

ANALYTICAL AND NUMERICAL SOLUTIONS FOR A CLASS OF OPTIMIZATION PROBLEMS IN ELASTICITY

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ABSTRACT. The subject of topology optimization methods in structural design has increased rapidly since the publication of [?], where some ideas from homogenization theory were put into practice. Since then, several engineering applications have been developed successfully. However, in the literature, there is a lack of analytical solutions, even for simple cases, which might help in the validation of the numerical results. In this work, we develop analytical solutions for simple minimum compliance problems, in the framework of elasticity theory. We compare these analytical solutions with numerical results obtained via an algorithm proposed in [?].

1. Introduction. The efficient use of materials is one of the major goals of engineering sciences. In the field of structural optimization, we can identify three main types of design problems (independent of the objective function): to determine the optimal thickness distribution, where the domain of the design and state variable is *a priori* known and fixed in the optimization process — size problem; to find the optimal shape of the design — shape problem; to determine the number and location of holes and the connectivity of the domain — topology problem. During the last decades several powerful algorithms were developed in order to address all these types of problems.

One of the algorithms that proved to be applicable to practical problems in the area of topology optimization was introduced in [?], giving a new impulse to the most complex design problem. There, the first introduced novelty was the transformation of the initial problem into a material distribution problem, where composite materials were used as the base material. Another novelty was the application of Homogenization Theory (*eg.*, [?], [?]) to determine the macroscopic material properties from the microscopic material constituents (*cf.* [?], [?], [?], [?]).

In this work we are interested in studying some analytical solutions of that method, in order to provide a basis for comparison with the numerical results. To do so, we use laminates because for this type of composite materials there is an explicit dependence of the homogenized coefficients on the design variables.

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The type of problems we wish to address may be described by the following abstract setting: let V be a Hilbert space (state space), Λ be a Banach space and Φ be an open subset of Λ (control space), where we give the functionals

$$\begin{aligned} a : \quad \Phi \times V \times V &\longrightarrow \mathbb{R} \\ (\tau, u, v) &\longmapsto a(\tau; u, v), \\ \\ \ell : \quad \Phi \times V &\longrightarrow \mathbb{R} \\ (\tau, v) &\longmapsto \ell(\tau; v), \\ \\ J : \quad \Phi \times V &\longrightarrow \mathbb{R} \\ (\tau, v) &\longmapsto J(\tau; v). \end{aligned}$$

For each τ , $a(\tau; \cdot, \cdot)$ is supposed to be bilinear, continuous, symmetric and coercive in u and v and $\ell(\tau; \cdot)$ is supposed to be linear and continuous in v . Both are supposed to be of class \mathcal{C}^1 with respect to τ in the spaces of continuous bilinear functionals and continuous linear functionals, respectively. As for J , it is supposed to be of class \mathcal{C}^1 with respect to the pair (τ, v) .

By application of Lax-Milgram's Lemma, the hypotheses imposed in functionals a and ℓ guarantee that, for a fixed τ , the state equation

$$a(\tau; u^\tau, v) = \ell(\tau; v), \forall v \in V$$

possesses a unique solution. Therefore, we may define the functional

$$\begin{aligned} j : \quad \Phi &\longrightarrow \mathbb{R} \\ \tau &\longmapsto j(\tau) = J(\tau; u^\tau). \end{aligned}$$

The problem we are studying is formulated as

$$\begin{aligned} \min_{\tau} \quad &j(\tau) \\ \text{st :} \quad &a(\tau; u^\tau, v) = \ell(\tau; v), \forall v \in V, \\ &\tau \in \Phi. \end{aligned}$$

In order to determine the necessary stationarity conditions, we need to differentiate $j(\tau)$ with respect to τ . We have the classical result ([?]):

THEOREM 1.1. *Under the above conditions, the functions*

$$\begin{aligned} u : \quad \Phi &\longrightarrow V & j : \quad \Phi &\longrightarrow \mathbb{R} \\ \tau &\longmapsto u^\tau, & \tau &\longmapsto j(\tau) \end{aligned}$$

are of class \mathcal{C}^1 . Moreover, we have

$$\frac{dj}{d\tau}(\tau) \cdot \delta\tau = \frac{\partial J}{\partial \tau}(\tau; u^\tau) \cdot \delta\tau - \frac{\partial a}{\partial \tau}(\tau; u^\tau, p^\tau) \cdot \delta\tau - \frac{\partial \ell}{\partial \tau}(\tau; p^\tau) \cdot \delta\tau, \forall \delta\tau \in \Lambda,$$

where $p^\tau \in V$ is the adjoint state variable, which is given as the unique solution of the equation

$$a(\tau; w, p^\tau) = \frac{\partial J}{\partial v}(\tau; u^\tau) \cdot w, \forall w \in V.$$

□

This result can be generalized to the case where there is more than one control (design) variable.

Although this abstract setting is capable of describing a large variety of problems in one dimension (*eg.*, extension and bending of a rod), in two dimensions (*eg.*, heat transfer and torsion of a bar) and in three dimensions (elasticity), in this work and without any loss of generality we address the most complex problems (elasticity case), but in order to keep the analytical solutions to a reasonable size we restrict ourselves to the two dimensional plane stress case.

In the next section we state the problem we wish to address together with some classical results on the subject. In section 3 we prove the existence of a solution to the problem together with some fixed point results that will be useful in the numerical applications. In section 4 we describe the numerical procedure employed with the adaptations introduced. Finally, in section 5, we present several examples comparing the analytical and the numerical solutions.

2. Problem definition. During all this paper, we will use the usual notation in Elasticity theory, where latin indices take the values 1, 2 and 3 while greek indices take the values 1 and 2. The summation convention on repeated indices will also be assumed.

Let us consider a solid occupying volume Ω , an open bounded simply-connected subset of \mathbb{R}^n , $n = 2, 3$, with surface $\partial\Omega$. Moreover, consider that the body is fixed in a part of its surface, Γ_0 , and that we have $\partial\Omega = \overline{\Gamma_0} \cup \overline{\Gamma_1}$, $\Gamma_0 \cap \Gamma_1 = \emptyset$. Let $f = (f_i)$ and $g = (g_i)$, denote the force per unit volume and the force per unit surface area applied to the body, respectively. Denoting the independent variable by $x = (x_i)$, the displacement field by $u = (u_i)$, the strain tensor by $e = (e_{ij})$, where $e_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$, and the stress tensor by $\sigma = (\sigma_{ij})$, where $\sigma_{ij} = E_{ijkl}e_{kl}$, E_{ijkl} being the elasticity coefficients, the elasticity boundary value problem can be written as

$$\begin{aligned} -\frac{\partial}{\partial x_j} \left(E_{ijkl} e_{kl}(u) \right) &= f_i && \text{in } \Omega, \\ u_i &= 0 && \text{on } \Gamma_0, \\ \sigma_{ij} n_j &= g_i && \text{on } \Gamma_1, \end{aligned}$$

where $n = (n_i)$ is the outward-pointing normal to Γ_1 .

From now on, we will only consider the two-dimensional case. It should be noted, however, that everything that will follow has an immediate generalization to the three-dimensional case.

Let us assume, now, that the material the solid is made of possess a laminated microstructure formed by two materials. These materials are supposed to be homogeneous and isotropic of Young's modulus E^+ and E^- and with specific mass ρ^+ and ρ^- , respectively, both with Poisson's ratio ν . We will assume that the materials are well ordered, that is, $E^+ > E^-$ and $\rho^+ > \rho^-$.

