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Mathematica Tools for Quaternionic Polynomials

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Abstract

In this paper we revisit the ring of (left) one-sided quaternionic polynomials with special focus on its zero structure. This area of research has attracted the attention of several authors and therefore it is natural to develop computational tools for working in this setting. The main contribution of this paper is a Mathematica collection of functions `QPolynomial` for solving polynomial problems that we frequently find in applications.

1 Introduction

In this paper we are going to study polynomials in one formal variable x whose coefficients are quaternions located on the left side of the powers of x , i.e. polynomials of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad a_k \in \mathbb{H}. \quad (1)$$

If $Q(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0$ is another polynomial of the same kind, then one can define the operation:

$$P(x) + Q(x) = (a_n + b_n)x^n + (a_{n-1} + b_{n-1})x^{n-1} + \cdots + (a_1 + b_1)x + a_0 + b_0,$$

so that the polynomials (1) form an additive abelian group. There are several ways of defining the multiplication such that this group is made a ring. For details on the theory of non-commutative polynomials, we refer the reader to the well-known work of Ore [23]. In the classical case and in many practical applications (e.g. [11, 25, 26]), one assumes that the variable x commutes with the coefficients, i.e. the multiplication of two polynomials P and Q of degrees n and m , respectively, is defined as

$$P(x)Q(x) = \sum_{k=0}^{m+n} \left(\sum_{j=0}^k a_j b_{k-j} \right) x^k, \quad (2)$$

with the implicit assumption that $a_k = 0$, if $k > n$ and $b_k = 0$, if $k > m$. In other words, the set of polynomials of the form (1) endowed with the aforementioned addition and multiplication is a polynomial ring over \mathbb{H} and

is denoted by $\mathbb{H}[x]$. In the literature, $\mathbb{H}[x]$ is usually referred to as the ring of (left) one-sided, unilateral, basic, simple or standard polynomials and the determination and classification of their zeros have attracted the attention of several authors over the years (see e.g. [1, 2, 6, 12, 13, 14, 17, 18, 22, 27, 29, 30, 31]) due to the appearance of new interesting algebraic and computational properties.

General polynomials are defined as finite sums of noncommutative monomials of type $a_0 x a_1 x \dots x a_n$. General polynomials have been also considered in several works ([7, 14, 28]) specially the cases of two-sided or quadratic polynomials (e.g. [16, 20, 32]).

In this paper we describe a Mathematica tool for polynomial manipulation in $\mathbb{H}[x]$. The paper is organized as follows. In Sect. 2 we recall well-known concepts and review important results on quaternionic polynomials. In Sect. 3 we focus on the zeros of a polynomial and the relation to its factorization(s) in linear terms. Finally, in Sect. 4 the new Mathematica tools are presented.

2 The Ring of Quaternionic Polynomials

In 1843, Hamilton introduced numbers of the form

$$x = x_0 + \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3, \quad x_i \in \mathbb{R}, \quad (3)$$

where \mathbf{i} , \mathbf{j} and \mathbf{k} satisfy the multiplication rules

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1 \text{ and } \mathbf{ij} = -\mathbf{ji} = \mathbf{k}. \quad (4)$$

This non-commutative product generates the algebra of real quaternions \mathbb{H} (named after Hamilton). For a quaternion x of the form (3) we can define, in analogy with the complex case, the real part of x , $\text{Re } x := x_0$, the vector part of x , $\text{Vec } x := \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3$ and the conjugate of x , $\bar{x} := x_0 - \mathbf{i}x_1 - \mathbf{j}x_2 - \mathbf{k}x_3$. The norm of x is given by $|x| := \sqrt{x\bar{x}} = \sqrt{\bar{x}x} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$. It immediately follows that each non-zero $x \in \mathbb{H}$ has an inverse given by $x^{-1} = \frac{\bar{x}}{|x|^2}$ and therefore \mathbb{H} is a non-commutative division ring or a skew field.

Let us review now some well-known facts on quaternionic polynomials and fix some notation. In what follows, q denotes a quaternion, P is a left one-sided polynomial and \mathbf{Z}_P denotes the zero-set of P .

Definition 1. (*Similarity*) Two quaternions q and q' are called *similar*, $q \sim q'$, if there exists $h \neq 0$ such that $q' = hqh^{-1}$. Similarity is an equivalence relation and the *similarity class* of q , denoted by $[q]$, is the set $[q] = \{q' \in \mathbb{H} : q \sim q'\}$.

