invertible along a in a semigroup (or ring). For example, we can easily see that a is invertible along a^* if and only if a^* is invertible along a if and only if a is Moore-Penrose invertible, where $a \in S$, S is a $*$-monoid.

In [14], the authors gave an existence criterion of the one-side inverse along pmq (see [14, Theorem 3.2]). In addition, D. S. Rakić etc. [13] proved $a^{\oplus} = a^{\|aa^*}$ and $a_{\oplus} = a^{\|a^*a}$ under the condition $a \in R^{\dagger}$. According to these facts. In section 2, we further consider the inverse along a product pmq and generalize some results of [14]. Conversely, we consider that pmq is invertible along a. Also, we prove that a regular element $a \in R$ is Moore-Penrose invertible if and only if $(aa^*)^k + 1 - aa^-$ is left invertible if and only $(aa^*)^k + 1 - aa^-$ is right invertible, where k is an arbitrary given positive integer. In section 3, we mainly obtain that $a \in S^{\dagger} \bigcap S^{\#}$ if and only if $(a^*a)^k$ is invertible along a if and only if $(aa^*)^k$ is invertible along a, where k is also an arbitrary given positive integer. In section 4, we give that $a \in S^{\oplus}$ if and only if a is invertible along aa^* if and only if $a \in S^{\dagger} \cap S^{\sharp}$, under the condition $a \in S^{\{1,4\}}$.

Let $a \in R$, by a_l^{-1} and a_r^{-1} we denote a left inverse and a right inverse of a, respectively. First, we state some auxiliary results we will rely on.

Lemma 1.1. [8, Exercise 1.6] *Let $a, b \in R$.*

(1) *If $1+ab$ is left invertible, then $1+ba$ is left invertible and $(1+ba)_l^{-1} = 1-b(1+ab)_l^{-1}a$.*

(2) *If $1+ab$ is right invertible, then $1+ba$ is right invertible and $(1+ba)_r^{-1} = 1-b(1+ab)_r^{-1}a$.*

(3) *If $1+ab$ is invertible, then $1+ba$ is invertible and $(1+ba)^{-1} = 1-b(1+ab)^{-1}a$.*

Lemma 1.2. *Let $a, d \in S$. Then*

(1) [14, Theorem 2.3] *a is left invertible along d if and only if $Sd = Sdad$. In this case, ud is a left inverse of a along d, where d=$udad$, $u \in S$.*

(2) [14, Theorem 2.4] *a is right invertible along d if and only if $dS = dadS$. In this case, dv is a right inverse of a along d, where d=$dadv$, $v \in S$.*

(3) [10, Theorem 2.2] *a is invertible along d if and only if $Sd = Sdad$ and $dS = dadS$.*

(4) *a is invertible along d with inverse y if and only if a is right invertible along d with a right inverse x and a is left invertible along d with a left inverse z. In this case y=x=z.*

Proof. (4) We only need prove $y = x = z$. Suppose a is invertible along d with inverse y, then $yad = d$, $Sy \subset Sd$. From $Sy \subset Sd$, it follows that there exists $t_1 \in S$ such that $y = t_1 d$. Since x is a right inverse of a along d, we get $dax = d$ and $xS \subset dS$, which implies $x = dt_2$ for some $t_2 \in S$. Hence, $y = t_1 d = t_1 dax = yax$, and $x = dt_2 = yadt_2 = yax$. So, $y = x$ holds. Similarly, we have $y = z$.

Lemma 1.3. *Let $a, d \in R$ with d regular. Then*

(1) [14, Corollary 3.3] *a is left invertible along d if and only if $u = da + 1 - dd^-$ is left invertible if and only if $v = ad + 1 - d^-d$ is left invertible. In this case, $u_l^{-1}d$ is a left inverse of a along d.*

(2) [14, Corollary 3.5] *a is right invertible along d if and only if $u = da + 1 - dd^-$ is right invertible if and only if $v = ad + 1 - d^-d$ is right invertible. In this case, dv_r^{-1} is a right inverse of a along d.*

(3) [10, Theorem 3.2] *a is invertible along d if and only if $u = da + 1 - dd^-$ is invertible if and only if $v = ad + 1 - d^-d$ is invertible. In this case, $a^{\|d} = u^{-1}d = dv^{-1}$.*

Lemma 1.4. [14, Theorem 2.16] *Let S be a $*$-monoid and let $a \in S$. Then a is Moore-Penrose invertible if and only if $a \in aa^*aS$ if and only if $a \in Saa^*a$.*

Lemma 1.5. *Let S be a $*$-monoid and let $a \in S$.*
(1) [14, Theorem 2.19] *If $a = aa^*ax$ for some $x \in S$, then $a \in R^\dagger$ and $a^\dagger = a^*ax^2a^*$.*
(2) [14, Theorem 2.20] *If $a = yaa^*a$ for some $y \in S$, then $a \in R^\dagger$ and $a^\dagger = a^*y^2aa^*$.*

Lemma 1.6. [5, Theorem 1] *Let $a \in S$. Then $a \in S^\#$ if and only if $a = a^2x = ya^2$ for some $x, y \in S$. In this case, $a^\# = yax = y^2a = ax^2$.*

Lemma 1.7. [4] *Let R be a $*$-ring and let $a, x, y \in R$. Then*
(1) *x is a $\{1,3\}$-inverse of a if and only if $a = x^*a^*a$.*
(2) *y is a $\{1,4\}$-inverse of a if and only if $a = aa^*y^*$.*

Lemma 1.8. [7, Lemma 5.1] *Let R be a $*$-ring and let $a \in R$. Then $a \in R^\dagger$ if and only if there exist $x, y \in R$ such that $axa = a = aya, (ax)^* = ax, (ya)^* = ya$. In this case, $a^\dagger = yax$.*

Next Lemma is proved in a $*$-ring (see [1, Proposition 2.1]). Indeed, it is true in a $*$-monoid.