We will consider two kinds of microstructure: in the first one, the material represented by the pair (E^+, ρ^+) is vertically intercalated with the material represented by the pair (E^-, ρ^-) , in the proportions τ and $1 - \tau$, respectively, with $0 \leq \tau \leq 1$ — this is called the rank-1 microstructure (left-hand side of Figure ??); in the second one, the material represented by the pair (E^+, ρ^+) is again vertically intercalated with proportions τ and $1 - \tau$, but this time with a new material, that we identify by the pair (E^{H^-}, ρ^{H^-}) ; this new material is a rank-1, formed once again with the two base materials with proportions μ and $1 - \mu$ ($0 \leq \mu \leq 1$), being the two

scales of layers orthogonal — this is called the rank–2 microstructure (right-hand side of Figure ??).

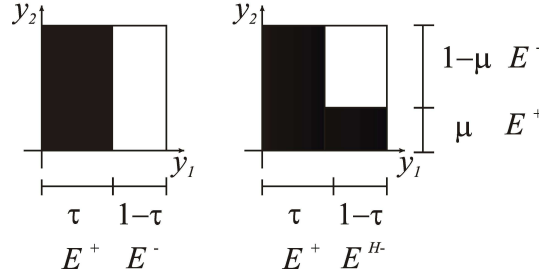


FIGURE 1. rank–1 and rank–2 layered microstructure, where the system of axes Oy_1y_2 represents the microscopic level

If we apply the Homogenization Theory ([?]), the non-null homogenized elasticity coefficients have the following expressions:

$$\begin{aligned}
 \bar{E}_{1111}^H &= \frac{E_{1111}^+ E_{1111}^-}{\tau E_{1111}^- + (1 - \tau) E_{1111}^+}, \\
 \bar{E}_{1122}^H &= \left(\tau \frac{E_{1122}^+}{E_{1111}^+} + (1 - \tau) \frac{E_{1122}^-}{E_{1111}^-} \right) \frac{E_{1111}^+ E_{1111}^-}{\tau E_{1111}^- + (1 - \tau) E_{1111}^+}, \\
 \bar{E}_{2222}^H &= \tau E_{2222}^+ + (1 - \tau) E_{2222}^- - \left(\tau \frac{(E_{1122}^+)^2}{E_{1111}^+} + (1 - \tau) \frac{(E_{1122}^-)^2}{E_{1111}^-} \right) + \\
 &\quad + \left(\tau \frac{E_{1122}^+}{E_{1111}^+} + (1 - \tau) \frac{E_{1122}^-}{E_{1111}^-} \right)^2 \frac{E_{1111}^+ E_{1111}^-}{\tau E_{1111}^- + (1 - \tau) E_{1111}^+}, \\
 \bar{E}_{1212}^H &= \frac{E_{1212}^+ E_{1212}^-}{\tau E_{1212}^- + (1 - \tau) E_{1212}^+},
 \end{aligned} \tag{1}$$

where, in plane stress, we have for the non-null coefficients $E_{\alpha\beta\gamma\delta}^+$ (both for rank–1 and rank–2 microstructures):

$$E_{1111}^+ = \frac{E^+}{1 - \nu^2}, \quad E_{1122}^+ = \frac{\nu E^+}{1 - \nu^2}, \quad E_{2222}^+ = \frac{E^+}{1 - \nu^2}, \quad E_{1212}^+ = \frac{E^+}{2(1 + \nu)}, \tag{2}$$

and for the non-null coefficients $E_{\alpha\beta\gamma\delta}^-$ the expressions:

$$E_{1111}^- = \frac{E^-}{1 - \nu^2}, \quad E_{1122}^- = \frac{\nu E^-}{1 - \nu^2}, \quad E_{2222}^- = \frac{E^-}{1 - \nu^2}, \quad E_{1212}^- = \frac{E^-}{2(1 + \nu)}, \tag{3}$$

for a rank–1 microstructure and

$$E_{1111}^- = I_2 + \frac{\nu^2 I_1}{1 - \nu^2}, \quad E_{1122}^- = \frac{\nu I_1}{1 - \nu^2}, \quad E_{2222}^- = \frac{I_1}{1 - \nu^2}, \quad E_{1212}^- = \frac{I_1}{2(1 + \nu)}, \tag{4}$$

for a rank-2 microstructure, and where

$$I_1 = \frac{E^+ E^-}{\mu E^- + (1 - \mu) E^+} \text{ and } I_2 = \mu E^+ + (1 - \mu) E^-.$$

If we consider an angle θ made by the microstructure reference axes Oy_1y_2 with respect to the macroscopic axes Ox_1x_2 , the homogenized elasticity coefficients are given by ([?])

$$E_{\alpha\beta\gamma\delta}^H(\tau, \mu, \theta) = \bar{E}_{\varepsilon\zeta\eta\xi}^H(\tau, \mu) R_{\alpha\varepsilon} R_{\beta\zeta} R_{\gamma\eta} R_{\delta\xi}$$

where

$$R = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.$$

The homogenized specific mass is given by

$$\rho^H(\tau) = \tau \rho^+ + (1 - \tau) \rho^- \quad (5)$$

if a rank-1 microstructure is considered, or by

$$\rho^H(\tau, \mu) = \rho^+(\tau + (1 - \tau)\mu) + \rho^-(1 - \tau)(1 - \mu) \quad (6)$$

if rank-2 microstructure is considered.

If instead of plane stress, we have plane deformation, we just have to alter the expressions (??), (??) and (??), but qualitatively there is no difference between these two situations.

The problem we are addressing considers as objective function the work of the applied forces plus a term which penalizes the stiffest material, supposed to be more expensive than the weakest material. If we consider a rank-2 microstructure with rotation, we want to determine the functions τ , μ and θ (the design variables), which minimize the objective function, subject to the equilibrium equation and to the lower and upper bounds in τ and μ . If we write the equilibrium equation in its variational form, the problem can be stated as

$$\begin{aligned} \min_{\tau, \mu, \theta} \quad & j(\tau, \mu, \theta) \\ \text{st :} \quad & a(\tau, \mu, \theta; u^{\tau\mu\theta}, v) = \ell(\tau, \mu, \theta; v), \forall v \in V, \\ & 0 \leq \tau \leq 1, \\ & 0 \leq \mu \leq 1, \end{aligned}$$

where

$$j(\tau, \mu, \theta) = \int_{\Omega} f_{\alpha} u_{\alpha}^{\tau\mu\theta} dx + \int_{\Gamma_1} g_{\alpha} u_{\alpha}^{\tau\mu\theta} ds + k \int_{\Omega} \rho^H(\tau, \mu) dx,$$

$$a(\tau, \mu, \theta; u, v) = \int_{\Omega} E_{\alpha\beta\gamma\delta}^H(\tau, \mu, \theta) e_{\gamma\delta}(u) e_{\alpha\beta}(v) dx,$$

$$\ell(\tau, \mu, \theta; v) = \int_{\Omega} f_{\alpha} v_{\alpha} dx + \int_{\Gamma_1} g_{\alpha} v_{\alpha} ds,$$

$$V = \{v \in [H^1(\Omega)]^2 : v_{\alpha} = 0 \text{ on } \Gamma_0\},$$

where constant k represents the work done in order to add to the solid a unit of mass and V denotes the space of kinematically admissible displacement fields.