It can easily be shown (see, e.g. [3]) that

$$[q] = \{q' \in \mathbb{H} : \text{Re } q = \text{Re } q' \text{ and } |q| = |q'|\}.$$

Example 1. The similarity class of $1 + \mathbf{i} + \mathbf{j} + \mathbf{k}$ is the three-dimensional sphere in the hyperplane $\{(1, x, y, z) \in \mathbb{R}^4\}$, with center $(1, 0, 0, 0)$ and radius $\sqrt{3}$, i.e.

$$[1 + \mathbf{i} + \mathbf{j} + \mathbf{k}] = \{1 + \mathbf{i}x + \mathbf{j}y + \mathbf{k}z \in \mathbb{H} : x^2 + y^2 + z^2 = 3\}.$$

Definition 2. (*Special polynomials*)

1. The *conjugate* of P is the polynomial

$$\bar{P}(x) := \bar{a}_n x^n + \bar{a}_{n-1} x^{n-1} + \dots + \bar{a}_1 x + \bar{a}_0.$$

2. The *characteristic polynomial* of q is the polynomial

$$\Psi_q(x) := (x - q)(x - \bar{q}) = x^2 - 2\text{Re}(q)x + |q|^2. \quad (5)$$

3. The *companion polynomial* of P is the real polynomial

$$\mathcal{C}_P(x) := P(x)\bar{P}(x) = \bar{P}(x)P(x). \quad (6)$$

Remark 1. The characteristic polynomial of q depends only on the real part and norm of q and, if $q \in \mathbb{H} \setminus \mathbb{R}$, then Ψ_q is an irreducible quadratic trinomial from the ring of polynomials $\mathbb{R}[x]$.

Remark 2. In [17], Janovská and Opfer had called the polynomial

$$\sum_{k=0}^{2n} \left(\sum_{j=\max\{0, k-n\}}^{\min\{k, n\}} \bar{a}_j a_{k-j} \right) x^k$$

the companion polynomial of the quaternionic polynomial P , without mentioning that this is nothing else than the usual product of \bar{P} and P in $\mathbb{H}[x]$ (cf. (2)). As pointed out by these authors, the idea of constructing such a polynomial came from the work of Niven [22], where the construction of the polynomial is described as follows: By replacing the quaternions a_k by their conjugates \bar{a}_k we obtain a polynomial \bar{P} . We multiply this on the right by $P(x)$, and allow x to be commutative with the coefficients. Thus we obtain a polynomial with coefficients in \mathbb{R} .

In the literature one can find different designations to the polynomial (6), namely basic polynomial, semi-norm or symmetrization of a polynomial. We prefer the designation companion polynomial due to its connection (not recognized explicitly in [17]) to the characteristic polynomial of the companion matrix which was already apparent from the work of [29] (see also [4]).

Example 2. For $P(x) = x^2 + (2 - \mathbf{i})x + \mathbf{j} + \mathbf{k}$ and $q = \mathbf{i} + \mathbf{j}$ we have

$$\begin{aligned} \Psi_q(x) &= x^2 + 2, \\ \mathcal{C}_P(x) &= (x^2 + (2 - \mathbf{i})x + \mathbf{j} + \mathbf{k})(x^2 + (2 + \mathbf{i})x - \mathbf{j} - \mathbf{k}) = x^4 + 4x^3 + 5x^2 + 2. \end{aligned}$$

Definition 3. (Evaluation map and zeros)

1. The *evaluation* of P at q is defined as $P(q) := a_n q^n + a_{n-1} q^{n-1} + \cdots + a_1 q + a_0$.
2. A quaternion q is a *zero* (or a *root*) of P , if $P(q) = 0$.
3. A zero q is called an *isolated zero* of P , if $[q]$ contains no other zeros of P .
4. A zero q is called a *spherical zero* of P , if q is not an isolated zero; $[q]$ is referred to as a sphere of zeros.¹

Remark 3. We recall here the well-known fact that the evaluation map (at a quaternion q) is not an algebra homomorphism. In fact, if $P(x) = L(x)R(x)$, then (see e.g. [19])

$$P(q) = \begin{cases} 0, & \text{if } R(q) = 0, \\ L(\tilde{q})R(q), & \text{if } R(q) \neq 0, \end{cases} \quad \text{where } \tilde{q} = R(q)q(R(q))^{-1}. \quad (7)$$

In particular, if q is a zero of P which is not a zero of R , then \tilde{q} is a zero of L . Next example illustrates this remark.

Example 3. Consider the polynomial P and the quaternion q of last example. If $Q(x) = x - q$, then we have

$$P(q) = -1 + 2\mathbf{i} + 3\mathbf{j} \quad \text{and} \quad Q(q) = 0.$$

On the other hand, since

$$(QP)(x) = x^3 + (2 - 2\mathbf{i} - \mathbf{j})x^2 + (-1 - 2\mathbf{i} - \mathbf{j})x + 1 - \mathbf{i} + \mathbf{j} - \mathbf{k},$$

we remark that

$$(QP)(q) = -2\mathbf{k} \neq 0 = Q(q)P(q).$$

¹In such case, it can be proved that all quaternions in $[q]$ are in fact zeros of P and therefore the choice of the term spherical to designate this type of zeros is natural.

