

# Reverse order law for the inverse along an element

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## Abstract

In this paper, we introduce a new concept called left (right) g-MP inverse in a  $*$ -semigroup. The relations of this type of generalized inverse with left (right) inverse along an element are investigated. Also, the reverse order law for the inverse along an element is studied. Then, the existence criteria and formulae of the inverse along an element of triple elements are investigated in a semigroup. Finally, we further study left and right g-MP inverses, the inverse along an element, core and dual core inverses in the context of rings.

*Keywords:*

(von Neumann) regularity, Moore-Penrose inverse, (Left, Right) Inverse along an element, Green's preorders, Semigroups, Reverse order law

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## 1. Introduction

There are many types of generalized inverses in mathematical literature, such as, Drazin inverses, group inverses [2], Moore-Penrose inverses [7], (left, right) inverse along an element [8, 11, 12], core and dual core inverses [1, 10] and so on. Many properties of these generalized inverses were considered in different settings. In particular, a large amount of work has been devoted to the study of the reverse order law for Moore-Penrose inverses, group inverses, core and dual core inverses. However, few results have been presented

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concerning the reverse order law for the inverse along an element since it was introduced.

In this article, we introduce a new concept called left (right) g-MP inverse in a  $*$ -semigroup  $S$ . An element  $a \in S$  is called left (resp., right) g-MP invertible if  $Sa = Sa^2 = Saa^*a$  (resp.,  $aS = a^2S = aa^*aS$ ). The relations of this type of generalized inverse with left (right) inverses along an element will be considered in semigroups. Also, the reverse order law for the inverse along an element is studied. Then, the existence criteria and formulae of the inverse along an element of triple elements are investigated in a semigroup. Finally, we further study left and right g-MP inverses, the inverse along an element, core and dual core inverses in rings.

An element  $a$  in a semigroup  $S$  is called (von Neumann) regular if there exists  $x \in S$  such that  $a = axa$ . Such  $x$  is called an inner inverse (von Neumann inverse) of  $a$ , and is denoted by  $a^{(1)}$ . The symbol  $a\{1\} := \{x \in S : axa = a\}$  means the set of all inner inverses of  $a \in S$ .

Let  $*$  be an involution on  $S$ , that is the involution  $*$  satisfies  $(x^*)^* = x$  and  $(xy)^* = y^*x^*$  for all  $x, y \in S$ . We call  $S$  a  $*$ -semigroup if there exists an involution on  $S$ . Recall that an element  $a \in S$  (with involution) is Moore-Penrose invertible (see [7]) if there exists  $x \in S$  satisfying the following equations

$$(i) \ axa = a \quad (ii) \ xax = x \quad (iii) \ (ax)^* = ax \quad (iv) \ (xa)^* = xa.$$

Any element  $x$  satisfying the equations above is called a Moore-Penrose inverse of  $a$ . If such  $x$  exists, then it is unique and is denoted by  $a^\dagger$ . If  $x$  satisfies the conditions (i) and (iii), then  $x$  is called a  $\{1, 3\}$ -inverse of  $a$ , and is denoted by  $a^{(1,3)}$ . If  $x$  satisfies the conditions (i) and (iv), then  $x$  is called a  $\{1, 4\}$ -inverse of  $a$ , and is denoted by  $a^{(1,4)}$ . Recall that  $a^\dagger$  exists if and only if both  $a^{(1,3)}$  and  $a^{(1,4)}$  exist. In this case,  $a^\dagger = a^{(1,4)}aa^{(1,3)}$ . If  $x$  in conditions (i) and (ii) satisfies  $ax = xa$ , then  $a$  is group invertible. Moreover, the group inverse of  $a$  is unique if it exists, and is denoted by  $a^\#$ . By  $S^\dagger$  and  $S^\#$  we denote the sets of all Moore-Penrose invertible and group invertible elements in  $S$ , respectively.

Green's preorders (see [3]) in a monoid semigroup  $S$  are defined by: (i)  $a \leq_{\mathcal{L}} b \Leftrightarrow Sa \subseteq Sb \Leftrightarrow a = xb$  for some  $x \in S$ ; (ii)  $a \leq_{\mathcal{R}} b \Leftrightarrow aS \subseteq bS \Leftrightarrow a = bx$  for some  $x \in S$ ; (iii)  $a \leq_{\mathcal{H}} b \Leftrightarrow a \leq_{\mathcal{L}} b$  and  $a \leq_{\mathcal{R}} b$ .