In order to solve the problem under consideration, and for a rank-2 microstructure with rotation, we construct the Lagrangian

$$\begin{aligned} \mathcal{L} = & j(\tau, \mu, \theta) + \lambda \left(a(\tau, \mu, \theta; u^{\tau\mu\theta}, v) - \ell(\tau, \mu, \theta; v) \right) + \\ & + \int_{\Omega} \tau^+ (\tau - 1) dx - \int_{\Omega} \tau^- \tau dx + \int_{\Omega} \mu^+ (\mu - 1) dx - \int_{\Omega} \mu^- \mu dx, \end{aligned} \tag{7}$$

where τ^+ , τ^- , μ^+ , μ^- and λ are the Lagrange multipliers associated to constraints $\tau \leq 1$, $\tau \geq 0$, $\mu \leq 1$, $\mu \geq 0$ and the equilibrium equation, respectively. From the necessary conditions of stationarity one obtains, in Ω :

$$v = \frac{1}{\lambda} u^{\tau\mu\theta}, \tag{8}$$

$$a(\tau, \mu, \theta; u^{\tau\mu\theta}, v) = \ell(\tau, \mu, \theta; v), \tag{9}$$

$$\tau^+ \geq 0, \tau^- \geq 0, \tau^+(\tau - 1) = 0, \tau^- \tau = 0, \tag{10}$$

$$\mu^+ \geq 0, \mu^- \geq 0, \mu^+(\mu - 1) = 0, \mu^- \mu = 0, \tag{11}$$

$$k \frac{\partial \rho^H}{\partial \tau} - \frac{\partial E_{\alpha\beta\gamma\delta}^H}{\partial \tau} e_{\gamma\delta}(u^{\tau\mu\theta}) e_{\alpha\beta}(u^{\tau\mu\theta}) + \tau^+ - \tau^- = 0, \tag{12}$$

$$k \frac{\partial \rho^H}{\partial \mu} - \frac{\partial E_{\alpha\beta\gamma\delta}^H}{\partial \mu} e_{\gamma\delta}(u^{\tau\mu\theta}) e_{\alpha\beta}(u^{\tau\mu\theta}) + \mu^+ - \mu^- = 0, \tag{13}$$

$$\frac{\partial E_{\alpha\beta\gamma\delta}^H}{\partial \theta} e_{\gamma\delta}(u^{\tau\mu\theta}) e_{\alpha\beta}(u^{\tau\mu\theta}) = 0, \tag{14}$$

where the last three equations are the result of applying a generalized version of Theorem 1.1.

Now, let

$$\chi_{\tau} = k \frac{\partial \rho^H}{\partial \tau} - \frac{\partial E_{\alpha\beta\gamma\delta}^H}{\partial \tau} e_{\gamma\delta}(u^{\tau\mu\theta}) e_{\alpha\beta}(u^{\tau\mu\theta}), \tag{15}$$

$$\chi_{\mu} = k \frac{\partial \rho^H}{\partial \mu} - \frac{\partial E_{\alpha\beta\gamma\delta}^H}{\partial \mu} e_{\gamma\delta}(u^{\tau\mu\theta}) e_{\alpha\beta}(u^{\tau\mu\theta}). \tag{16}$$

Then, equations (??) and (??) can be rewritten as

$$\chi_{\tau} + \tau^+ - \tau^- = 0 \text{ and } \chi_{\mu} + \mu^+ - \mu^- = 0. \tag{17}$$

Therefore, considering (??), from the first equation of (??), we have

$$\begin{cases} \chi_{\tau} < 0 & \Rightarrow \tau^+ > 0, \tau^- = 0 & \Rightarrow \tau = 1 \\ \chi_{\tau} = 0 & \Rightarrow \tau^+ = 0, \tau^- = 0 & \Rightarrow \tau \in [0, 1] \\ \chi_{\tau} > 0 & \Rightarrow \tau^+ = 0, \tau^- > 0 & \Rightarrow \tau = 0, \end{cases} \tag{18}$$

where the same reasoning can be applied to χ_μ . So, if the cost k is sufficiently low we just have the stiffest material; if the cost is too high, we just have the weakest material; for intermediate values of k , we have a real laminate with a mixture of the two base materials.

Following [?], if α denotes the angle formed by the principal strain axes with the macroscopic system of axes Ox_1x_2 and ψ the angle of rotation of the material frame Oy_1y_2 with respect to the principal strain axes (we denote by e_I and e_{II} the principal strains), that is if $\theta = \alpha + \psi$, then, from (??), it can be shown that, if $e_I = e_{II}$, then ψ can take any value (in this case, we consider $\theta = 0$); if $e_I \neq e_{II}$, then

$$\sin(2\psi) = 0 \text{ or } \cos(2\psi) = \frac{e_I + e_{II}}{e_I - e_{II}}, \tag{19}$$

and one should choose the value that maximizes the strain energy

$$\frac{1}{2} E_{\alpha\beta\gamma\delta}^H e_{\alpha\beta} e_{\gamma\delta}. \tag{20}$$

3. Analytical formulation. We shall now address the question of existence of a solution to the problem under consideration. Simultaneously, we shall also prepare the iterative algorithm to be used later on.

In order to simplify the notation, the proof will be done only for the case of a single design variable, namely, τ . Thus, we specifically have the Hilbert space $V = \{v \in [H^1(\Omega)]^2 : v_\alpha = 0 \text{ on } \Gamma_0\}$, the Banach space $\Lambda = L^\infty(\Omega)$ and the subset $\Phi = L^\infty(\Omega;]0, 1[)$, together with the following functionals

$$\begin{aligned} a : \quad \Phi \times V \times V &\longrightarrow \mathbb{R} \\ (\tau, u, v) &\longmapsto a(\tau; u, v) = \int_{\Omega} E_{\alpha\beta\gamma\delta}^H(\tau) e_{\gamma\delta}(u) e_{\alpha\beta}(v) \, dx, \\ \ell : \quad V &\longrightarrow \mathbb{R} \\ v &\longmapsto \ell(v) = \int_{\Omega} f_\alpha v_\alpha \, dx + \int_{\Gamma_1} g_\alpha v_\alpha \, ds, \end{aligned}$$

where $f_\alpha, g_\alpha \in L^2(\Omega)$ and $E_{\alpha\beta\gamma\delta}^H \in L^\infty(\Omega)$, whose expressions were given in the previous section.

The fact that for a fixed control τ the elasticity problem possesses a unique solution is stated in the following result:

LEMMA 3.1. *Let $\tau \in \Phi$ be given. Then, $\exists^1 u^\tau \in V : a(\tau; u^\tau, v) = \ell(v), \forall v \in V$.*

Proof.

The proof relies on standard results in elasticity theory. In fact for a given $\tau \in \Phi$, functional $a(\tau; \cdot, \cdot)$ is bilinear, continuous and coercive, due to Korn's inequality and the fact that the homogenized elasticity coefficients $E_{\alpha\beta\gamma\delta}^H(\tau)$, as given by (??), (??) and (??), are strictly increasing functions of τ . On the other hand, functional $\ell(\cdot)$ is linear and continuous and the conclusion now follows from Lax-Milgram's Lemma. □

Let us now consider the functionals,

$$\begin{aligned}
 J : \Phi \times V &\longrightarrow \mathbb{R} \\
 (\tau, v) &\longmapsto J(\tau; v) = \ell(v) + k \int_{\Omega} \rho^H(\tau) \, dx, \\
 \\
 j : \Phi &\longrightarrow \mathbb{R} \\
 \tau &\longmapsto j(\tau) = J(\tau; u^\tau),
 \end{aligned}$$

where $k > 0$ represents the cost of the work done in order to add a unit of mass to the system, $\rho^H \in L^\infty(\Omega)$, whose expression was given in the previous section, and u^τ stands for the unique solution of the elasticity problem for a fixed τ . Thus the problem we wish to address is the following:

$$\begin{aligned}
 \min_{\tau} \quad & j(\tau) \\
 \text{st :} \quad & a(\tau; u^\tau, v) = \ell(v), \forall v \in V, \\
 & \tau \in \Phi.
 \end{aligned}$$

In order to prove the existence of a solution for this problem we start by establishing the following result:

LEMMA 3.2. *Let $\tau_n, \tau_0 \in \Phi$, $\tau_0 \geq \tau_n$ ae. in Ω . Then, $\ell(u^{\tau_n}) - \ell(u^{\tau_0}) \geq 0$.*

Proof.