3 Zeros and Factor-Terms in $\mathbb{H}[x]$

We list now the main results concerning the zero-structure and the factorization of polynomials in $\mathbb{H}[x]$. For the proofs and other details we refer the reader to [14, 19, 23].

We recall first that it can be proved (see e.g. [15]) that $\mathbb{H}[x]$ is a principal ideal domain and therefore also left and right division algorithms can be defined.

Theorem 1 (Euclidean division). *If $A(x)$ and $B(x)$ are polynomials in $\mathbb{H}[x]$ (with $\deg B \leq \deg A$ and $B \neq 0$), then there exist unique $Q(x)$, $Q'(x)$, $R(x)$ and $R'(x)$ such that*

$$A(x) = Q(x)B(x) + R(x) \quad (8)$$

and

$$A(x) = B(x)Q'(x) + R'(x) \quad (9)$$

with $\deg R < \deg B$ and $\deg R' < \deg B$.

If $R(x) = 0$ (resp. $R'(x) = 0$) in (8) (resp. (9)), then $B(x)$ is called a right (resp. left) divisor of $A(x)$. The greatest common (right and left) divisor polynomial of two polynomials can be computed using the Euclidean algorithm, by a basic procedure similar to the one used in the complex setting. In this context we also refer to [5] where issues of division within $\mathbb{H}[x]$ are addressed. In what follows, if nothing is specified, right divisors are always assumed.

Concerning the zero structure, the theory of quaternionic polynomials is very different from that of complex polynomials. Nevertheless, in 1941, Niven [22] proved the Fundamental Theorem of Algebra for quaternionic polynomials, establishing that a non-constant polynomial P in $\mathbb{H}[x]$ has a zero in \mathbb{H} . As in the classical case, it is also possible to write P as a product of linear factors; however the link between zeros and factors is not straightforward.

Theorem 2 (Factorization into linear terms [14, 19]). *Any monic polynomial P of degree $n \in \mathbb{N}$ in $\mathbb{H}[x]$ admits a factorization into linear factors, i.e. there exist $x_1, \dots, x_n \in \mathbb{H}$, such that*

$$P(x) = (x - x_n)(x - x_{n-1}) \cdots (x - x_1). \quad (10)$$

Definition 4. (*Chain*) In a factorization of P of the form (10), the quaternions x_1, \dots, x_n will be called *factor-terms* of P and the n -uple (x_1, \dots, x_n) will be called a *factor-terms chain associated with P* or simply a *chain of P* .

Theorem 3 ([19, 30]). *Let (x_1, \dots, x_n) be a chain of a polynomial P . Then every zero of P is similar to some factor-term x_k in the chain and reciprocally every factor-term x_k is similar to some zero of P .*

Remark 4. *If (y_1, \dots, y_n) is another chain of P , then there exists a permutation π of $(1, \dots, n)$ and $h_1, \dots, h_n \in \mathbb{H}$, such that $y_{\pi(i)} = h_i x_i h_i^{-1}$; $i = 1, \dots, n$, i.e. $[y_{\pi(i)}] = [x_i]$. The explicit expression of h_i will be given later on. In such cases we write $(x_1, \dots, x_n) \sim (y_1, \dots, y_n)$ and say that the chains (x_1, \dots, x_n) and (y_1, \dots, y_n) are equivalent or similar.*

The explicit relation between factor-terms and zeros of a quaternionic polynomial is addressed in the following results. The first result is useful if one knows a factorization of the polynomial; for example in [10] a numerical method for computing a Weierstrass factorization of a quaternionic polynomial is proposed and the non-spherical zeros are obtained via (11). The second result plays an important role in the construction of polynomials with prescribed zeros.

Theorem 4 (Zeros from factors [19]). *Consider a chain (x_1, \dots, x_n) of a polynomial P . If the similarity classes $[x_k]$ are distinct, then P has exactly n zeros ζ_k which are related to the factor-terms x_k as follows:*

$$\zeta_k = \overline{\mathcal{P}_k(x_k)} x_k (\overline{\mathcal{P}_k(x_k)})^{-1}; \quad k = 1, \dots, n, \quad (11)$$

where

$$\mathcal{P}_k(x) := \begin{cases} 1, & \text{if } k = 1, \\ (x - x_{k-1}) \cdots (x - x_1), & \text{otherwise.} \end{cases} \quad (12)$$

Theorem 5 (Factors from zeros [1]). *If ζ_1, \dots, ζ_n are quaternions such that the similarity classes $[\zeta_k]$ are distinct, then there is a unique polynomial P of degree n with zeros ζ_1, \dots, ζ_n , which can be constructed from the chain (x_1, \dots, x_n) , where*

$$x_k = \mathcal{P}_k(\zeta_k) \zeta_k (\mathcal{P}_k(\zeta_k))^{-1}; \quad k = 1, \dots, n \quad (13)$$

and \mathcal{P}_k is the polynomial (12).