Let  $a, d \in S$ . An element  $a$  is called left (resp., right) invertible along  $d$  [11] if there exists  $b \in S$  such that  $bad = d$  (resp.,  $dab = b$ ) and  $b \leq_{\mathcal{L}} d$  (resp.,  $b \leq_{\mathcal{R}} d$ ). It is known [11] that  $a$  is both left and right invertible along  $d$  if

and only if it is invertible along  $d$  if and only if  $d \leq_{\mathcal{H}} dad$ . In this case, the inverse of  $a$  along  $d$  is unique, and is denoted by  $a^{\parallel d}$ .

## 2. Left (right) g-MP inverse and reverse order law in semigroups

**Definition 2.1.** *Let  $S$  be a  $*$ -semigroup and let  $a \in S$ . We call a left g-MP invertible if  $Sa = Sa^2 = Saa^*a$ .*

We next give several examples of left g-MP invertible elements.

**Example 2.2.** (i) The unity 1 in  $S$  is left g-MP invertible.

(ii) An EP element  $a$  (i.e.,  $a \in S^{\#} \cap S^{\dagger}$  and  $a^{\#} = a^{\dagger}$ ) is left g-MP invertible. Indeed,  $a = a^{\#}a^2 \in Sa^2$ , i.e.,  $Sa = Sa^2$ . Also,  $a = aa^{\dagger}a = (a^{\dagger})^*a^*a = (a^{\dagger})^*a^{\dagger}aa^*a \in Saa^*a$ . So,  $Sa = Sa^2 = Saa^*a$ .

(iii) Let  $S = M_2(\mathbb{C})$  be the semigroup of  $2 \times 2$  complex matrices and  $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \in S$ . Then  $A$  is left g-MP invertible since  $A = I \cdot A^2 = \frac{1}{2}AA^*A$ .

Next, we give the definition of right g-MP inverse in a  $*$ -semigroup.

**Definition 2.3.** *Let  $S$  be a  $*$ -semigroup and let  $a \in S$ . We call a right g-MP invertible if  $aS = a^2S = aa^*aS$ .*

**Lemma 2.4.** [11, Theorem 2.3] *Let  $a, d \in S$ . Then*

- (i)  *$a$  is left invertible along  $d$  if and only if  $d \leq_{\mathcal{L}} dad$ .*
- (i)  *$a$  is right invertible along  $d$  if and only if  $d \leq_{\mathcal{R}} dad$ .*

The following theorem characterizes the relations between  $Sa = Saa^*a$  and left inverse along an element in a  $*$ -semigroup.

**Theorem 2.5.** *Let  $S$  be a  $*$ -semigroup and let  $a \in S$ . Then following conditions are equivalent:*

- (i)  $Sa = Saa^*a$ .
- (ii)  $a^*$  is left invertible along  $a$ .

PROOF. (i)  $\Rightarrow$  (ii) Suppose  $Sa = Saa^*a$ . Then  $a \leq_{\mathcal{L}} aa^*a$ . Hence,  $a^*$  is left invertible along  $a$  by Lemma 2.4.

(ii)  $\Rightarrow$  (i) If  $a^*$  is left invertible along  $a$ , it follows from Lemma 2.4 that  $a \leq_{\mathcal{L}} aa^*a$ , i.e.,  $Sa \subseteq Saa^*a$ . So, it follows  $Sa = Saa^*a$ .  $\square$

Dually, we have the following result:

**Theorem 2.6.** *Let  $S$  be a  $*$ -semigroup and let  $a \in S$ . Then following conditions are equivalent:*

- (i)  $aS = aa^*aS$ .
- (ii)  $a^*$  is right invertible along  $a$ .

It follows from [11, Theorem 2.16] that  $aS = aa^*aS$  if and only if  $Sa = Saa^*a$  if and only if  $a$  is Moore-Penrose invertible. Hence, we get

**Corollary 2.7.** *Let  $S$  be a  $*$ -semigroup and let  $a \in S$ . Then following conditions are equivalent:*

- (i)  $a$  is Moore-Penrose invertible.
- (ii)  $a^*$  is left invertible along  $a$ .
- (iii)  $a^*$  is right invertible along  $a$ .

*In this case,  $(a^\dagger)^*$  is a left (right) inverse of  $a^*$  along  $a$ .*

From [8, Theorem 11] and Corollary 2.7,  $a \in S^\dagger$  if and only if  $a$  is invertible along  $a^*$  if and only if  $a^*$  is invertible along  $a$ . Moreover,  $a^{\parallel a^*} = a^\dagger$  and  $(a^*)^{\parallel a} = (a^\dagger)^*$ . Moreover, left (right) g-MP invertibility can be presented as between Moore-Penrose invertibility and left (right) regularity. If  $a \in S$  is both left and right g-MP invertible, then  $a \in S^\# \cap S^\dagger$ .