Since the homogenized coefficients, $E_{\alpha\beta\gamma\delta}^H(\tau)$, are positive and strictly increasing functions of its argument, one has:

$$\exists c > 0 : \tau_2 \geq \tau_1 \Rightarrow (E_{\alpha\beta\gamma\delta}^H(\tau_2) - E_{\alpha\beta\gamma\delta}^H(\tau_1)) \geq c(\tau_2 - \tau_1).$$

So, we have:

$$\begin{aligned}
 \ell(u^{\tau_n}) - \ell(u^{\tau_0}) &= \ell(u^{\tau_n}) - 2\ell(u^{\tau_0}) + \ell(u^{\tau_0}) + a(\tau_n; u^{\tau_0}, u^{\tau_0}) - a(\tau_n; u^{\tau_0}, u^{\tau_0}) \\
 &= a(\tau_n; u^{\tau_n}, u^{\tau_n}) - a(\tau_n; u^{\tau_n}, u^{\tau_0}) - a(\tau_n; u^{\tau_0}, u^{\tau_n}) + \\
 &\quad + a(\tau_0; u^{\tau_0}, u^{\tau_0}) + a(\tau_n; u^{\tau_0}, u^{\tau_0}) - a(\tau_n; u^{\tau_0}, u^{\tau_0}) \\
 &= a(\tau_n; u^{\tau_n} - u^{\tau_0}, u^{\tau_n} - u^{\tau_0}) + a(\tau_0; u^{\tau_0}, u^{\tau_0}) - a(\tau_n; u^{\tau_0}, u^{\tau_0}) \\
 &\geq a(\tau_0; u^{\tau_0}, u^{\tau_0}) - a(\tau_n; u^{\tau_0}, u^{\tau_0}) \\
 &\geq \int_{\Omega} c(\tau_0 - \tau_n) \sum_{\alpha, \beta, \gamma, \delta} e_{\gamma\delta}(u^{\tau_0}) e_{\alpha\beta}(u^{\tau_0}) \, dx \\
 &= \int_{\Omega} c(\tau_0 - \tau_n) \left(\sum_{\alpha, \beta} e_{\alpha\beta}(u^{\tau_0}) \right)^2 \, dx \\
 &\geq 0.
 \end{aligned}$$

□

We are now in a position of proving a lower semicontinuity type result for functional $j(\cdot)$.

LEMMA 3.3. Let $\tau_0 \in \Phi$, $(\tau_n) \subset \Phi$, $\tau_0 \geq \tau_n$ ae. in Ω , such that $\tau_n \xrightarrow{w^*} \tau_0$. Then, $\liminf_n j(\tau_n) \geq j(\tau_0)$.

Proof.

From Lemma ?? and the fact that from the definition of the homogenized density $\rho^H(\tau)$, defined by (??), one has $\int_{\Omega} \rho^H(\tau_n) dx \rightarrow \int_{\Omega} \rho^H(\tau_0) dx$, we have:

$$\begin{aligned} \liminf_n j(\tau_n) &= j(\tau_0) + \liminf_n \left(j(\tau_n) - j(\tau_0) \right) \\ &= j(\tau_0) + \liminf_n \left(\ell(u^{\tau_n}) - \ell(u^{\tau_0}) \right) + \\ &\quad + k \liminf_n \int_{\Omega} \left(\rho^H(\tau_n) - \rho^H(\tau_0) \right) dx \\ &\geq j(\tau_0). \end{aligned}$$

□

From these preliminary results we now deduce the existence of a solution for the problem under study.

THEOREM 3.1. Under the previous conditions there exists $\tau_0 \in \Phi$ such that $j(\tau_0) = \min_{\tau \in \Phi} j(\tau)$.

Proof.

Let $M \geq 0$, be such that $M = \inf_{\tau \in \Phi} j(\tau)$, which always exists since $j(\tau) \geq 0$, $\forall \tau \in \Phi$. Then, there exists a subsequence, $(\bar{\tau}_n) \subset \Phi$ such that $j(\bar{\tau}_n) \rightarrow M$. But since $(\bar{\tau}_n)$ is bounded in $L^\infty(\Omega)$, by Alaoglu's theorem, there exists a subsequence $\bar{\tau}_{n'} \prec \bar{\tau}_n$ and an element $\tau_0 \in \Phi$ such that $\bar{\tau}_{n'} \xrightarrow{w^*} \tau_0$.

Since this subsequence $(\bar{\tau}_{n'})$ is bounded in Ω we consider the following:

- i) If $(\bar{\tau}_{n'})$ is increasing we define $\tau_n = \bar{\tau}_{n'}$,
- ii) If $(\bar{\tau}_{n'})$ admits an increasing subsequence $(\bar{\tau}_{n''})$, we define $\tau_n = \bar{\tau}_{n''}$,
- iii) If $(\bar{\tau}_{n'})$ does not admit an increasing subsequence, we take a decreasing subsequence $(\bar{\tau}_{n''})$, which in this case always exists. Since this subsequence is also bounded we take $\tau_n = \lim_{n''} \bar{\tau}_{n''}$.

In this way we have an increasing sequence $(\tau_n) \subset \Phi$, such that $\tau_n \xrightarrow{w^*} \tau_0$ and that $j(\tau_n) \rightarrow M$. Moreover, $\liminf_n j(\tau_n) \geq j(\tau_0)$, by Lemma ??. Thus $M \geq j(\tau_0)$, but since $M = \inf_{\tau \in \Phi} j(\tau)$ one must have $j(\tau_0) = M$, that is, the infimum is attained.

□

We shall now give a different characterization of the minimum of the problem under study in such a way that is suitable to apply a numerical algorithm of the type described in [?].

THEOREM 3.2. *Let*

$$\mathcal{F}(\tau) = \begin{cases} 1 & \text{if } \bar{\chi}_\tau(\tau) > 1, \\ \tau & \text{if } \bar{\chi}_\tau(\tau) = 0, \\ 0 & \text{if } \bar{\chi}_\tau(\tau) < 1 \end{cases} \quad \text{with} \quad \bar{\chi}_\tau(\tau) = \frac{1}{k \frac{d\rho^H}{d\tau}(\tau)} \frac{dE_{\alpha\beta\gamma\delta}^H}{d\tau}(\tau) e_{\gamma\delta}(u^\tau) e_{\alpha\beta}(u^\tau).$$

Then, $j(\tau_0) = \min_{\tau \in \Phi} j(\tau) \Rightarrow \tau_0 = \mathcal{F}(\tau_0)$ in Ω .

Proof.