In [1, 13] more general forms of Theorems 4 and 5 can be found, where the notion of multiplicity of a zero plays a fundamental role. The following result motivates the definition of multiplicity we are going to adopt (in this context we refer also to [1, 2, 9, 24]).

Theorem 6 ([1, 24]). *Let P be a quaternionic polynomial with degree n . Then $x_1 \in \mathbb{H} \setminus \mathbb{R}$ is the unique zero of P if and only if P admits a unique chain (x_1, \dots, x_n) which has the property²*

$$x_l \in [x_1] \quad \text{and} \quad x_l \neq \bar{x}_{l-1}, \quad (14)$$

for all $l = 2, \dots, n$.

Definition 5. (*Multiplicity*) The *multiplicity of a zero q* of P , $m_P(q)$, is defined as the maximum degree of the right factors of P having q as their unique zero. The *multiplicity of a sphere of zeros $[q]$* of P , $m_P([q])$, is the largest $k \in \mathbb{N}_0$ for which Ψ_q^k divides P .

Example 4. The polynomial $P(x) = (x - \mathbf{k})(x - \mathbf{j})(x - \mathbf{i})$ has \mathbf{i} as its unique zero of multiplicity 3. Concerning the polynomial $Q(x) = (x + \mathbf{i})(x - \mathbf{i})(x - \mathbf{i})$ we observe that $m_Q(\mathbf{i}) = 2$ while $m_Q([\mathbf{i}]) = 1$. In [9] such a zero has been called a *mixed zero*.³

The final results of this section are presented/rewritten by taking into account the purpose of the present paper.

Theorem 7. *Consider a chain $(x_1, \dots, x_{l-1}, x_l, \dots, x_n)$ of a polynomial P .*

1. *If $h := \bar{x}_l - x_{l-1} \neq 0$ then*

$$(x_1, \dots, h^{-1}x_l h, h^{-1}x_{l-1}h, \dots, x_n) \sim (x_1, \dots, x_{l-1}, x_l, \dots, x_n). \quad (15)$$

2. *If the chain (x_1, \dots, x_n) has the property (14) and Q is a polynomial of degree m in $\mathbb{H}[x]$ such that $y_1 \in \mathbb{H} \setminus \mathbb{R}$ is its unique zero and $y_1 \notin [x_1]$, then the polynomial of degree $n + m$, QP , has only two zeros, namely x_1 and $\bar{P}(y_1)y_1(\bar{P}(y_1))^{-1}$.*

Proof. The first result follows by simple manipulation (see also [30] for a different, but equivalent result).

By Theorem 6, the polynomial Q can be written as

$$Q(x) = (x - y_m) \dots (x - y_2)(x - y_1), \quad y_l \in [y_1], \quad y_l \neq \bar{y}_{l-1}, \quad l = 2, \dots, m.$$

and therefore

$$T(x) := Q(x)P(x) = (x - y_m) \dots (x - y_2)(x - y_1)(x - x_n) \dots (x - x_2)(x - x_1).$$

It is clear that x_1 is a zero of T with multiplicity n . Multiplying both sides of last expression by $\bar{P}(x)$ and recalling (5) and (14) we obtain

$$\begin{aligned} T(x)\bar{P}(x) &= (x - y_m) \dots (x - y_2)(x - y_1)\Psi_{x_1}^n(x) \\ &= \Psi_{x_1}^n(x)(x - y_m) \dots (x - y_2)(x - y_1). \end{aligned}$$

Since $\Psi_{x_1}(q) \neq 0$ if $q \notin [x_1]$, we conclude that y_1 is the only zero of $T(x)\bar{P}(x)$ which is not in $[x_1]$. From $\bar{P}(y_1) \neq 0$ and (7), it follows at once that the only zero of T apart from x_1 is $\bar{P}(y_1)y_1(\bar{P}(y_1))^{-1}$. □

²In [2], such a chain was called a *spherical chain*.

³More precisely, q is a mixed zero of P if $m_P([q]) > 0$ and $m_P(q) > m_P(q')$, for all $q' \in [q]$.

The repeated use of Theorem 7 allows to classify (and identify) the zeros of a polynomial from one of its factorization.