The relations between left g-MP inverse and the recently introduced notion called left inverse along an element will be given in Theorem 2.10 below. Herein, we first give a relation between  $\{1,3\}$ -inverse and  $\{1,4\}$ -inverse in a  $*$ -semigroup.

**Theorem 2.8.** *Let  $S$  be a  $*$ -semigroup and let  $a \in S$ . If  $a = xaa^*a$  for some  $x \in S$ , then  $(xa)^*$  is both a  $\{1,3\}$ -inverse and a  $\{1,4\}$ -inverse of  $a$  and  $a^\dagger = (xa)^*a(xa)^*$ .*

PROOF. From [13, Lemma 2.2], we know that  $(xa)^*$  is a  $\{1,3\}$ -inverse of  $a$ .

Note that

$$\begin{aligned} (xa)^*a &= a^*x^*a = (xaa^*a)^*x^*a = a^*aa^*(x^*)^2a \\ &= a^*(xaa^*a)a^*(x^*)^2a = a^*x(xaa^*a)a^*aa^*(x^*)^2a \\ &= a^*x^2(aa^*)^3(x^2)^*a. \end{aligned}$$

Hence,  $[(xa)^*a]^* = (xa)^*a$ , that is,  $(xa)^*$  is a  $\{1,4\}$ -inverse of  $a$ . Hence,  $a^\dagger = a^{(1,4)}aa^{(1,3)} = (xa)^*a(xa)^*$ .  $\square$

**Theorem 2.9.** *Let  $S$  be a  $*$ -semigroup and let  $a \in S$ . If  $a = aa^*ay$  for some  $y \in S$ , then  $(ay)^*$  is both a  $\{1, 3\}$ -inverse and a  $\{1, 4\}$ -inverse of  $a$  and  $a^\dagger = (ay)^*a(ay)^*$ .*

It follows from Lemma 2.4 that  $Sa = Sa^2$  if and only if  $a$  is left invertible along  $a$ . Also, Theorem 2.5 ensures that  $Sa = Saa^*a$  if and only if  $a^*$  is left invertible along  $a$ . Suppose  $Sa = Sa^2 = Saa^*a$ . Then there exist  $s, t \in S$  such that  $a = sa^2 = taa^*a$ . So,  $a = t(sa^2)a^*a$ , which means  $a \leq_{\mathcal{L}} a^2a^*a$  and hence  $aa^*$  is left invertible along  $a$  from Lemma 2.4. One may guess whether the converse holds? that is, if  $aa^*$  is left invertible along  $a$  implies  $Sa = Sa^2 = Saa^*a$ ? Theorem 2.10 below illustrates this fact.

**Theorem 2.10.** *Let  $S$  be a  $*$ -semigroup and let  $a \in S$ . Then  $a$  is left  $g$ -MP invertible if and only if  $aa^*$  is left invertible along  $a$ .*

PROOF. We need only to prove “if” part.

Suppose that  $aa^*$  is left invertible along  $a$ . It follows from Lemma 2.4 that  $a \leq_{\mathcal{L}} a^2a^*a$ , which leads to  $Sa = Saa^*a$ .

Also,  $a \leq_{\mathcal{L}} a^2a^*a$  means  $a = ba^2a^*a$  for some  $b \in S$ . Moreover,  $(ba^2)^*$  is a  $\{1, 4\}$ -inverse of  $a$  from Theorem 2.8. Therefore,  $a = a(ba^2)^*a = a[(ba^2)^*a]^* = aa^*ba^2$ , which yields  $Sa = Sa^2$ .

So,  $a$  is left  $g$ -MP invertible.  $\square$

The following theorem can be proved similarly.

**Theorem 2.11.** *Let  $S$  be a  $*$ -semigroup and let  $a \in S$ . Then  $a$  is right  $g$ -MP invertible if and only if  $a^*a$  is right invertible along  $a$ .*

Further, we have

**Theorem 2.12.** *Let  $S$  be a  $*$ -semigroup and let  $a \in S$ . Then the following conditions are equivalent:*

- (i)  $a$  is both left and right  $g$ -MP invertible.
- (ii)  $aa^*$  is left invertible along  $a$  and  $a^*a$  is right invertible along  $a$ .
- (iii)  $a^*a$  is invertible along  $a$ .
- (iv)  $aa^*$  is invertible along  $a$ .

*In this case,  $(a^*a)^{\parallel a} = a^\#(a^\dagger)^*$  and  $(aa^*)^{\parallel a} = (a^\dagger)^*a^\#$ .*

PROOF. (i)  $\Leftrightarrow$  (ii) follows from Theorems 2.10 and 2.11.