Lemma 1.9. *Let S be a $*$-monoid and let $a \in S$. Then*
(1) *$a \in S^{\oplus}$ if and only if $a \in S^\sharp \cap S^{\{1,3\}}$. In this case, $a^{\oplus} = a^\sharp aa^{(1,3)}$.*
(2) *$a \in S_{\oplus}$ if and only if $a \in S^\sharp \cap S^{\{1,4\}}$. In this case, $a_{\oplus} = a^{(1,4)}aa^\sharp$.*

2 The one-sided inverse along the product pmq

In this section, we give a new existence criterion for the one-side inverse along a product pmq in a ring R, which covers [14, Theorem 3.2].

Theorem 2.1. *Let $a, m, p, p', q, q' \in R$ with m regular and $k \geq 1$. If $p'pm = m = mqq'$, then the following are equivalent:*
(1) *a is left invertible along pmq;*
(2) *$u = (qapm)^k + 1 - m^-m$ is left invertible;*
(3) *$v = (mqap)^k + 1 - mm^-$ is left invertible.*
In this case, $pv_l^{-1}(mqap)^{k-1}mq$ is a left inverse of a along pmq.

Proof. (2) $\Leftrightarrow$ (3) Since $u = (qapm)^{k-1}qapm + 1 - m^-m = 1 + ((qapm)^{k-1}qap - m^-)m$, according to Lemma 1.1(1), we have u is left invertible, i.e. $1 + m((qapm)^{k-1}qap - m^-) = v$ is left invertible.

(1) $\Rightarrow$ (2) Suppose that a is left invertible along pmq, by Lemma 1.2(1), we get $pmq = xpmqapmq$ for some $x \in R$. Multiplying the previous equality by q' from the right side and using the equality $mqq' = m$, we have $pm = xpmqapm$. Repeatedly use the equality

$pm = xpmqapm$, we have $pm = x(pm)qapm = x(xpmqapm)qapm = x^2pm(qapm)^2 = \cdots = x^k(pm)(qapm)^k$. Then, note that $p'pm = m$, we get

$$\begin{aligned} & (mm^-p'x^kpmm^- + 1 - mm^-)(m(qapm)^km^- + 1 - mm^-) \\ = \; & mm^-p'(x^kpmm^-m(qapm)^k)m^- + 1 - mm^- \\ = \; & mm^-p'pmm^- + 1 - mm^- \\ = \; & 1, \end{aligned}$$

which implies $m(qapm)^km^- + 1 - mm^- = 1 + m((qapm)^k - 1)m^-$ is left invertible. Applying Lemma 1.1(1), we deduce that $1+((qapm)^k-1)m^-m = (qapm)^km^-m+1-m^-m = (qapm)^k + 1 - m^-m = u$ is left invertible.

(3) $\Rightarrow$ (1) If $v = (mqap)^k + 1 - mm^-$ is left invertible, then there exists $s \in R$ such that $s((mqap)^k + 1 - mm^-) = 1$. Multiplying the previous equation by m from the right side yields $m = s(mqap)^km$. Let $b = ps(mqap)^{k-1}mq$, then $ba(pmq) = ps(mqap)^{k-1}mqapmq = ps(mqap)^kmq = pmq$. Since $p'pm = m$, we get $b = ps(mqap)^{k-1}mq = ps(mqap)^{k-1}p'pmq$, which implies $Rb \subset Rpmq$. Therefore, b is a left inverse of a along pmq. □

As special cases of Theorem 2.1, we get the following results.

Corollary 2.2. [14, Theorem 3.2] *Let $a, m, p, p', q, q' \in R$ with m regular. If $p'pm = m = mqq'$, then the following are equivalent:*

(1) *a is left invertible along pmq;*

(2) *$u = qapm + 1 - m^-m$ is left invertible;*

(3) *$v = mqap + 1 - mm^-$ is left invertible.*

In this case, $pv_l^{-1}mq$ is a left inverse of a along pmq.

Corollary 2.3. *Let $a, m \in R$ with m regular and $k \geq 1$. Then the following are equivalent:*

(1) *a is left invertible along m;*

(2) *$u = (am)^k + 1 - m^-m$ is left invertible;*

(3) *$v = (ma)^k + 1 - mm^-$ is left invertible.*

In this case, $v_l^{-1}(ma)^{k-1}m$ is a left inverse of a along m.

Corollary 2.4. *Let $a \in R$ be regular and $k \geq 1$. Then the following are equivalent:*

(1) *$Ra = Ra^2$;*

(2) *1 is left invertible along a;*

(3) *$u = a^k + 1 - a^-a$ is left invertible;*

(4) *$v = a^k + 1 - aa^-$ is left invertible.*

In this case, $v_l^{-1}a^k$ is a left inverse of 1 along a.

Proof. (1) $\Leftrightarrow$ (2) By Lemma 1.2(1), we have (1) $\Leftrightarrow$ (2).

(2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) In Corollary 2.3, take $a = 1$, $m = a$. then (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4). □

Dually, we have the following results.

Theorem 2.5. *Let $a, m, p, p', q, q' \in R$ with m regular and $k \geq 1$. If $p'pm = m = mqq'$, then the following are equivalent:*

(1) *a is right invertible along pmq;*
(2) *$u = (qapm)^k + 1 - m^- m$ is right invertible;*
(3) *$v = (mqap)^k + 1 - mm^-$ is right invertible.*
In this case, $pm(qapm)^{k-1}u_r^{-1}q$ is a right inverse of a along pmq.

Corollary 2.6. [14, Theorem 3.4] *Let $a, m, p, p', q, q' \in R$ with m regular. If $p'pm = m = mqq'$, then the following are equivalent:*
(1) *a is right invertible along pmq;*
(2) *$u = qapm + 1 - m^- m$ is right invertible;*
(3) *$v = mqap + 1 - mm^-$ is right invertible.*
In this case, $pmu_r^{-1}q$ is a right inverse of a along pmq.