Example 5. Consider the polynomial P associated with the chain

$$(5\mathbf{j} - 5\mathbf{k}, 5\mathbf{j}, -7\mathbf{j} + \mathbf{k}, 1 + \mathbf{i}, -3\mathbf{i} - 4\mathbf{j}, 3\mathbf{j} - 4\mathbf{k}).$$

The repeated use of (15) allows to state that this chain is equivalent to the following ones:

$$(5\mathbf{j} - 5\mathbf{k}, -5\mathbf{j} + 5\mathbf{k}, 3\mathbf{j} - 4\mathbf{k}, 1 + \mathbf{i}, -3\mathbf{i} - 4\mathbf{j}, 3\mathbf{j} - 4\mathbf{k}).$$

$$(5\mathbf{j} - 5\mathbf{k}, -5\mathbf{j} + 5\mathbf{k}, 1 - \frac{23}{27}\mathbf{i} - \frac{2}{27}\mathbf{j} - \frac{14}{27}\mathbf{k}, \frac{50}{27}\mathbf{i} + \frac{83}{27}\mathbf{j} - \frac{94}{27}\mathbf{k}, -3\mathbf{i} - 4\mathbf{j}, 3\mathbf{j} - 4\mathbf{k}).$$

Therefore

- $5\mathbf{j} - 5\mathbf{k}$ is a spherical zero of P with multiplicity 1;
- $1 - \frac{23}{27}\mathbf{i} - \frac{2}{27}\mathbf{j} - \frac{14}{27}\mathbf{k}$ is an isolated zero of P with multiplicity 1;
- $3\mathbf{j} - 4\mathbf{k}$ is an isolated zero of P with multiplicity 3.

Finally, one can construct a polynomial with assigned zeros, by the repeated use of the following result.

Theorem 8. A polynomial having ζ_1 and ζ_2 as its isolated zeros of multiplicity n and m , respectively, and a sphere of zeros $[\zeta_s]$ with multiplicity k , can be constructed through the chain

$$\underbrace{(\zeta_1, \dots, \zeta_1)}_n, \underbrace{(\tilde{\zeta}_2, \dots, \tilde{\zeta}_2)}_m, \underbrace{(\zeta_s, \overline{\zeta_s}, \dots, \zeta_s, \overline{\zeta_s})}_{2k}, \quad (16)$$

where $\tilde{\zeta}_2 = \mathcal{Q}(\zeta_2) \zeta_2 (\mathcal{Q}(\zeta_2))^{-1}$ and $\mathcal{Q}(x) = (x - \zeta_1)^n$.

Proof. The use of Theorem 2.8 in [13], adapted to the case of left one-sided polynomials, allows the construction of a family of polynomials with the assigned zeros, namely $P = \Psi_{\zeta_s}^k(x)(x - y_m) \dots (x - y_1)(x - x_n) \dots (x - x_1)$, where

- $x_1 = \zeta_1$ and x_l are chosen in $[x_1]$, so that $x_l \neq \bar{x}_{l-1}$, $l = 2, \dots, n$. In particular, one can take $x_n = \dots = x_1 = \zeta_1$ in line with the classical definition of multiplicity;
- $\tilde{\zeta}_2 = y_1 = \mathcal{P}_{n+1}(\zeta_2) \zeta_2 (\mathcal{P}_{n+1}(\zeta_2))^{-1}$, where \mathcal{P}_{n+1} is given by (12) (cf. (13));
- y_k , $k = 2, \dots, m$ are chosen in $[y_1]$, so that $y_k \neq \bar{y}_{k-1}$. The particular choice⁴ $y_m = y_{m-1} = \dots = y_1 = \tilde{\zeta}_2$ leads to the final result.

□

Example 6. An example of a polynomial P which has $x_1 = \mathbf{i}$ as a zero of multiplicity 3, $x_2 = -1 + \mathbf{j} + \mathbf{k}$ as a zero of multiplicity 2 and $[2 + \mathbf{i}]$ as a sphere of zeros with multiplicity 2 is:

$$P(x) = \Psi_{2+\mathbf{i}}^2(x - \tilde{x}_2)^2(x - x_1)^3,$$

where $\tilde{x}_2 = \mathcal{P}_4(x_2)x_2(\mathcal{P}_4(x_2))^{-1}$ and $\mathcal{P}_4(x) = (x - x_1)^3$, i.e.

$$P(x) = (x - 2 - \mathbf{i})^2(x - 2 + \mathbf{i})^2(x + 1 + \frac{7}{5}\mathbf{i} + \frac{1}{5}\mathbf{j})^2(x - \mathbf{i})^3. \quad (17)$$

It follows immediately from the above considerations that the polynomial

$$Q(x) = (x - 2 - \mathbf{i})^2(x - 2 + \mathbf{i})^2(x + 1 - \mathbf{j} - \mathbf{k})(x + 1 + \frac{7}{5}\mathbf{i} + \frac{1}{5}\mathbf{j})(x - \mathbf{k})(x - \mathbf{j})(x - \mathbf{i})$$

solves the same problem.