Let $\tau_0 \in \Phi$ be such that $j(\tau_0) = \min_{\tau \in \Phi} j(\tau)$. Then,

$$\frac{dj}{d\tau}(\tau_0) \cdot (\tau - \tau_0) \geq 0, \forall \tau \in \Phi.$$

Applying Theorem 1.1 one obtains:

$$\frac{dj}{d\tau}(\tau) \cdot \delta_\tau = \int_\Omega \left(k \frac{d\rho^H}{d\tau}(\tau) - \frac{dE_{\alpha\beta\gamma\delta}^H}{d\tau}(\tau) e_{\gamma\delta}(u^\tau) e_{\alpha\beta}(u^\tau) \right) \cdot \delta_\tau dx, \forall \delta_\tau \in \Phi,$$

from which one gets:

$$\chi_\tau(\tau_0) \cdot (\tau - \tau_0) \geq 0, \forall \tau \in \Phi, \text{ with } \chi_\tau(\tau_0) = k \frac{d\rho^H}{d\tau}(\tau_0) - \frac{dE_{\alpha\beta\gamma\delta}^H}{d\tau}(\tau_0) e_{\gamma\delta}(u^{\tau_0}) e_{\alpha\beta}(u^{\tau_0}),$$

which implies that

- i) $\bar{\chi}_\tau(\tau_0) > 1 \Leftrightarrow \chi_\tau(\tau_0) < 0 \Rightarrow (\tau - \tau_0) \leq 0, \tau \in \Phi \Rightarrow \tau_0 = 1;$
- ii) $\bar{\chi}_\tau(\tau_0) = 1 \Leftrightarrow \chi_\tau(\tau_0) = 0 \Rightarrow (\tau - \tau_0) \in \Phi \Rightarrow \tau_0 \in \Phi;$
- iii) $\bar{\chi}_\tau(\tau_0) < 1 \Leftrightarrow \chi_\tau(\tau_0) > 0 \Rightarrow (\tau - \tau_0) \geq 0, \tau \in \Phi \Rightarrow \tau_0 = 0,$

that is to say,

$$\tau_0 = \begin{cases} 1 & \text{if } \bar{\chi}_\tau(\tau_0) > 1, \\ \tau_0 & \text{if } \bar{\chi}_\tau(\tau_0) = 1, \\ 0 & \text{if } \bar{\chi}_\tau(\tau_0) < 1 \end{cases} \Leftrightarrow \tau_0 = \mathcal{F}(\tau_0),$$

where we have chosen τ_0 whenever $\bar{\chi}_\tau(\tau_0) = 1$. □

From the numerical point of view it is more suitable to use the algorithm described in the next result.

THEOREM 3.3. *Let*

$$\mathcal{F}(\tau) = \begin{cases} 1 & \text{se } \bar{\chi}_\tau(\tau) > 1, \\ \tau & \text{se } \bar{\chi}_\tau(\tau) = 1, \\ 0 & \text{se } \bar{\chi}_\tau(\tau) < 1, \end{cases}$$

$$\mathcal{G}(\tau) = \begin{cases} \min\{(1 + \xi)\xi_0, 1\} & \text{if } \min\{(1 + \xi)\tau, 1\} < \tau \bar{\chi}_\tau(\tau) \text{ and } \tau = 0, \\ \min\{(1 + \xi)\tau, 1\} & \text{if } \min\{(1 + \xi)\tau, 1\} < \tau \bar{\chi}_\tau(\tau) \text{ and } \tau \neq 0, \\ \tau \bar{\chi}_\tau(\tau) & \text{if } (1 - \xi)\tau \leq \tau \bar{\chi}_\tau(\tau) \leq \min\{(1 + \xi)\tau, 1\}, \\ (1 - \xi)\tau & \text{if } \tau \bar{\chi}_\tau(\tau) < (1 - \xi)\tau, \end{cases}$$

where $\xi, \xi_0 \in (0, 1)$ and $\bar{\chi}_\tau : \Phi \rightarrow L^\infty(\Omega)$. Then, for all $\tau_0 \in \Phi$, $\tau_0 = \mathcal{F}(\tau_0)$ in $\Omega \Leftrightarrow \tau_0 = \mathcal{G}(\tau_0)$ in Ω .

Proof.

- (\Rightarrow) i) first branch of \mathcal{F} : let $\bar{\chi}_\tau(\tau_0) > 1$; then, $\tau_0 = \mathcal{F}(\tau_0) = 1$; looking at function \mathcal{G} , one has:
- first branch of \mathcal{G} : impossible condition since $\tau_0 \neq 0$;
 - second branch of \mathcal{G} : since $\tau_0 \neq 0$ and $\min\{(1 + \xi)\tau_0, 1\} < \tau_0\bar{\chi}_\tau(\tau_0) \Leftrightarrow 1 < \bar{\chi}_\tau(1)$, one has $\mathcal{G}(\tau_0) = \min\{(1 + \xi)\tau_0, 1\} = 1$;
 - third branch of \mathcal{G} : impossible condition since $(1 - \xi)\tau_0 \leq \tau_0\bar{\chi}_\tau(\tau_0) \leq \min\{(1 + \xi)\tau_0, 1\} \Leftrightarrow (1 - \xi) \leq \bar{\chi}_\tau(1) \leq 1$;
 - fourth branch of \mathcal{G} : impossible condition since $\tau_0\bar{\chi}_\tau(\tau_0) < (1 - \xi)\tau_0 \Leftrightarrow \bar{\chi}_\tau(1) < 1$;
- ii) second branch of \mathcal{F} : let $\bar{\chi}_\tau(\tau_0) = 1$; then, $\mathcal{F}(\tau_0) = \tau_0$; looking at function \mathcal{G} , one has:
- first branch of \mathcal{G} : impossible condition since $(\min\{(1 + \xi)\tau_0, 1\} < \tau_0\bar{\chi}_\tau(\tau_0) \text{ and } \tau_0 = 0) \Leftrightarrow 0 < 0$;
 - second branch of \mathcal{G} : impossible condition since
 - if $(1 + \xi)\tau_0 < 1$, then $(\min\{(1 + \xi)\tau_0, 1\} < \tau_0\bar{\chi}_\tau(\tau_0) \text{ and } \tau_0 \neq 0) \Leftrightarrow (\xi\tau_0 < 0 \text{ and } \tau_0 \neq 0)$;
 - if $1 \leq (1 + \xi)\tau_0$, then $(\min\{(1 + \xi)\tau_0, 1\} < \tau_0\bar{\chi}_\tau(\tau_0) \text{ and } \tau_0 \neq 0) \Leftrightarrow 1 < \tau_0$;
 - third branch of \mathcal{G} : since
 - if $(1 + \xi)\tau_0 < 1$, then $(1 - \xi)\tau_0 \leq \tau_0\bar{\chi}_\tau(\tau_0) \leq \min\{(1 + \xi)\tau_0, 1\} \Leftrightarrow (1 - \xi)\tau_0 \leq \tau_0 \leq (1 + \xi)\tau_0 \Leftrightarrow 0 < \tau_0 < 1$;
 - if $1 \leq (1 + \xi)\tau_0$, then $(1 - \xi)\tau_0 \leq \tau_0\bar{\chi}_\tau(\tau_0) \leq \min\{(1 + \xi)\tau_0, 1\} \Leftrightarrow (1 - \xi)\tau_0 \leq \tau_0 \leq 1 \Leftrightarrow 0 < \tau_0 \leq 1$;
 one obtains $\mathcal{G}(\tau_0) = \tau_0$;
 - fourth branch of \mathcal{G} : impossible condition since $\tau_0\bar{\chi}_\tau(\tau_0) < (1 - \xi)\tau_0 \Leftrightarrow \xi\tau_0 < 0$;
- iii) third branch of \mathcal{F} : let $\bar{\chi}_\tau(\tau_0) < 1$; then, $\tau_0 = \mathcal{F}(\tau_0) = 0$; looking at function \mathcal{G} , one has:
- first branch of \mathcal{G} : impossible condition since $\min\{(1 + \xi)\tau_0, 1\} < \tau_0\bar{\chi}_\tau(\tau_0) \text{ and } \tau_0 = 0 \Leftrightarrow 0 < 0$;
 - second branch of \mathcal{G} : impossible condition since $\tau_0 = 0$;
 - third branch of \mathcal{G} : since $(1 - \xi)\tau_0 \leq \tau_0\bar{\chi}_\tau(\tau_0) \leq \min\{(1 + \xi)\tau_0, 1\} \Leftrightarrow 0 \leq 0 \leq 0$, one obtains $\mathcal{G}(\tau_0) = \tau_0\bar{\chi}_\tau(\tau_0) = 0$;
 - fourth branch of \mathcal{G} : impossible condition since $\tau_0\bar{\chi}_\tau(\tau_0) < (1 - \xi)\tau_0 \Leftrightarrow 0 < 0$;

thus, we have, $\tau_0 = \mathcal{G}(\tau_0)$;