Corollary 2.7. *Let $a, m \in R$ with m regular and $k \geq 1$. Then the following are equivalent:*
(1) *a is right invertible along m;*
(2) *$u = (am)^k + 1 - m^- m$ is right invertible;*
(3) *$v = (ma)^k + 1 - mm^-$ is right invertible.*
In this case, $m(am)^{k-1}u_r^{-1}$ is a right inverse of a along m.

Corollary 2.8. *Let $a \in R$ be regular and $k \geq 1$. Then the following are equivalent:*
(1) *$aR = a^2R$;*
(2) *1 is right invertible along a;*
(3) *$u = a^k + 1 - a^- a$ is right invertible;*
(4) *$v = a^k + 1 - aa^-$ is right invertible.*
In this case, $a^k u_r^{-1}$ is a right inverse of a^{k-1} along a.

According to Corollary 2.3, Corollary 2.7 and Lemma 1.2(4), we have the following result, which generalize [10, Theorem 3.2].

Corollary 2.9. *Let $a, m \in R$ with m regular and $k \geq 1$. Then the following are equivalent:*
(1) *a is invertible along m;*
(2) *$u = (am)^k + 1 - m^- m$ is invertible;*
(3) *$v = (ma)^k + 1 - mm^-$ is invertible.*
In this case, $a^{\parallel m} = v^{-1}(ma)^{k-1}m = m(am)^{k-1}u^{-1}$.

We know that 1 is invertible a if and only if $a \in R^{\#}$ (see [10, Corollary 3.4]). By Corollary 2.9, we get

Corollary 2.10. *Let $a \in R$ be regular and $k \geq 1$. Then the following are equivalent:*
(1) *$a \in R^{\#}$;*
(2) *$u = a^k + 1 - a^- a$ is invertible;*
(3) *$v = a^k + 1 - aa^-$ is invertible.*
In this case, $a^{\#} = a^k u^{-1} = v^{-1}a^k$.

In [14], H. H. Zhu etc. showed $a \in R^{\dagger}$ if and only if a is left(or right) invertible along a^*. In [11], P. Patrício proved that $a \in R^{\dagger}$ if and only if $aa^* + 1 - aa^-$ is invertible if and only if $a^*a + 1 - a^-a$ is invertible. In the following theorem, we characterize the existence of $a^{\dagger}$ by means of one-side invertibility.

Theorem 2.11. *Let $a \in R$ be regular and $k \geq 1$. Then the following are equivalent:*
(1) *a is Moore-Penrose invertible;*
(2) *$u = (aa^*)^k + 1 - aa^-$ is left invertible;*
(3) *$u = (aa^*)^k + 1 - aa^-$ is right invertible;*
(4) *$v = (a^*a)^k + 1 - a^-a$ is left invertible;*
(5) *$v = (a^*a)^k + 1 - a^-a$ is right invertible.*
In this case,

$$\begin{aligned} a^{\dagger} &= a^*(u_l^{-1}(aa^*)^{k-1})^2 aa^* = a^*a((a^*a)^{k-1}v_r^{-1})^2 a^* \\ &= a^*(aa^*)^{k-1}(u_l^{-1})^* = (v_r^{-1})^*(a^*a)^{k-1}a^*. \end{aligned}$$

Proof. (1) $\Leftrightarrow$ (2) Since a is regular, then a^* is regular and $(a^*)^- = (a^-)^*$. In Corollary 2.7, let $m = a^*$. Then we have that a is right invertible along a^* if and only if $(aa^*)^k + 1 - (a^*)^-a^* = (aa^*)^k + 1 - (a^-)^*a^* = (aa^*)^k + 1 - (aa^-)^* = u^*$ is right invertible. Note that u is left invertible if and only if u^* is right invertible. In addition, $(u^*)_r^{-1} = (u_l^{-1})^*$. Thus, we get (1) $\Leftrightarrow$ (2). In this case, $a^*(aa^*)^{k-1}(u_l^{-1})^*$ is a right inverse of a along a^*. Applying Lemma 1.2(4), we get $a^{\dagger} = a^*(aa^*)^{k-1}(u_l^{-1})^*$.

(1) $\Leftrightarrow$ (5) Similar to the proof of (1) $\Leftrightarrow$ (2). Also, we can have $a^{\dagger} = (v_r^{-1})^*(a^*a)^{k-1}a^*$.

(2) $\Leftrightarrow$ (4) and (3) $\Leftrightarrow$ (5) Applying Lemma 1.1.

Next, we give the expression for $a^{\dagger}$. Since u is left invertible, there exists $r \in R$ such that $ru = 1$, which implies $rua = a$. Thus, $a = rua = r((aa^*)^k + 1 - aa^-)a = r(aa^*)^{k-1}aa^*a$, by Lemma 1.5(2), we get $a^{\dagger} = a^*(u_l^{-1}(aa^*)^{k-1})^2 aa^*$. Similarly, we can prove another expression for $a^{\dagger}$. □

Take $k = 1$ in Theorem 2.11, then we obtain the following corollary.

Corollary 2.12. *Let $a \in R$ be regular. Then the following are equivalent:*
(1) *a is Moore-Penrose invertible;*
(2) *$u = aa^* + 1 - aa^-$ is left invertible;*
(3) *$u = aa^* + 1 - aa^-$ is right invertible;*
(4) *$v = a^*a + 1 - a^-a$ is left invertible;*
(5) *$v = a^*a + 1 - a^-a$ is right invertible.*
In this case, $a^{\dagger} = a^*u_l^{-2}aa^* = a^*av_r^{-2}a^* = a^*(u_l^{-1})^* = (v_r^{-1})^*a^*$.

In [9], X. Mary showed that $a \in R$ is invertible along a^k if and only if a is Drazin invertible. Naturally, we next consider when a^k is invertible along a.