⁴In [2] the same problem is also addressed and one can find a particular (different) choice of \tilde{x}_l ($l = 2, \dots, n$) and y_l ($l = 2, \dots, m$).

4 QPolynomial

Some years ago, two of the authors of this paper introduced some new tools on the Mathematica standard package `Quaternions` for implementing Hamilton's Quaternion Algebra [8]. Later on, a new add-on `QuaternionAnalysis` [21] was developed to provide also tools for manipulating regular quaternion valued functions. This package can be downloaded at the Wolfram Library archive and full description of the `QuaternionAnalysis` functions as well as illustrative examples can be found in the tutorial included in [21].

`QPolynomial` is a collection of Mathematica functions, depending on the package `QuaternionAnalysis`, for treating usual problems in $\mathbb{H}[x]$: evaluation, euclidean division, greatest common divisor polynomial, construction of a polynomial with prescribed zeros, etc. In this context, a quaternionic polynomial is an object of the form `Polynomial[an, an-1, ..., a1, a0]` accordingly to (1). For such objects, rules as `Plus`, `NonCommutativeMultiply`, `Power` and functions as `Conjugate`, `Eval`, `CharacteristicPolynomial`, `CompanionPolynomial`, etc., are defined. We summarize in Table 1 the most important functions included in `QPolynomial`. Auxiliary functions which come with its own usefulness are also listed in the same table. The use of these functions is illustrated by several examples.

Example 7. Rules on the object `Polynomial`

```
In[1]:= P1 = Polynomial[1, Quaternion[2, -1, 0, 0]];
        P2 = Polynomial[Quaternion[0, 0, 0, 1], Quaternion[0, 0, 1, 1]];
        α = Quaternion[0, 1, 1, 0];
In[4]:= P1 + P2
Out[4]= Polynomial[Quaternion[1, 0, 0, 1], Quaternion[2, -1, 1, 1]]
In[5]:= α ** P1
Out[5]= Polynomial[Quaternion[0, 1, 1, 0], Quaternion[1, 2, 2, 1]]
In[6]:= P1 ** P2
Out[6]= Polynomial[Quaternion[0, 0, 0, 1], Quaternion[0, 0, 2, 3], Quaternion[0, 0, 3, 1]]
```

Example 8. Polynomial Functions (cf. Example 2)

```
In[1]:= P = Polynomial[1, Quaternion[2, -1, 0, 0], Quaternion[0, 0, 1, 1]];
        q = Quaternion[0, 1, 1, 0];
In[3]:= CharacteristicPolynomial[q]
Out[3]= Polynomial[1, 0, 2]
In[4]:= CompanionPolynomial[P]
Out[4]= Polynomial[1, 4, 5, 0, 2]
In[5]:= P ** Conjugate[P] // PSimplify
Out[5]= Polynomial[1, 4, 5, 0, 2]
```

Example 9. Evaluation and zeros (cf. Example 3)

```
In[6]:= Q = Polynomial[1, -q]
Out[6]= Polynomial[1, Quaternion[0, -1, -1, 0]]
In[7]:= Eval[P][q]
Out[7]= Quaternion[-1, 2, 3, 0]
In[8]:= Eval[Q][q]
Out[8]= 0
In[9]:= Eval[Q ** P][q]
Out[9]= Quaternion[0, 0, 0, -2]
```

Next example addresses issues of division in $\mathbb{H}[x]$. Both left and right Euclidean division are implemented in `QPolynomial` (see Theorem 1).

Table 1: qPolynomial functions

Algebraic Operations on Polynomials	
Conjugate [P]	gives the conjugate of P
CharacteristicPolynomial [q]	gives the characteristic polynomial of q
Eval [P] [q]	gives P(q)
CompanionPolynomial [P]	gives the companion polynomial of P
PolynomialDivision [P ₁ , P ₂]	gives a list with the quotient and remainder {Q,R} of left division i.e. P ₁ =P ₂ Q+R
PolynomialDivisionR [P ₁ , P ₂]	gives a list with the quotient and remainder {Q,R} of right division i.e. P ₁ =QP ₂ +R
PolynomialGCDL [P ₁ , ..., P _n]	gives the greatest common left divisor of P ₁ , ..., P _n
PolynomialGCDR [P ₁ , ..., P _n]	gives the greatest common right divisor of P ₁ , ..., P _n
ZeroesFromChain [{x ₁ , ..., x _n }]	gives the (isolated) zeros from a chain of a polynomial
ChainFromZeros [{r ₁ , ..., r _n }]	gives a chain of a polynomial with prescribed (simple) zeros
ChainFromZeros [{r ₁ , m ₁ }, ..., {r _p , m _p }, {{s ₁ , n ₁ }, ..., {s _q , n _q }}]	gives a polynomial with isolated zeros r _i and spherical zeros s _j , with multiplicities m _i and n _j , respectively