- (\Leftarrow) i) first branch of \mathcal{G} : impossible condition since $(\min\{(1 + \xi)\tau_0, 1\} < \tau_0\bar{\chi}_\tau(\tau_0) \text{ and } \tau_0 = 0) \Leftrightarrow 0 < 0$;
- ii) second branch of \mathcal{G} :
- if $(1 + \xi)\tau_0 < 1$, then this is impossible since $((\min\{(1 + \xi)\tau_0, 1\} < \tau_0\bar{\chi}_\tau(\tau_0) \text{ and } \tau_0 \neq 0) \Rightarrow \tau_0 = \min\{(1 + \xi)\tau_0, 1\}) \Leftrightarrow (((1 + \xi)\tau_0 < \tau_0\bar{\chi}_\tau(\tau_0) \text{ and } \tau_0 \neq 0) \Rightarrow \tau_0 = 0)$;
 - if $1 \leq (1 + \xi)\tau_0$, then, since $((\min\{(1 + \xi)\tau_0, 1\} < \tau_0\bar{\chi}_\tau(\tau_0) \text{ and } \tau_0 \neq 0) \Rightarrow \tau_0 = \min\{(1 + \xi)\tau_0, 1\}) \Leftrightarrow ((1 < \tau_0\bar{\chi}_\tau(\tau_0) \text{ and } \tau_0 \neq 0) \Rightarrow \tau_0 = 1) \Leftrightarrow \bar{\chi}_\tau(\tau_0) > 1$, one has $\mathcal{F}(\tau_0) = 1$;
- iii) third branch of \mathcal{G} :

- if $(1 + \xi)\tau_0 < 1$, then, since $((1 - \xi)\tau_0 \leq \tau_0\bar{\chi}_\tau(\tau_0) \leq \min\{(1 + \xi)\tau_0, 1\} \Rightarrow \tau_0 = \tau_0\bar{\chi}_\tau(\tau_0)) \Leftrightarrow ((1 - \xi)\tau_0 \leq \tau_0\bar{\chi}_\tau(\tau_0) \leq (1 + \xi)\tau_0 \Rightarrow \tau_0 = \tau_0\bar{\chi}_\tau(\tau_0)) \Leftrightarrow \bar{\chi}_\tau(\tau_0) = 1$, one has $\mathcal{F}(\tau_0) = \tau_0$;
 - if $1 \leq (1 + \xi)\tau_0$, then, since $((1 - \xi)\tau_0 \leq \tau_0\bar{\chi}_\tau(\tau_0) \leq \min\{(1 + \xi)\tau_0, 1\} \Rightarrow \tau_0 = \tau_0\bar{\chi}_\tau(\tau_0)) \Leftrightarrow ((1 - \xi)\tau_0 \leq \tau_0\bar{\chi}_\tau(\tau_0) \leq 1 \Rightarrow \tau_0 = \tau_0\bar{\chi}_\tau(\tau_0)) \Leftrightarrow \bar{\chi}_\tau(\tau_0) = 1$, one has $\mathcal{F}(\tau_0) = \tau_0$;
- iv) fourth branch of \mathcal{G} : impossible condition since $(\tau_0\bar{\chi}_\tau(\tau_0) < (1 - \xi)\tau_0 \Rightarrow \tau_0 = (1 - \xi)\tau_0) \Leftrightarrow (\tau_0\bar{\chi}_\tau(\tau_0) < (1 - \xi)\tau_0 \Rightarrow \tau_0 = 0) \Leftrightarrow 0 < 0$;
- thus, $\tau_0 = \mathcal{F}(\tau_0)$;

□

4. Numerical solution. In order to solve the problem numerically, we use the finite element method to solve the boundary value problem. We start by discretizing the domain Ω in a finite element mesh, where we assume a constant value for the design variables in each finite element. As an iterative algorithm, an initial approximation for the design variables in each finite element should be given, after what we can compute the homogenized elastic properties. Then, we determine an approximation for the displacement field. A new approximation of the design variables is calculated in each element and a stopping criteria, based in the infinity norm of the difference in two consecutive iterations of the design variables, is tested. If the criteria is satisfied, the iterative process is stopped. Otherwise, the process restarts. Following [?] and considering Theorem ??, the next fixed-point update algorithm for $\tau_{e,p}$ — the value of τ in iteration p and at element e —, is proposed:

$$\tau_{e,p} = \begin{cases} \min\{(1 + \xi)\xi_0, 1\} & \text{if } A < \tau_{e,p-1}(\bar{\chi}_{e,p-1})^\eta \text{ and } \tau_{e,p-1} = 0, \\ A & \text{if } A < \tau_{e,p-1}(\bar{\chi}_{e,p-1})^\eta \text{ and } \tau_{e,p-1} \neq 0, \\ \tau_{e,p-1}(\bar{\chi}_{e,p-1})^\eta & \text{if } (1 - \xi)\tau_{e,p-1} \leq \tau_{e,p-1}(\bar{\chi}_{e,p-1})^\eta \leq A, \\ (1 - \xi)\tau_{e,p-1} & \text{if } \tau_{e,p-1}(\bar{\chi}_{e,p-1})^\eta < (1 - \xi)\tau_{e,p-1}, \end{cases}$$

where $A = \min\{(1 + \xi)\tau_{e,p-1}; 1\}$, η is a weighting factor, ξ is a move limit in order to control design changes between iterations and ξ_0 is a reference value (in our computations we consider $\eta = 0.8$, $\xi = 0.5$ and $\xi_0 = 0.5$). The variable μ in each finite element is updated in a similar manner and θ in each iteration and in each element is updated according to (??) and (??). In order to avoid checkerboard patterns, which are usual in this type of problems when four-node quadrilateral elements are involved, we use nine-node quadrilateral elements ([?]).

5. Examples. In this section we will present three examples, all in plane stress, where we tested different loading type conditions and geometries. For each example, we tested different values of k , as well as different types of the microstructure. We identify each tested case by a sequence formed of three parts, separated by a dot: the first one refers to the load conditions and the geometry; the second indicates the order in the sequence of the values of k ; the last one identifies the microstructure type: suffix 10 if rank-1 without rotation, suffix 11 if rank-1 with rotation, suffix 20 if rank-2 without rotation, suffix 21 if rank-2 with rotation. For each example, we also indicate the number of iterations the process took to reach convergence (n) and the value the objective function attained (j). In all the examples, the considered base materials have the following properties: $E^+ = 2$, $\rho^+ = 2$, $E^- = 1$, $\rho^- = 1$, $\nu = 0.25$, since the major role is played by the ratio between homologous quantities. However different values could have been considered.

5.1. **Example 1.** Consider a square plate with principal axes Ox_1 and Ox_2 subjected to uniform loads on its boundary — $\sigma_{11} = \bar{\sigma}_{11}$, $\sigma_{22} = \bar{\sigma}_{22}$ and $\sigma_{12} = \bar{\sigma}_{12}$ — in a plane stress state (Figure ??).