Theorem 2.13. *Let $a \in S$ and $k \geq 0$. Then the following are equivalent:*
(1) *a^k is left invertible along a;*
(2) *$Sa = Sa^2$.*

Proof. (1) ⇒ (2) Suppose that a^k is left invertible along a, by Lemma 1.2(1), we have $Sa \subset Sa^k a^2 \subset Sa^2$, which implies $Sa = Sa^2$.

(2) ⇒ (1) Assume $Sa = Sa^2$, then there exists $r \in S$ such that $a = ra^2$. Thus, $a = ra^2 = r^2a^3 = \cdots = r^{k+1}a^{k+2} \in Saa^ka$. According to Lemma 1.2(1) again, we get a^k is left invertible along a. □

Dually, we have

Theorem 2.14. *Let $a \in S$ and $k \geq 0$. Then the following are equivalent:*

(1) *a^k is right invertible along a;*

(2) *$aS = a^2S$.*

Using Theorem 2.13 and Theorem 2.14, we obtain

Corollary 2.15. *Let $a \in S$ and $k \geq 0$. Then the following are equivalent:*

(1) *a^k is invertible along a;*

(2) *$a \in S^{\#}$.*

We next consider when the product paq is invertible along d under certain condition.

Theorem 2.16. *Let $a, d, p, p', q, q' \in S$. If $q'qd = d = dpp'$, then the following are equivalent:*

(1) *paq is invertible along d with inverse y;*

(2) *pa is right invertible along qd with a right inverse x and aq is left invertible along dp with a left inverse z.*

In this case, y=zax.

Proof. (1) ⇒ (2) Suppose paq is invertible along d, by Lemma 1.2(3), we have $dpaqdS = dS$ and $Sdpaqd = Sd$, which imply $qdpaqdS = qdS$ and $Sdpaqdp = Sdp$. According to Lemma 1.2(1)(2), we have pa is right invertible along qd and aq is left invertible along dp.

(2) ⇒ (1) Suppose pa is right invertible along qd with a right inverse x, then $qdpax = qd$ and $xS \subset qdS$. From $xS \subset qdS$, it follows that $x = qdt_1$ for suitable $t_1 \in S$. Hence $qdpaqdt_1 = qd$. Multiplying the previous equation by q' from the left side, we get $q'qdpaqdt_1 = q'qd$. Using the equation $q'qd = d$, we obtain $dpaqdt_1 = d$.

Similarly, since aq is left invertible along dp with a left inverse z, then $zaqdp = dp$ and $Sz \subset Sdp$. From $Sz \subset Sdp$, we get $z = t_2dp$ for some $t_2 \in S$. Therefore, $t_2dpaqdp = dp$, which implies $t_2dpaqdpp' = dpp'$. Since $dpp' = d$, then $t_2dpaqd = d$.

Let $u = zax$. We will prove u is the inverse of paq along d. Then, from above equations, we have

$$upaqd = zaxpaqd = t_2dpaqdt_1paqd = t_2dpaqd = d$$

and

$$dpaqu = dpaqzax = dpaqt_2dpaqdt_1 = dpaqdt_1 = d.$$

Also, $u = zax = t_2(dpaqdt_1) = t_2d = (t_2dpaqd)t_1 = dt_1$ implies $uS \subset dS$ and $Su \subset Sd$. Thus, u is the inverse of paq along d. □

Note that, Theorem 2.16 is in general false without the condition $q'qd = d = dpp'$:

Example 2.17. *Let S be the algebra $M_2(\mathbb{F})$ of all 2×2 matrices over a field $\mathbb{F}$. Take*

$$p = a = q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad d = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then, we can see that paq is not invertible, so paq is not invertible along d. However, $pa^{\|qd} = aq^{\|dp} = a$.

3 When a^*a (or aa^*) is invertible along a

In this section, we mainly consider the relation between the (left, right) inverse of $aa^*(a^*a)$ along a and the classical generalized inverses in a $*$-monoid. In what follows, R always denotes a $*$-ring and S denotes a $*$-monoid.

Theorem 3.1. *Let $a \in S$ and $k \geq 1$. Then the following are equivalent:*
(1) *$a \in S^\dagger$ and $aS = a^2S$;*
(2) *$(a^*a)^k$ is right invertible along a.*

Proof. (1) $\Rightarrow$ (2) From the condition $a \in S^\dagger$ and by Lemma 1.4, it follows that $a \in aa^*aS$, which imply $a = aa^*ah$ for some $h \in S$. Then, we have $a = aa^*ah = a(a^*a)^2h^2 = \cdots = a(a^*a)^kh^k$. According to the equality $aS = a^2S$, there exists $s \in S$ such that $a = a^2s$. Then, we have $a = a(a^*a)^kh^k = a(a^*a)^{k-1}a^*ah^k = a(a^*a)^{k-1}a^*a^2sh^k = a(a^*a)^kash^k \in a(a^*a)^kaS$. Applying Lemma 1.2(2), we can deduce that $(a^*a)^k$ is right invertible along a.