Auxiliary Functions

PSimplify [P]	gives P with simplified coefficients
PNormalize [P]	gives the normalized (monic) polynomial
SimilarQ [q ₁ , q ₂]	gives True if [q ₁]=[q ₂]
SimilarChainQ [c ₁ , c ₂]	gives True if the chains c ₁ and c ₂ are similar
SphericalQ [P, q]	gives True if q is a spherical zero of P
FactorShift [{q ₁ , q ₂ }]	gives a list {q̃ ₂ , q̃ ₁ } such that (x-q ₁)(x-q ₂)=(x-q̃ ₂)(x-q̃ ₁)
PolynomialFromChain [x ₁ , ..., x _n]	gives the Polynomial object corresponding to the polynomial (x-x _n)... (x-x ₁)

Example 10. Euclidean division and GCD

```
In[1]:= P1=Polynomial[1,Quaternion[1,0,2,1],-2,Quaternion[1,1,1,0]];
        P2=Polynomial[1,1,Quaternion[1,0,-1,1]];
In[3]:= {quotientR,remainderR} = PolynomialDivisionR[P1,P2]
Out[3]= {Polynomial[1,Quaternion[0,0,2,1]],Polynomial[Quaternion[-3,0,-1,-2],Quaternion[0,-2,-1,-1]]}
In[4]:= P1 - quotientR ** P2 - remainderR // PSimplify
Out[4]= 0
In[5]:= P1 - P2 ** quotientR - remainderR // PSimplify
Out[5]= Polynomial[Quaternion[0,-6,0,0]]
In[6]:= {quotientL,remainderL} = PolynomialDivisionL[P1,P2]
Out[6]= {Polynomial[1,Quaternion[0,0,2,1]],Polynomial[Quaternion[-3,0,-1,-2],Quaternion[0,4,-1,-1]]}
In[7]:= P1 - quotientL ** P2 - remainderL // PSimplify
Out[7]= Polynomial[Quaternion[0,6,0,0]]
In[8]:= P1 - P2 ** quotientL - remainderL // PSimplify
Out[8]= 0
```

In the quaternion case, greatest common left and right divisors are computed by a procedure similar to the classical Euclidean algorithm via `PolynomialGCDL` and `PolynomialGCDR`, respectively.

```
In[1]:= P=Polynomial[1,Quaternion[1,0,1,0]];
        Q=Polynomial[1,Quaternion[2,1,0,1]];
        R=Polynomial[1,Quaternion[-1,1,0,1]];
        S=Polynomial[1,Quaternion[0,0,1,0]];
In[5]:= PolynomialGCDR[P**Q**R,S**Q**R] // PSimplify
Out[5]= Polynomial[1,Quaternion[1,2,0,2],Quaternion[-4,1,0,1]]
In[6]:= PSimplify[PolynomialGCDL[P**Q**R,S**Q**R]]
Out[6]= 1
In[7]:= PolynomialGCDR[Q**P**R,S**Q**R]
Out[7]= Polynomial[1,Quaternion[-1,1,0,1]]
In[8]:= PolynomialGCDL[Q**P**R,S**Q**R] // PSimplify
Out[8]= 1
```

For complex arguments, `PolynomialDivisionL` and `PolynomialDivisionR` coincide with the `PolynomialQuotientRemainder` built-in function and the same is true for `PolynomialGCDL`, `PolynomialGCDR` and `PolynomialGCD`.

```
In[1]:= P1=Polynomial[1,1]**Polynomial[1,2]**Polynomial[1,3];
        P2=Polynomial[1,-1]**Polynomial[1,2];
In[3]:= {quotientR,remainderR}=PolynomialDivisionR[P1,P2]
Out[3]= {Polynomial[1,5],Polynomial[8,16]}
In[4]:= {quotientL,remainderL}=PolynomialDivisionL[P1,P2]
Out[4]= {Polynomial[1,5],Polynomial[8,16]}
In[5]:= PolynomialQuotientRemainder[(x+1)(x+2)(x+3),(x-1)(x+2),x]
Out[5]= {x+5,8x+16}
In[6]:= PolynomialGCDR[P1,P2] // PSimplify
Out[6]= Polynomial[1,2]
In[7]:= PolynomialGCDL[P1,P2] // PSimplify
Out[7]= Polynomial[1,2]
In[8]:= PolynomialGCD[(x+1)(x+2)(x+3),(x-1)(x+2)]
Out[8]= x+2
```

The functions `ZerosFromChain` and `ChainFromZeros` are based on the use of Theorem 4 and Theorem 5, respectively.