(ii)  $\Rightarrow$  (iii) We need only to prove that  $a^*a$  is left invertible along  $a$ . The condition (ii) implies  $a \in S^\# \cap S^\dagger$ . Then  $a = a^\#(a^\dagger)^*a^\dagger aa^*a^2$ , i.e.,  $a \leq_{\mathcal{L}} aa^*a^2$ . Hence,  $a^*a$  is invertible along  $a$ .

(iii)  $\Rightarrow$  (ii) If  $a^*a$  is invertible along  $a$ , then we have  $a \leq_{\mathcal{L}} aa^*a^2$ , which implies  $a = ya^2$  for some  $y \in S$ . Furthermore,  $a \leq_{\mathcal{R}} aa^*a^2$  means  $a = aa^*a^2x \in aa^*aS$  for some  $x \in S$ . So, it follows from [11, Theorem 2.19] that  $a \in S^\dagger$  and hence  $a = (a^\dagger)^*a^\dagger aa^*a = (a^\dagger)^*a^\dagger(ya^2)a^*a \leq_{\mathcal{L}} a^2a^*a$ . Therefore,  $aa^*$  is left invertible along  $a$  by Lemma 2.4.

(ii)  $\Leftrightarrow$  (iv) can be proved similarly.

Once given the formula, it is easy to check  $(a^*a)^{\parallel a} = a^\#(a^\dagger)^*$  and  $(aa^*)^{\parallel a} = (a^\dagger)^*a^\#$ .  $\square$

We next consider some relations between left and right g-MP inverses, under certain conditions.

**Theorem 2.13.** *Let  $S$  be a  $*$ -semigroup and  $aS = a^*S$ . Then the following conditions are equivalent:*

- (i)  $a$  is left g-MP invertible.
  - (ii)  $a$  is right g-MP invertible.
- In this case,  $a$  is EP.*

PROOF. (i)  $\Rightarrow$  (ii) Suppose that  $a$  is left g-MP invertible. Then  $a$  is Moore-Penrose invertible and hence  $aS = aa^*aS$ . On the other hand,  $aS = aa^*(a^\dagger)^*S \subseteq aa^*S = a^2S$  since  $aS = a^*S$ . Therefore,  $a$  is right g-MP invertible.

(ii)  $\Rightarrow$  (i) It is similar to the proof of (i)  $\Rightarrow$  (ii).

The EP-ness follows from [6, Proposition 2(1)].  $\square$

It is well known that the reverse order law holds for the classical inverse in any monoid semigroup  $S$ . More precisely,  $(ab)^{-1} = b^{-1}a^{-1}$  for any invertible elements  $a$  and  $b$  in  $S$ . However,  $(ab)^{\parallel d} = b^{\parallel d}a^{\parallel d}$  does not hold in general in  $S$ . For instance, in the semigroup of 2 by 2 complex matrices, take  $a = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$  and  $d = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ , then  $a^{\parallel d} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{pmatrix}$ . However,  $(a^2)^{\parallel d} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{pmatrix} \neq a^{\parallel d}a^{\parallel d} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ 0 & 0 \end{pmatrix}$ .

Next, we consider the reverse order law for the product of the inverse along an element under certain conditions.

**Lemma 2.14.** [8, Theorem 10] *Let  $a, d \in S$  with  $ad = da$ . If  $a^{\parallel d}$  exists, then  $a^{\parallel d}$  commutes with  $a$  and  $d$ .*

**Theorem 2.15.** *Let  $a, b, d \in S$  with  $ad = da$ . If  $a^{\parallel d}$  and  $b^{\parallel d}$  exist, then  $(ab)^{\parallel d}$  exists and  $(ab)^{\parallel d} = b^{\parallel d}a^{\parallel d}$ .*

PROOF. Since  $b^{\parallel d}$  can be written as  $xd$  for some  $x \in S$ , we have

$$b^{\parallel d}a^{\parallel d}abd = b^{\parallel d}aa^{\parallel d}bd = x(daa^{\parallel d})bd = xdbd = b^{\parallel d}bd = d \text{ and } dabb^{\parallel d}a^{\parallel d} = adbb^{\parallel d}a^{\parallel d} = ada^{\parallel d} = daa^{\parallel d} = d.$$

As  $a^{\parallel d} \leq_{\mathcal{H}} d$  and  $b^{\parallel d} \leq_{\mathcal{H}} d$ , then  $b^{\parallel d}a^{\parallel d} \leq_{\mathcal{H}} d$ .