(2) $\Rightarrow$ (1) Suppose that $(a^*a)^k$ is right invertible along a, by Lemma 1.2(2), there exists $t \in S$ such that $a = a(a^*a)^kat$ and hence $a^* = t^*a^*(a^*a)^ka^*$. Since $(a^*a)^kat = a^*a(a^*a)^{k-1}at = t^*a^*(a^*a)^ka^*a(a^*a)^{k-1}at = t^*a^*(a^*a)^{2k}at$, then we have $((a^*a)^kat)^* = (a^*a)^kat$. Next, we will prove that $(a(a^2t)^*)^* = a(a^2t)^*$. Since

$$\begin{aligned}
& a(a^2t)^* \\
= \ & at^*(a^2)^* = at^*a^*a^* \\
= \ & at^*a^*t^*a^*(a^*a)^ka^* \\
= \ & a(tat)^*a^*(a^*a)^{k-1}a^*aa^* \\
= \ & a(tat)^*a^*(a^*a)^{k-1}a^*a(a^*a)^kata^* \\
= \ & a(tat)^*a^*(a^*a)^{2k-1}a^*aata^* \\
= \ & a(tat)^*a^*(a^*a)^{2k-1}a^*a(a^*a)^katata^* \\
= \ & a(tat)^*a^*(a^*a)^{3k}a(tat)a^*,
\end{aligned}$$

it follows that $(a(a^2t)^*)^* = a(a^2t)^*$. Therefore, we get $a = a(a^*a)^kat = a((a^*a)^{k-1}a^*a^2t)^* = a(a^2t)^*a(a^*a)^{k-1} = (a(a^2t)^*)^*a(a^*a)^{k-1} = a^2t(a^*a)^k \in a^2S$, which implies $aS = a^2S$.

Also, from the equality $a = a(a^*a)^kat = aa^*a(a^*a)^{k-1}at \in aa^*aS$, by Lemma 1.4, we deduce that $a \in S^\dagger$. $\square$

Remark 3.2. *Note that $a \in S^\dagger$ and $aS = a^2S$ can not imply $a \in S^\#$. For example, take S to be the ring of both row-finite and column-finite infinite matrices over a field $\mathbb{F}$. Let involution $*$ be the transpose. Take $a = \sum_{i=1}^{\infty} e_{i,i+1}$, where $e_{i,j}$ denotes the infinite matrix whose (i,j)-entry is 1 and others are zero. Then $aa^* = I$. Hence, we have $a^\dagger = a^*$, and $aS = a^2S$. However, $Sa \neq Sa^2$, which implies a is not group invertible.*

Applying the previous theorem in a $*$-ring R, we have the following corollary.

Corollary 3.3. *Let $a \in R$ be regular and $k \geq 1$. Then the following are equivalent:*
(1) *$a \in R^\dagger$ and $aR = a^2R$;*
(2) *$(a^*a)^k$ is right invertible along a;*
(3) *$u = a(a^*a)^k + 1 - aa^-$ is right invertible;*
(4) *$v = (a^*a)^k a + 1 - a^-a$ is right invertible.*
In this case, $a^\dagger = a^*a((a^*a)^{k-1}av_r^{-1})^2a^*$.

Proof. (1) $\Leftrightarrow$ (2) By Theorem 3.1.
(2) $\Leftrightarrow$ (3) By Lemma 1.3.
(3) $\Leftrightarrow$ (4) By Lemma 1.1(2).
Next, we give the expression for the Moore-Penrose inverse $a^\dagger$. Since v is right invertible, we have $vv_r^{-1} = 1$, which implies $a = a(a^*a)^k av_r^{-1} = aa^*a(a^*a)^{k-1}av_r^{-1}$. By Lemma 1.5(1), we obtain $a^\dagger = a^*a((a^*a)^{k-1}av_r^{-1})^2a^*$. □

Dually, we have the following results.

Theorem 3.4. *Let $a \in S$ and $k \geq 1$. Then the following are equivalent:*
(1) *$a \in S^\dagger$ and $Sa = Sa^2$;*
(2) *$(aa^*)^k$ is left invertible along a.*

Corollary 3.5. *Let $a \in R$ with a regular and $k \geq 1$. Then the following are equivalent:*
(1) *$a \in R^\dagger$ and $Ra = Ra^2$;*
(2) *$(aa^*)^k$ is left invertible along a;*
(3) *$u = a(aa^*)^k + 1 - aa^-$ is left invertible;*
(4) *$v = (aa^*)^k a + 1 - a^-a$ is left invertible.*
In this case, $a^\dagger = a^*(u_l^{-1}a(aa^*)^{k-1})^2aa^*$.

□

In the following theorem, we consider when a^*a (resp. aa^*) is left (resp. right) invertible along a under the condition $a \in S^\dagger$.

Theorem 3.6. *Let $a \in S^\dagger$ and $k \geq 1$. Then*
(1) *$Sa = Sa^2$ if and only if $(a^*a)^k$ is left invertible along a.*
(2) *$aS = a^2S$ if and only if $(aa^*)^k$ is right invertible along a.*

Proof. (1) Suppose $Sa = Sa^2$, we have $a = sa^2$ for some $s \in S$. According to the condition $a \in S^\dagger$ and Lemma 1.4, there exists $r \in S$ such that $a = raa^*a$. Hence, we deduce that $a = sa^2 = s(raa^*a)a = srraa^*aa^*aa = sr^2a(a^*a)^2a = \cdots = sr^ka(a^*a)^ka \in Sa(a^*a)^ka$. By Lemma 1.2(1), we get $(a^*a)^k$ is left invertible along a.

Conversely, suppose that $(a^*a)^k$ is left invertible along a. Using Lemma 1.2(1) again, there exists $t \in S$ such that $a = ta(a^*a)^ka = t(aa^*)^ka^2$, which implies $Sa = Sa^2$.

(2) This statement can be proved in the same manner as (1). □

Note that, in the proof of sufficiency of Theorem 3.6, we need not $a \in S^\dagger$. So, we have the following questions.

Question 3.7. *Suppose that a^*a is left invertible along a, does $a \in S^\dagger$ hold? In addition, assume that aa^* is right invertible along a, does $a \in S^\dagger$ hold?*

We now give the relations of these inverses, such as the inverse of a^*a along a, the inverse of aa^* along a, Moore-Penrose inverse and group inverse.