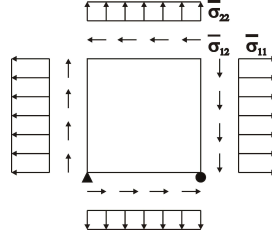


FIGURE 2. Example 1 — geometry and boundary conditions

For a homogeneous solution, if we consider a rank-1 layered microstructure without rotation and from the constitutive equations $\sigma_{\alpha\beta} = E_{\alpha\beta\gamma\delta}e_{\gamma\delta}$, the associated strain field is given by

$$e_{11}(u^\tau) = \bar{\sigma}_{11} \left(\frac{1-\nu^2}{I_1} + \frac{\nu^2}{I_2} \right) - \frac{\bar{\sigma}_{22}\nu}{I_2}, \quad e_{22}(u^\tau) = \frac{\bar{\sigma}_{22}}{I_2} - \frac{\bar{\sigma}_{11}\nu}{I_2}, \quad e_{12}(u^\tau) = -\frac{\bar{\sigma}_{12}(1+\nu)}{I_1},$$

where $I_1 = \frac{E^+E^-}{\tau E^- + (1-\tau)E^+}$ and $I_2 = \tau E^+ + (1-\tau)E^-$. So, we have

$$\chi_\tau = k(\rho^+ - \rho^-) - \frac{E^+ - E^-}{E^+E^-} \left((1-\nu^2)\bar{\sigma}_{11}^2 + E^+E^- \left(\frac{\bar{\sigma}_{22} - \nu\bar{\sigma}_{11}}{I_2} \right)^2 + 2(1+\nu)\bar{\sigma}_{12}^2 \right),$$

which will enable us to determine the analytical solution.

In this subsection we will present two cases corresponding to two different loading conditions. Tables 1 and 2 present the analytical solution for rank-1 microstructure without rotation (represented by $\bar{\tau}$). Tables 3 to 6 illustrate the corresponding numerical solutions. These are represented just by one element if the design variables take the same value in all elements, where we also indicate the respective value. Some examples where the numerical solution is not constant all over its domain, are detailed for finer meshes. The values of k were chosen taking into consideration the qualitative different parts of the presented analytical solution, with particular care to the transition points.

$$\bar{\tau}(x) = \begin{cases} 0 & \text{if } 0.484375 \leq k \\ \sqrt{\frac{2}{32k-15}} - 1 & \text{if } 0.53125 \leq k \leq 0.484375 \\ 1 & \text{if } k \leq 0.53125 \end{cases}$$

TABLE 1. Example 1-100: $\bar{\sigma}_{11} = 1, \bar{\sigma}_{22} = 0, \bar{\sigma}_{12} = 0$

$$\bar{\tau}(x) = \begin{cases} 0 & \text{if } 9.75 \leq k \\ \sqrt{\frac{4}{4k-35}} - 1 & \text{if } 9. \leq k \leq 9.75 \\ 1 & \text{if } k \leq 9. \end{cases}$$

TABLE 2. Example 1-421: $\bar{\sigma}_{11} = 4, \bar{\sigma}_{22} = 2, \bar{\sigma}_{12} = 1$

































ex1-100.1.10 $n = 13/j = 2.00$ $\tau = 0, \bar{\tau} = 0$ 	ex1-100.1.11 $n = 35/j = 2.00$ $\tau = 0.02/\theta = 90$ 	ex1-100.1.20 $n = 34/j = 2.00$ $\tau = 0/\mu = 0.02$ 	ex1-100.1.21 $n = 34/j = 2.00$ $\tau = 0/\mu = 0.02/\theta = 0$ 
ex1-100.2.10 $n = 95/j = 1.54$ $\tau = 0.03, \bar{\tau} = 0$ 	ex1-100.2.11 $n = 10/j = 1.47$ $\tau = 0.36/\theta = 90$ 	ex1-100.2.20 $n = 20/j = 1.47$ $\tau = 0/\mu = 0.36$ 	ex1-100.2.21 $n = 20/j = 1.47$ $\tau = 0/\mu = 0.36/\theta = 0$ 
ex1-100.3.10 $n = 106/j = 1.53$ $\tau = 0.05, \bar{\tau} = 0$ 	ex1-100.3.11 $n = 10/j = 1.46$ $\tau = 0.37/\theta = 90$ 	ex1-100.3.20 $n = 20/j = 1.46$ $\tau = 0/\mu = 0.37$ 	ex1-100.3.21 $n = 20/j = 1.46$ $\tau = 0/\mu = 0.37/\theta = 0$ 
ex1-100.4.10 $n = 107/j = 1.53$ $\tau = 0.06, \bar{\tau} = 0.01$ 	ex1-100.4.11 $n = 10/j = 1.46$ $\tau = 0.37/\theta = 90$ 	ex1-100.4.20 $n = 20/j = 1.46$ $\tau = 0/\mu = 0.37$ 	ex1-100.4.21 $n = 20/j = 1.46$ $\tau = 0/\mu = 0.37/\theta = 0$ 
ex1-100.5.10 $n = 175/j = 1.43$ $\tau = \dots, \bar{\tau} = 0.41$ (see table 4)	ex1-100.5.11 $n = 9/j = 1.41$ $\tau = 0.41/\theta = 90$ 	ex1-100.5.20 $n = 22/j = 1.41$ $\tau = 0/\mu = 0.41$ 	ex1-100.5.21 $n = 22/j = 1.41$ $\tau = 0/\mu = 0.41/\theta = 0$ 
ex1-100.6.10 $n = 170/j = 1.42$ $\tau = \dots, \bar{\tau} = 0.71$	ex1-100.6.11 $n = 9/j = 1.40$ $\tau = 0.43/\theta = 90$ 	ex1-100.6.20 $n = 23/j = 1.40$ $\tau = 0/\mu = 0.43$ 	ex1-100.6.21 $n = 23/j = 1.40$ $\tau = 0/\mu = 0.43/\theta = 0$ 
ex1-100.7.10 $n = 214/j = 1.41$ $\tau = \dots, \bar{\tau} = 1$	ex1-100.7.11 $n = 9/j = 1.39$ $\tau = 0.44/\theta = 90$ 	ex1-100.7.20 $n = 23/j = 1.39$ $\tau = 0/\mu = 0.44$ 	ex1-100.7.21 $n = 23/j = 1.39$ $\tau = 0/\mu = 0.44/\theta = 0$ 
ex1-100.8.10 $n = 105/j = 1.41$ $\tau = \dots, \bar{\tau} = 1$	ex1-100.8.11 $n = 9/j = 1.39$ $\tau = 0.44/\theta = 90$ 	ex1-100.8.20 $n = 23/j = 1.39$ $\tau = 0/\mu = 0.44$ 	ex1-100.8.21 $n = 23/j = 1.39$ $\tau = 0/\mu = 0.44/\theta = 0$ 
ex1-100.9.10 $n = 3/j = 0.70$ $\tau = 1, \bar{\tau} = 1$ 	ex1-100.9.11 $n = 3/j = 0.70$ $\tau = 1/\theta = 0$ 	ex1-100.9.20 $n = 3/j = 0.70$ $\tau = 1/\mu = 1$ 	ex1-100.9.21 $n = 3/j = 0.70$ $\tau = 1/\mu = 1/\theta = 0$ 

TABLE 3. Example 1-100: $\bar{\sigma}_{11} = 1, \bar{\sigma}_{22} = 0, \bar{\sigma}_{12} = 0$ — numerical solution with $k = 1, k = 0.54, k = 0.53125, k = 0.53, k = 0.5, k = 0.49, k = 0.484375, k = 0.48$ and $k = 0.1$ (top to bottom)