Therefore,  $ab$  is invertible along  $d$  and  $(ab)^{\parallel d} = b^{\parallel d}a^{\parallel d}$ .  $\square$

Similarly, we have

**Theorem 2.16.** *Let  $a, b, d \in S$  with  $bd = db$ . If  $a^{\parallel d}$  and  $b^{\parallel d}$  exist, then  $(ab)^{\parallel d}$  exists and  $(ab)^{\parallel d} = b^{\parallel d}a^{\parallel d}$ .*

We next consider the existence criteria and formulae of the inverse along an element of triple elements.

**Theorem 2.17.** *Let  $S$  be a semigroup and let  $a, b, d \in S$ . If  $a$  is invertible along  $d$ , then the following conditions are equivalent:*

- (i)  $b$  is invertible along  $d$ .
- (ii)  $adb$  and  $bda$  are both invertible along  $d$ .

*In this case,  $(adb)^{\parallel d} = b^{\parallel d}d^{(1)}a^{\parallel d}$  and  $(bda)^{\parallel d} = a^{\parallel d}d^{(1)}b^{\parallel d}$  for all choices  $d^{(1)} \in \{1\}$ .*

PROOF. (i)  $\Rightarrow$  (ii) Suppose that  $b$  is invertible along  $d$ . Then  $d \leq_{\mathcal{H}} dbd$  by Lemma 2.4, i.e,  $d = sdbd = dbdt$  for some  $s, t \in S$ . Also, as  $a$  is invertible along  $d$ , then  $d = xdad = dady$  for some  $x, y \in S$ . Hence,  $d = x(sdbd)ad = db(dady)t$ , namely,  $d \leq_{\mathcal{H}} dbdad$ . So,  $bda$  invertible along  $d$  from Lemma 2.4.

Similarly,  $d = s(xdad)bd = da(dbdt)y$  and hence  $d \leq_{\mathcal{H}} dadbd$ . Thus,  $adb$  invertible along  $d$ .

(ii)  $\Rightarrow$  (i) As  $adb$  is invertible along  $d$ , then  $d \leq_{\mathcal{H}} d(adb)d$ , which implies  $d \leq_{\mathcal{L}} dbd$ . On the other hand, since  $bda$  is both invertible along  $d$ , it follows that  $d \leq_{\mathcal{R}} dbd$ . Hence,  $d \leq_{\mathcal{H}} dbd$ . So,  $d$  is invertible along  $d$ .

It is well known that  $a$  is invertible along  $d$  implies that  $d$  is regular (see [9]). We next show that  $m = b^{\parallel d}d^{(1)}a^{\parallel d}$  is the inverse of  $adb$  along  $d$ .

We have  $a^{\parallel d}ad = d = daa^{\parallel d}$  and  $a^{\parallel d} = x_1d = dx_2$  for some  $x_1, x_2 \in S$ . Also,  $b^{\parallel d}bd = d = dbb^{\parallel d}$  and  $b^{\parallel d} = y_1d = dy_2$  for some  $y_1, y_2 \in S$ .

It follows that

$$\begin{aligned}
maddb &= b^{\parallel d} d^{(1)} (a^{\parallel d} ad) bd = b^{\parallel d} d^{(1)} dbd \\
&= y_1 d d^{(1)} dbd = y_1 dbd = b^{\parallel d} bd \\
&= d.
\end{aligned}$$

and

$$\begin{aligned}
dadbm &= da(dbb^{\parallel d})d^{(1)}a^{\parallel d} = dadd^{(1)}a^{\parallel d} \\
&= dadd^{(1)}dx_2 = dadx_2 = daa^{\parallel d} \\
&= d.
\end{aligned}$$

As  $m = b^{\parallel d} d^{(1)} a^{\parallel d} = dy_2 d^{(1)} a^{\parallel d} = b^{\parallel d} d^{(1)} x_1 d$ , then  $m \leq_{\mathcal{H}} d$ .

Hence,  $(adb)^{\parallel d} = b^{\parallel d} d^{(1)} a^{\parallel d}$ .

Similarly, we can check  $(bda)^{\parallel d} = a^{\parallel d} d^{(1)} b^{\parallel d}$ .  $\square$

**Remark 2.18.** If  $S$  is a ring, then the condition (ii) in Theorem 2.17 can be weakened to: (ii)'  $adb$  is invertible along  $d$  or (ii)''  $bda$  is invertible along  $d$  (see Theorem 3.7 below).

The following result can be seen as a generalization of the reverse order law for the inverse along an element of triple elements.