Theorem 3.8. *Let $a \in S$ and $k \geq 1$. Then the following are equivalent:*

(1) *$a \in S^\dagger \bigcap S^\#$;*

(2) *$(a^*a)^k$ is right invertible along a and $(aa^*)^k$ is left invertible along a;*

(3) *$(a^*a)^k$ is invertible along a;*

(4) *$(aa^*)^k$ is invertible along a.*

In this case,

$$a^\dagger = a^*a((a^*a)^{k-1}((a^*a)^k)^{\|a})^2a^* = a^*(((aa^*)^k)^{\|a}(aa^*)^{k-1})^2aa^*,$$
$$a^\# = (((a^*a)^k)^{\|a}(a^*a)^{k-1}a^*)^2a = a(a^*(aa^*)^{k-1}((aa^*)^k)^{\|a})^2,$$
$$((a^*a)^k)^{\|a} = aa^\#(a^\dagger(a^\dagger)^*)^k \text{ and } ((aa^*)^k)^{\|a} = ((a^\dagger)^*a^\dagger)^ka^\#a.$$

Proof. (1) $\Leftrightarrow$ (2) By Theorem 3.1 and 3.4.

(1) $\Rightarrow$ (3) According to the condition $a \in S^\dagger \bigcap S^\#$ and Theorem 3.6, we get $(a^*a)^k$ is left invertible along a. Applying Theorem 3.1, $(a^*a)^k$ is right invertible along a. Hence, $(a^*a)^k$ is invertible along a.

(3) $\Rightarrow$ (2) Suppose that $(a^*a)^k$ is invertible along a, by Theorem 3.1, then $a \in S^\dagger$. Note that $(a^*a)^k$ is left invertible along a, by Lemma 1.2(1), we have $a \in Sa(a^*a)^ka = Sa(a^*a)^{k-1}a^*a^2 \subset Sa^2$. By Theorem 3.4, we get $(aa^*)^k$ is left invertible along a.

(1) $\Rightarrow$ (4) $\Rightarrow$ (2) It is similar to the proof of (1) $\Rightarrow$ (3) $\Rightarrow$ (2).

Next, we give representations of $a^\dagger$, $a^\#$, $((a^*a)^k)^{\|a}$ and $((aa^*)^k)^{\|a}$. Since $(a^*a)^k$ is invertible along a, we have

$$a = a(a^*a)^k((a^*a)^k)^{\|a} = aa^*a(a^*a)^{k-1}((a^*a)^k)^{\|a}$$

and

$$a = ((a^*a)^k)^{\|a}(a^*a)^ka = ((a^*a)^k)^{\|a}(a^*a)^{k-1}a^*a^2,$$

which imply $a^\dagger = a^*a((a^*a)^{k-1}((a^*a)^k)^{\|a})^2a^*$ and $a^\# = (((a^*a)^k)^{\|a}(a^*a)^{k-1}a^*)^2a$ by Lemma 1.5 and Lemma 1.6, respectively.

Similarly, we get $a^\dagger = a^*(((aa^*)^k)^{\|a}(aa^*)^{k-1})^2aa^*$ and $a^\# = a(a^*(aa^*)^{k-1}((aa^*)^k)^{\|a})^2$.

Note that $a = a(a^*a)^k aa^\#(a^\dagger(a^\dagger)^*)^k$, by Lemma 1.2, we have $((a^*a)^k)^{\|a} = aa^\#(a^\dagger(a^\dagger)^*)^k$. Similarly, from $a = ((a^\dagger)^*a^\dagger)^k a^\# a(aa^*)^k a$, it follows that $((aa^*)^k)^{\|a} = ((a^\dagger)^*a^\dagger)^k a^\# a$. □

Letting $k = 1$ in Theorem 3.8, we get

Corollary 3.9. *Let $a \in S$. Then the following are equivalent:*

(1) $a \in S^\dagger \bigcap S^\#$;

(2) *a^*a is right invertible along a and aa^* is left invertible along a;*

(3) *a^*a is invertible along a;*

(4) *aa^* is invertible along a.*

In this case,

$$a^\dagger = a^*a((a^*a)^{\|a})^2a^* = a^*((aa^*)^{\|a})^2aa^*,$$
$$a^\# = ((a^*a)^{\|a}a^*)^2a = a(a^*(aa^*)^{\|a})^2,$$
$$(a^*a)^{\|a} = a^\#(a^\dagger)^* \quad \text{and} \quad (aa^*)^{\|a} = (a^\dagger)^*a^\#.$$

Applying Theorem 3.8, Lemma 1.3 and Lemma 1.9 in a $*$-ring R, we have the following corollary.

Corollary 3.10. *Let $a \in R$ be regular and $k \geq 1$. Then the following are equivalent:*

(1) $a \in R^\dagger \bigcap R^\#$;

(2) $a \in R^{\oplus} \bigcap R_{\oplus}$;

(3) *$u = a(a^*a)^k + 1 - aa^-$ is invertible;*

(4) *$v = (aa^*)^k a + 1 - a^-a$ is invertible;*

(5) *$s = (a^*a)^k a + 1 - a^-a$ is invertible;*

(6) *$t = a(aa^*)^k + 1 - aa^-$ is invertible.*

In this case,

$$a^{\oplus} = u^{-1}a(a^*a)^{k-1}a^*, a_{\oplus} = a^*(aa^*)^{k-1}av^{-1},$$
$$a^\dagger = (t^{-1}a(aa^*)^{k-1}a)^* = (a(a^*a)^{k-1}as^{-1})^*$$

and

$$a^\# = (u^{-1}a(a^*a)^{k-1}a^*)^2a = a(a^*(aa^*)^{k-1}av^{-1})^2.$$

Proof. We only need to prove the expressions of $a^{\oplus}$, $a_{\oplus}$, $a^\dagger$ and $a^\#$. Observe that $ua = a(a^*a)^k a = a(a^*a)^{k-1}a^*a^2$, which implies $a = u^{-1}a(a^*a)^{k-1}a^*a^2$. Since $a \in R^\#$, by Lemma 1.6, we have $a^\# = (u^{-1}a(a^*a)^{k-1}a^*)^2a$. Using Lemma 1.9, we obtain

$$\begin{aligned} a^{\oplus} &= a^\# aa^{(1,3)} = (u^{-1}a(a^*a)^{k-1}a^*)^2a^2a^{(1,3)} \\ &= u^{-1}a(a^*a)^{k-1}a^*(u^{-1}a(a^*a)^{k-1}a^*a^2)a^{(1,3)} \\ &= u^{-1}a(a^*a)^{k-1}a^*aa^{(1,3)} \\ &= u^{-1}a(a^*a)^{k-1}a^*. \end{aligned}$$

Similarly, we can get $a^\# = a(a^*(aa^*)^{k-1}av^{-1})^2$ and $a_{\oplus} = a^*(aa^*)^{k-1}av^{-1}$.