Example 11. Factors from zeros and zeros from factors

```
In[1]:= l = {Quaternion[1,1,0,0], Quaternion[2,0,1,0], Quaternion[3,0,0,1]};
In[2]:= r = ZerosFromChain[l]
Out[2]:= {Quaternion[1,1,0,0], Quaternion[2, 2/3, 1/3, 2/3], Quaternion[3, 9/11, 2/11, 6/11]}
In[3]:= P=PolynomialFromChain[l]
Out[3]:= Polynomial[1, Quaternion[-6, -1, -1, -1], Quaternion[11, 4, 5, 2], Quaternion[-7, -5, -5, 1]]
In[4]:= Table[Eval[P][r[[i]]], {i, 1, 3}]
Out[4]:= {0, 0, 0}
In[5]:= c=ChainFromZeros[RotateLeft[r]]
Out[5]:= {Quaternion[2, 2/3, 1/3, 2/3], Quaternion[3, 13/21, -4/21, 16/21], Quaternion[1, -2/7, 6/7, -3/7]}
In[6]:= SimilarChainQ[l, c]
Out[6]:= True
```

Observe that `ZerosFromChain/ChainFromZeros` can not be used when some of the arguments are similar. This problem can be overcome in the case of `ChainFromZeros` by using an alternative argument syntax (see Example 13).

```
In[1]:= l={Quaternion[1,1,0,1], Quaternion[0,1,0,0], Quaternion[0,1,0,0]};
In[2]:= ZerosFromChain[l]
Out[2]:= ZerosFromChain :: arg :arguments in the same similarity class.
In[3]:= ChainFromZeros[l]
Out[3]:= ChainFromZeros :: arg :arguments in the same similarity class.
```

Use an alternative syntax.

Next example shows how to use the function `FactorShift` for successive applications of (15).

Example 12. Identifying the zeros from a given chain (cf. Example 5)

```
In[1]:= x1=Quaternion[0,0,5,-5]; x2=Quaternion[0,0,5,0];
x3=Quaternion[0,0,-7,1]; x4=Quaternion[1,1,0,0];
x5=Quaternion[0,-3,-4,0]; x6=Quaternion[0,0,3,-4];
In[4]:= {y2, y3}=FactorShift[{x2, x3}];
In[5]:= TraditionalForm/@(c1={x1, y2, y3, x4, x5, x6})
Out[5]:= {5j - 5k, 5k - 5j, 3j - 4k, i + 1, -3i - 4j, 3j - 4k}
In[6]:= {z3, z4}=FactorShift[{y3, x4}];
In[7]:= TraditionalForm/@(c2={x1, y2, z3, z4, x5, x6})
Out[7]:= {5j - 5k, 5k - 5j, -23i/27 - 2j/27 - 14k/27 + 1, 50i/27 + 83j/27 - 94k/27, -3i - 4j, 3j - 4k}
In[8]:= (P=PolynomialFromChain[c1])==PolynomialFromChain[c2]
Out[8]:= True
In[9]:= Eval[P]/@{x1, -x1, z3, y3}
Out[9]:= {0, 0, 0, 0}
```

Our last example illustrates the use of the function `ChainFromZeros` which allows the construction of a polynomial with assigned zeros and corresponding multiplicities. It is based on the repeated use of Theorem 8.

Example 13. Polynomial with prescribed zeros (cf. Example 6)

```
In[1]:= r1=Quaternion[0,1,0,0]; m1=3;
r2=Quaternion[-1,0,1,1]; m2=2;
s1=Quaternion[2,1,0,0]; mS1=3;
In[4]:= c=ChainFromZeros[{{r1, m1}, {r2, m2}}, {{s1, mS1}}]
Out[4]:= {Quaternion[0,1,0,0], Quaternion[0,1,0,0], Quaternion[0,1,0,0], Quaternion[-1, -7/5, -1/5, 0],
Quaternion[-1, -7/5, -1/5, 0], Quaternion[2,1,0,0], Quaternion[2,-1,0,0], Quaternion[2,1,0,0],
Quaternion[2,-1,0,0], Quaternion[2,1,0,0], Quaternion[2,-1,0,0]}
In[5]:= P=PolynomialFromChain[c];
In[6]:= Eval[P]/@{r1, r2, s1, Conjugate[s1]}
Out[6]:= {0, 0, 0, 0}
```

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