**Corollary 2.19.** *Let  $S$  be a semigroup and let  $a, b, d \in S$ . If both  $a$  and  $d$  are invertible along  $d$ , then the following conditions are equivalent:*

(i)  $b$  is invertible along  $d$ .

(ii)  $adb$  and  $bda$  are both invertible along  $d$ .

In this case,  $(adb)^{\parallel d} = b^{\parallel d} d^{\parallel d} a^{\parallel d}$  and  $(bda)^{\parallel d} = a^{\parallel d} d^{\parallel d} b^{\parallel d}$ .

As special results of Theorem 2.17, it follows that

**Corollary 2.20.** *Let  $S$  be a semigroup and let  $b, d \in S$ . If  $1$  is invertible along  $d$ , then the following conditions are equivalent:*

(i)  $b$  is invertible along  $d$ .

(ii)  $bd$  and  $db$  are both invertible along  $d$ .

In this case,  $(bd)^{\parallel d} = d^{\parallel d} b^{\parallel d}$  and  $(db)^{\parallel d} = b^{\parallel d} d^{\parallel d}$ .

**PROOF.** Since  $1^{\parallel d} = dd^{\#}$ ,  $d^{\parallel d} = d^{\#}$  and  $b^{\parallel d} = dx$  for some  $x$ , we have  $(bd)^{\parallel d} = 1^{\parallel d} d^{(1)} b^{\parallel d} = d^{\#} dd^{(1)} dx = d^{\#} dx = d^{\#} b^{\parallel d} = d^{\parallel d} b^{\parallel d}$ .

We may use the same reasoning to obtain  $(db)^{\parallel d} = b^{\parallel d} d^{\parallel d}$ .  $\square$



Suppose  $a = d^*$  in Theorem 2.17. Then it follows that

**Corollary 2.21.** *Let  $S$  be a  $*$ -semigroup and let  $b, d \in S$ . If  $d^*$  is invertible along  $d$ , then the following conditions are equivalent:*

- (i)  $b$  is invertible along  $d$ .
- (ii)  $d^*db$  and  $bdd^*$  are both invertible along  $d$ .

*In this case,  $(d^*db)^{\parallel d} = b^{\parallel d}d^{\parallel d^*}(d^*)^{\parallel d}$  and  $(bdd^*)^{\parallel d} = (d^*)^{\parallel d}d^{\parallel d^*}b^{\parallel d}$ .*

PROOF. By Theorem 2.17, we have  $(d^*db)^{\parallel d} = b^{\parallel d}d^{(1)}(d^*)^{\parallel d}$ . Note that  $(d^*)^{\parallel d} = (d^\dagger)^*$  and  $b^{\parallel d}$  can be written as  $xd$  for an appropriate  $x$ . Hence,  $(d^*db)^{\parallel d} = xdd^{(1)}dd^\dagger(d^\dagger)^* = xdd^\dagger(d^\dagger)^* = b^{\parallel d}d^{\parallel d^*}(d^*)^{\parallel d}$ .

Similarly,  $(bdd^*)^{\parallel d} = (d^*)^{\parallel d}d^{\parallel d^*}b^{\parallel d}$ . □

### 3. Further results in rings

Let  $R$  be an associative unital ring. An involution  $*$  :  $R \rightarrow R$ ;  $a \mapsto a^*$  is an anti-isomorphism in  $R$  satisfying  $(a^*)^* = a$ ,  $(a + b)^* = a^* + b^*$  and  $(ab)^* = b^*a^*$  for all  $a, b \in R$ .

An element  $a \in R$  with involution is called core invertible [10] if there exists  $x \in R$  such that  $axa = a$ ,  $xR = aR$  and  $Rx = Ra^*$ . The core inverse of  $a$  is unique if it exists, and is denoted by  $a^\oplus$ . The dual core inverse of  $a$  when exists is defined as the unique  $a_\oplus$  such that  $aa_\oplus a = a$ ,  $a_\oplus R = a^*R$  and  $Ra_\oplus = Ra$ . By  $R^{-1}$ ,  $R^\oplus$  and  $R_\oplus$  we denote the sets of all invertible, core invertible and dual core invertible elements in  $R$ , respectively. It is known [10] that  $a \in R^\oplus \cap R_\oplus$  if and only if  $a \in R^\# \cap R^\dagger$ .

We next begin with two lemmas, which play an important role in the sequel.

**Lemma 3.1.** *Let  $a, b \in R$ . Then we have*

- (i) *If  $(1 + ab)x = 1$ , then  $(1 + ba)(1 - bxa) = 1$ .*
- (ii) *If  $y(1 + ab) = 1$ , then  $(1 - bya)(1 + ba) = 1$ .*

It follows from Lemma 3.1 that  $1 + ab \in R^{-1}$  if and only if  $1 + ba \in R^{-1}$ . In this case,  $(1 + ba)^{-1} = 1 - b(1 + ab)^{-1}a$ , which is known as Jacobson's Lemma.