From $as = a(a^*a)^k a$ and $ta = a(aa^*)^k a$, it follows that $a = aa^*a(a^*a)^{k-1}as^{-1}$ and $a = t^{-1}a(aa^*)^{k-1}aa^*a$. Applying Lemma 1.7 and Lemma 1.8, we have

$$\begin{aligned} a^\dagger &= (a(a^*a)^{k-1}as^{-1})^* a (t^{-1}a(aa^*)^{k-1}a)^* \\ &= (s^{-1})^* a^*(a^*a)^{k-1}a^*aa^*(aa^*)^{k-1}a^*(t^{-1})^* \\ &= (s^{-1})^*(a(a^*a)^k a)^*(aa^*)^{k-1}a^*(t^{-1})^* \\ &= (s^{-1})^*(as)^*(aa^*)^{k-1}a^*(t^{-1})^* \\ &= a^*(aa^*)^{k-1}a^*(t^{-1})^* \\ &= (t^{-1}a(aa^*)^{k-1}a)^*. \end{aligned}$$

Also, we can have $a^\dagger = (a(a^*a)^{k-1}as^{-1})^*$.

4 When a is invertible along aa^* (or a^*a)

In [13], D. S. Rakić etc. showed that the inverse of a along aa^* coincides with core inverse of a, under the condition $a \in R^\dagger$. Next, we will consider these kinds of inverses under weaker condition in a $*$-monoid.

It is well known that $a \in S^{\{1,4\}}$ if and only if $a \in aa^*S$. Under the hypothesis $a \in S^{\{1,4\}}$, we discuss the relation between the one-side inverse of a along aa^* and the one-side inverse of a^*a along a.

Theorem 4.1. *Let $a \in S^{\{1,4\}}$. Then the following are equivalent:*
(1) *a is left invertible along aa^*;*
(2) *a^*a is left invertible along a.*

Proof. (1) $\Rightarrow$ (2) Suppose that a is left invertible along aa^*, by Lemma 1.2(1), we have $aa^* \in Saa^*a^2a^*$, which implies $aa^* = t_1aa^*a^2a^*$ for some $t_1 \in S$. From the condition $a \in S^{\{1,4\}}$, there exists $t_2 \in S$ such that $a = aa^*t_2$. Hence, we deduce that $a = aa^*t_2 = t_1aa^*a^2a^*t_2 = t_1aa^*a^2$. According to Lemma 1.2(1) again, we get a^*a is left invertible along a.

(2) $\Rightarrow$ (1) Since a^*a is left invertible along a, by Lemma 1.2(1), we have $a = t_3aa^*a^2$ for some $t_3 \in S$. Multiplying the previous equation by a^* from the right side yields $aa^* = t_3aa^*a^2a^*$. Hence, a is left invertible along aa^*. □

Corollary 4.2. *Let $a \in R^{\{1,4\}}$. Then the following are equivalent:*
(1) *a is left invertible along aa^*;*
(2) *$u = aa^*a + 1 - aa^{(1,4)}$ is left invertible;*
(3) *$v = a^*a^2 + 1 - a^{(1,4)}a$ is left invertible;*
(4) *$f = (a^*)^2a + 1 - a^{(1,4)}a$ is right invertible;*
(5) *$g = a(a^*)^2 + 1 - aa^{(1,4)}$ is right invertible.*
In this case, $u_l^{-1}aa^*$ is a left inverse of a along aa^*.

Proof. (1) $\Leftrightarrow$ (2) Since $a \in R^{\{1,4\}}$ and by Lemma 1.3(1), we have a^*a is left invertible along a if and only if $aa^*a + 1 - aa^{(1,4)}$ is left invertible. By Theorem 4.1, it follows that (1) $\Leftrightarrow$ (2).

(3) $\Leftrightarrow$ (4) Note that $v = f^*$, then we get (3) $\Leftrightarrow$ (4).

(2) $\Leftrightarrow$ (3) and (4) $\Leftrightarrow$ (5) By Lemma 1.1(1)(2).

Suppose that u is left invertible, then $u_l^{-1}u = 1$, which implies $u_l^{-1}uaa^* = aa^*$. Note that $aa^* = u_l^{-1}uaa^* = u_l^{-1}(aa^*a + 1 - aa^{(1,4)})aa^* = u_l^{-1}aa^*aaa^*$. Hence, $u_l^{-1}aa^*$ is a left inverse of a along aa^* by Lemma 1.2(1). □

Similarly, we have the following results.

Theorem 4.3. *Let $a \in S^{\{1,4\}}$. Then the following are equivalent:*

(1) *a is right invertible along aa^*;*

(2) *a^*a is right invertible along a.*

Proof. Since $a \in S^{\{1,4\}}$, there exists $t_2 \in S$ such that $a = aa^*t_2$.

(1) $\Rightarrow$ (2) Note that $aa^* = aa^*a^2a^*t_1$ for some $t_1 \in S$ by Lemma 1.2(2). Thus, $a = aa^*t_2 = aa^*a^2a^*t_1t_2 \in a(a^*a)aS$, which implies a^*a is right invertible along a.

(2) $\Rightarrow$ (1) Suppose that a^*a is right invertible along a, there exists $t_3 \in S$ such that $a = a(a^*a)at_3 = aa^*a(aa^*t_2)t_3$. Then $aa^* = (aa^*)a(aa^*)t_2t_3a^* \in (aa^*)a(aa^*)S$, which gives a is right invertible along aa^*.

Corollary 4.4. *Let $a \in R^{\{1,4\}}$. Then the following are equivalent:*

(1) *a is right invertible along aa^*;*

(2) *$u = aa^*a + 1 - aa^{(1,4)}$ is right invertible;*

(3) *$v = a^*a^2 + 1 - a^{(1,4)}a$ is right invertible;*

(4) *$f = (a^*)^2a + 1 - a^{(1,4)}a$ is left invertible;*

(5) *$g = a(a^*)^2 + 1 - aa^{(1,4)}$ is left invertible.*

In this case, $aa^*(g_l^{-1})^*$ is a right inverse of a along aa^*.

Theorem 4.5. *Let $a \in S^{\{1,4\}}$. Then the following are equivalent:*

(1) *a is invertible along aa^*;*

(2) *a^*a is invertible along a;*

(3) *$a \in S^{\dagger} \cap S^{\sharp}$;*

(4) *$a \in S^{\oplus}$.*

In this case, $a^{\oplus} = a^{\|aa^*}$.

Proof. (1) $\Leftrightarrow$ (2) According to Theorem 4.1 and Theorem 4.3, we have (1) $\Leftrightarrow$ (2).

(2) $\Leftrightarrow$ (3) The equivalence of (2) and (3) can be obtained by Corollary 3.9.

(3) $\Leftrightarrow$ (4) Using Lemma 1.9 and $a \in S^{\{1,4\}}$, we have (3) $\Leftrightarrow$ (4).

Next, we will prove the inverse of a along aa^* coincides with core inverse of a under the condition $a \in S^{\{1,4\}}$. Since $a^{\oplus} = a^{\sharp}aa^{(1,3)}$, we have $a^{\oplus}a(aa^*) = a^{\sharp}aa^{(1,3)}a(aa^*) = aa^*$ and $a^{\oplus} = a^{\sharp}aa^{(1,3)} = a^{\#}(a^{(1,3)})^*a^{(1,4)}aa^* \in Saa^*$, which imply $a^{\oplus}$ is a left inverse of a along aa^*. According to Lemma 1.2 (4), we have $a^{\oplus} = a^{\|aa^*}$.

Remark 4.6. *Note that a is invertible along aa^* can not imply $a \in S^{\#}$ or $a \in S^{\{1,3\}}$ or $a \in S^{\{1,4\}}$. For example, let $S = Z_4$ and $x^* = x$ for any $x \in S$. Take $a = 2$, then $aa^* = 0$ and $a^{\|aa^*} = 0$. But a is not regular, so $a \notin S^{\#}$, $a \notin S^{\{1,3\}}$ and $a \notin S^{\{1,4\}}$.*

Remark 4.7. *Under the condition $a \in S^{\dagger}$, we can not have the conclusion a is left(right) invertible along aa^*. For example, let $S = M_2(\mathbb{H})$ and the involution be the conjugate transpose, where $\mathbb{H}$ denotes the division ring of quaternions. We know that any element in S is Moore-Penrose invertible. Take $a = \begin{bmatrix} i-j & 1-k \\ 1+k & -i-j \end{bmatrix}$. Then $d =: aa^* = 4\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$, $aa^*a = 8a$ and $dad = aa^*aaa^* = 8aaa^* = 0$. Hence, $d \notin Sdad$ ($d \notin dadS$), which imply a is not left(right) invertible along aa^*.*

Remark 4.8. *We have seen that a is left invertible along a^* if and only if a is right invertible along a^*. However, the following example shows that a is left invertible along aa^* is not equivalent to a is right invertible along aa^* in general.*

Example 4.9. *Let S be the ring which is the same as the infinite matrix ring in Remark 3.2 and let $a = \sum_{i=1}^{\infty} e_{i+1,i}$. Then, $d =: aa^* = \sum_{i=2}^{\infty} e_{i,i}$ and $dad = \sum_{i=2}^{\infty} e_{i+1,i}$. We can easily see that $d \notin dadS$, which implies a is not right invertible along d. While, $d = (\sum_{i=2}^{\infty} e_{i,i+1})dad \in Sdad$, we deduce that a is left invertible along d.*

Remark 4.10. *In Theorem 4.5, we can not replace $a \in S^{\{1,4\}}$ with $a \in S^{\{1,3\}}$. For example, let $S = M_2(\mathbb{C})$ and the involution is the transpose. Take $a = \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$. Then $a \in Sa^*a$, which implies $a \in S^{\{1,3\}}$. Note that $aa^* = 0$, a is invertible along aa^*. But, $a \notin aa^*S$, which yields $a \notin S^{\{1,4\}}$ and $a \notin S^{\dagger}$.*

Similar to Theorem 4.5, we have the following result.

Theorem 4.11. *Let $a \in S^{\{1,3\}}$. Then the following are equivalent:*
(1) *a is invertible along a^*a;*
(2) *aa^* is invertible along a;*
(3) *$a \in S^{\dagger} \cap S^{\#}$;*
(4) *$a \in S_{\textcircled{\#}}$.*
In this case, $a_{\textcircled{\#}} = a^{\parallel a^*a}$.

According to Theorem 4.5 and Theorem 4.11, we get

Corollary 4.12. [13, Theorem 4.3] *Let $a \in R^{\dagger}$. Then*

(1) *a is core invertible if and only if a is invertible along aa^*. In this case, the inverse of a along aa^* coincides with core inverse of a.*

(2) *a is dual core invertible if and only if a is invertible along a^*a. In this case, the inverse of a along a^*a coincides with dual core inverse of a.*

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