**Lemma 3.2.** [11, Corollaries 3.3 and 3.5] *Let  $a, m \in R$  with  $m$  regular. Then the following conditions are equivalent:*

- (i)  *$a$  is (left, right) invertible along  $m$ .*
- (ii)  *$u = ma + 1 - mm^{(1)}$  is (left, right) invertible.*
- (iii)  *$v = am + 1 - m^{(1)}m$  is (left, right) invertible.*

For an element  $a \in R$ , the symbols  $a_l^{-1}$  and  $a_r^{-1}$  denote a left and a right inverse of  $a$ , respectively.

Applying Theorem 2.7 and Lemma 3.2, we derive the following results, which recover the classical existence criterion of Moore-Penrose inverse (see, e.g. [5, Theorem 1.2]) in rings.

**Theorem 3.3.** *Let  $R$  be a ring with involution and let  $a \in R$  be regular. Then the following conditions are equivalent:*

- (i)  $a \in R^\dagger$ .
- (ii)  $u = aa^* + 1 - aa^{(1)}$  is left invertible.
- (iii)  $v = a^*a + 1 - a^{(1)}a$  is left invertible.
- (iv)  $u' = aa^* + 1 - aa^{(1)}$  is right invertible.
- (v)  $v' = a^*a + 1 - a^{(1)}a$  is right invertible.

In this case, we have  $a^\dagger = (u_l^{-1}a)^*a(u_l^{-1}a)^* = [(v')_r^{-1}a]^*a[(v')_r^{-1}a]^*$ .

Also, we get

**Corollary 3.4.** *Let  $R$  be a ring with involution and let  $a \in R$  be regular. Then the following conditions are equivalent:*

- (i)  $a$  is left  $g$ -MP invertible.
- (ii)  $a^2a^* + 1 - aa^{(1)}$  is left invertible.
- (iii)  $aa^*a + 1 - a^{(1)}a$  is left invertible.

**Corollary 3.5.** *Let  $R$  be a ring with involution and let  $a \in R$  be regular. Then the following conditions are equivalent:*

- (i)  $a$  is right  $g$ -MP invertible.
- (ii)  $aa^*a + 1 - aa^{(1)}$  is right invertible.
- (iii)  $a^*a^2 + 1 - a^{(1)}a$  is right invertible.

The following theorem gives equivalences among left, right  $g$ -MP inverses, core and dual core inverses, which are characterized by units.

**Theorem 3.6.** *Let  $R$  be a ring with involution and let  $a \in R$  be regular. Then the following conditions are equivalent:*

- (i)  $a$  is both left and right  $g$ -MP invertible.
- (ii)  $a$  is both core and dual core invertible.
- (iii)  $u = aa^*a + 1 - aa^{(1)}$  is invertible.
- (iv)  $v = aa^*a + 1 - a^{(1)}a$  is invertible.

PROOF. (i)  $\Leftrightarrow$  (iii)  $\Leftrightarrow$  (iv) followed from Theorem 2.12 and Lemma 3.2.

(i)  $\Rightarrow$  (ii) Suppose that  $a$  is both left and right g-MP invertible. Then  $a \in R^\# \cap R^\dagger$ . So,  $a$  is both core and dual core invertible.

(ii)  $\Rightarrow$  (i) If  $a$  is both core and dual core invertible, then  $a \in S^\# \cap S^\dagger$  and hence  $a = a^\#a^2 = (a^\dagger)^*a^\dagger aa^*a$ . So,  $a$  is left g-MP invertible since  $Sa = Sa^2 = Saa^*a$ . Similarly,  $a = a^2a^\# = aa^*aa^\dagger(a^\dagger)^*$ , which implies  $aS = a^2S = aa^*aS$ . Hence,  $a$  is both left and right g-MP invertible.  $\square$

The following theorem considers the reverse order law for the inverse along an element in rings.

**Theorem 3.7.** *Let  $a, b, d \in R$  and let  $a$  be invertible along  $d$ . Then the following conditions are equivalent:*

(i)  $b$  is invertible along  $d$ .

(ii)  $adb$  is invertible along  $d$ .

(iii)  $bda$  is invertible along  $d$ .

In this case,  $(adb)^{\parallel d} = b^{\parallel d}d^{(1)}a^{\parallel d}$  and  $(bda)^{\parallel d} = a^{\parallel d}d^{(1)}b^{\parallel d}$ .

PROOF. (i)  $\Rightarrow$  (ii) and (i)  $\Rightarrow$  (iii) follow from Theorem 2.17.

(ii)  $\Rightarrow$  (i) Suppose that  $adb$  is invertible along  $d$ . Then  $dadb + 1 - dd^{(1)} = (dadd^{(1)} + 1 - dd^{(1)})(db + 1 - dd^{(1)}) \in R^{-1}$  by Lemma 3.2. As  $a$  is invertible along  $d$ , then we have  $da + 1 - dd^{(1)} \in R^{-1}$  from Lemma 3.2 and hence  $dadd^{(1)} + 1 - dd^{(1)} \in R^{-1}$  by Jacobson's Lemma. Hence,  $db + 1 - dd^{(1)} = (dadd^{(1)} + 1 - dd^{(1)})^{-1}(dadb + 1 - dd^{(1)}) \in R^{-1}$ . Again, Lemma 3.2 guarantees that  $b$  is invertible along  $d$ .

(iii)  $\Rightarrow$  (i) If  $bda$  is invertible along  $d$ , then  $bdad + 1 - d^{(1)}d = (bd + 1 - d^{(1)}d)(d^{(1)}dad + 1 - d^{(1)}d) \in R^{-1}$ . Also,  $a$  is invertible along  $d$ , then  $ad + 1 - d^{(1)}d \in R^{-1}$  and hence  $d^{(1)}dad + 1 - d^{(1)}d \in R^{-1}$  using Jacobson's Lemma. So,  $bd + 1 - d^{(1)}d \in R^{-1}$  and  $b$  is invertible along  $d$ .  $\square$

**Corollary 3.8.** *Let  $b, d \in R$  and let  $1$  be invertible along  $d$ . Then the following conditions are equivalent:*

(i)  $b$  is invertible along  $d$ .

(ii)  $bd$  is invertible along  $d$ .

(iii)  $db$  is invertible along  $d$ .

In this case,  $(bd)^{\parallel d} = d^{\parallel d}b^{\parallel d}$  and  $(db)^{\parallel d} = b^{\parallel d}d^{\parallel d}$ .

**Corollary 3.9.** *Let  $R$  be a ring with involution and let  $b, d \in R$ . If  $d^*$  is invertible along  $d$ , then the following conditions are equivalent:*

- (i)  $b$  is invertible along  $d$ .
  - (ii)  $d^*db$  is invertible along  $d$ .
  - (iii)  $bdd^*$  is invertible along  $d$ .
- In this case,  $(d^*db)^{\parallel d} = b^{\parallel d}d^{\parallel d^*}(d^*)^{\parallel d}$  and  $(bdd^*)^{\parallel d} = (d^*)^{\parallel d}d^{\parallel d^*}b^{\parallel d}$ .*

Setting  $b = 1$  in Corollary 3.9, we further obtain

**Corollary 3.10.** *Let  $R$  be a ring with involution and let  $d \in R$ . If  $d^*$  is invertible along  $d$ , then the following conditions are equivalent:*

- (i)  $1$  is invertible along  $d$ .
  - (ii)  $dd^*$  is invertible along  $d$ .
  - (iii)  $d^*d$  is invertible along  $d$ .
- In this case,  $(dd^*)^{\parallel d} = (d^*)^{\parallel d}d^{\parallel d}$  and  $(d^*d)^{\parallel d} = d^{\parallel d}(d^*)^{\parallel d}$ .*

We remark the fact that if  $d \in R^\oplus$  or  $d \in R_{\oplus}$ , then  $d \in R^\#$ . In particular,  $d \in R^\oplus \cap R_{\oplus}$  if and only if  $d \in R^\# \cap R^\dagger$ . Moreover,  $d^\oplus = d^\#dd^\dagger$  and  $d^\oplus = d^\dagger dd^\#$ . Note that  $1$  is invertible along  $d$  if and only if  $d^\#$  exists. Moreover,  $1^{\parallel d} = dd^\#$ . We can derive characterizations and presentations of core and dual core inverses by the inverse along an element.

**Corollary 3.11.** *Let  $R$  be a ring with involution and let  $d \in R^\dagger$ . Then the following conditions are equivalent:*

- (i)  $d \in R^\oplus$ .
  - (ii)  $d \in R_{\oplus}$ .
  - (iii)  $dd^*$  is invertible along  $d$ .
  - (iv)  $d^*d$  is invertible along  $d$ .
- In this case,  $d^\oplus = 1^{\parallel d}d^{\parallel d^*}$  and  $d_{\oplus} = d^{\parallel d^*}1^{\parallel d}$ .*

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