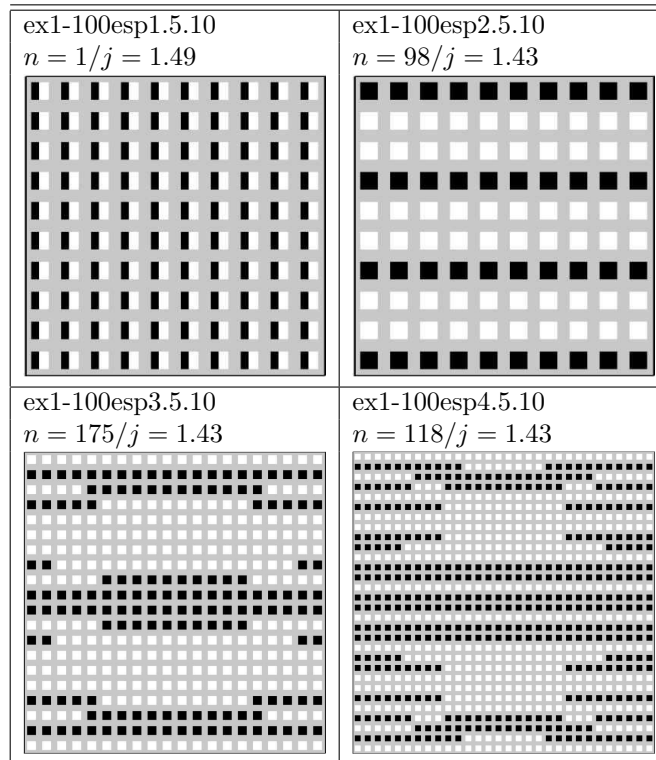


TABLE 4. Example 1-100.5.10: $\bar{\sigma}_{11} = 1, \bar{\sigma}_{22} = 0, \bar{\sigma}_{12} = 0$ — numerical solution with $k = 0.5$ and rank-1 microstructure without rotation; in cases *esp1* and *esp2* we considered 100 elements, in *esp3* we considered 400 elements and in *esp4* 900 elements. The initial approximation of τ for *esp1* was 0.41 (the analytical solution); for the other cases, the initial approximation was 0.5

We remark that for the transition values of k and for the case where the rank-1 laminates are not allowed to rotate (examples .10), the structure has a certain difficulty in transmitting the applied loads ($\sigma_{11} = \bar{\sigma}_{11}$) while, at the same time, minimizing the work done by the applied loads, as required. As can be seen from the above calculations, the numerical solution is no longer homogeneous and isotropic and the structure tries to spread the applied load through a region, as large as possible near the border, while, in the interior a rod like structure is developed in order to collect and transmit the applied loads. It might be interesting to study the ellipticity conditions associated to the stored energy functions of such limit cases as the number of elements grows. For a review of necessary and sufficient conditions leading to the ellipticity of stored energy functions that are isotropic we refer to [?]. We also remark that for the other cases (.11, .20, .21), where an alignment of the microstructure with the direction of the applied loads is possible, the solution is homogeneous according to the analytic result.

The next table corresponds to a general loading with all the components $\bar{\sigma}_{11}$, $\bar{\sigma}_{22}$ and $\bar{\sigma}_{12}$ different from zero.

































ex1-421.1.10 $n = 91/j = 28.51$ $\tau = 0.02, \bar{\tau} = 0$ 	ex1-421.1.11 $n = 12/j = 27.40$ $\tau = 0.35/\theta = 67.5$ 	ex1-421.1.20 $n = 30/j = 27.73$ $\tau = 0/\mu = 0.33$ 	ex1-421.1.21 $n = 22/j = 27.41$ $\tau = 0/\mu = 0.35/\theta = -22.5$ 
ex1-421.2.10 $n = 113/j = 28.26$ $\tau = 0.05, \bar{\tau} = 0$ 	ex1-421.2.11 $n = 11/j = 27.08$ $\tau = 0.37/\theta = 67.5$ 	ex1-421.2.20 $n = 32/j = 27.41$ $\tau = 0/\mu = 0.36$ 	ex1-421.2.21 $n = 23/j = 27.08$ $\tau = 0/\mu = 0.37/\theta = -22.5$ 
ex1-421.3.10 $n = 113/j = 28.25$ $\tau = 0.06, \bar{\tau} = 0$ 	ex1-421.3.11 $n = 11/j = 27.06$ $\tau = 0.37/\theta = 67.5$ 	ex1-421.3.20 $n = 32/j = 27.40$ $\tau = 0/\mu = 0.36$ 	ex1-421.3.21 $n = 23/j = 27.06$ $\tau = 0/\mu = 0.37/\theta = -22.5$ 
ex1-421.4.10 $n = 114/j = 28.24$ $\tau = 0.06, \bar{\tau} = 0.01$ 	ex1-421.4.11 $n = 11/j = 27.05$ $\tau = 0.37/\theta = 67.5$ 	ex1-421.4.20 $n = 32/j = 27.38$ $\tau = 0/\mu = 0.36$ 	ex1-421.4.21 $n = 23/j = 27.05$ $\tau = 0/\mu = 0.37/\theta = -22.5$ 
ex1-421.5.10 $n = 260/j = 26.53$ $\tau = \dots, \bar{\tau} = 0.69$ 	ex1-421.5.11 $n = 10/j = 26.15$ $\tau = 0.43/\theta = 67.5$ 	ex1-421.5.20 $n = 38/j = 26.49$ $\tau = 0/\mu = 0.43$ 	ex1-421.5.21 $n = 26/j = 26.15$ $\tau = 0/\mu = 0.43/\theta = -22.5$ 
ex1-421.6.10 $n = 257/j = 26.40$ $\tau = \dots, \bar{\tau} = 0.96$ (see table 6)	ex1-421.6.11 $n = 10/j = 26.02$ $\tau = 0.44/\theta = 67.5$ 	ex1-421.6.20 $n = 39/j = 26.36$ $\tau = 0/\mu = 0.44$ 	ex1-421.6.21 $n = 26/j = 26.03$ $\tau = 0/\mu = 0.44/\theta = -22.5$ 
ex1-421.7.10 $n = 172/j = 26.40$ $\tau = \dots, \bar{\tau} = 1$	ex1-421.7.11 $n = 10/j = 26.01$ $\tau = 0.44/\theta = 67.5$ 	ex1-421.7.20 $n = 39/j = 26.34$ $\tau = 0/\mu = 0.44$ 	ex1-421.7.21 $n = 27/j = 26.01$ $\tau = 0/\mu = 0.44/\theta = -22.5$ 
ex1-421.8.10 $n = 210/j = 26.38$ $\tau = \dots, \bar{\tau} = 1$	ex1-421.8.11 $n = 10/j = 25.99$ $\tau = 0.44/\theta = 67.5$ 	ex1-421.8.20 $n = 39/j = 26.33$ $\tau = 0/\mu = 0.44$ 	ex1-421.8.21 $n = 27/j = 26.00$ $\tau = 0/\mu = 0.44/\theta = -22.5$ 
ex1-421.9.10 $n = 3/j = 19.25$ $\tau = 1, \bar{\tau} = 1$ 	ex1-421.9.11 $n = 5/j = 19.25$ $\tau = 1/\theta = 0$ 	ex1-421.9.20 $n = 5/j = 19.25$ $\tau = 1/\mu = 1$ 	ex1-421.9.21 $n = 6/j = 19.25$ $\tau = 1/\mu = 1/\theta = 0$ 

TABLE 5. Example 1-421: $\bar{\sigma}_{11} = 4., \bar{\sigma}_{22} = 2., \bar{\sigma}_{12} = 1.$ — numerical solution with $k = 10, k = 9.76, k = 9.75, k = 9.74, k = 9.1, k = 9.01, k = 9, k = 8.99$ and $k = 5$ (top to bottom)

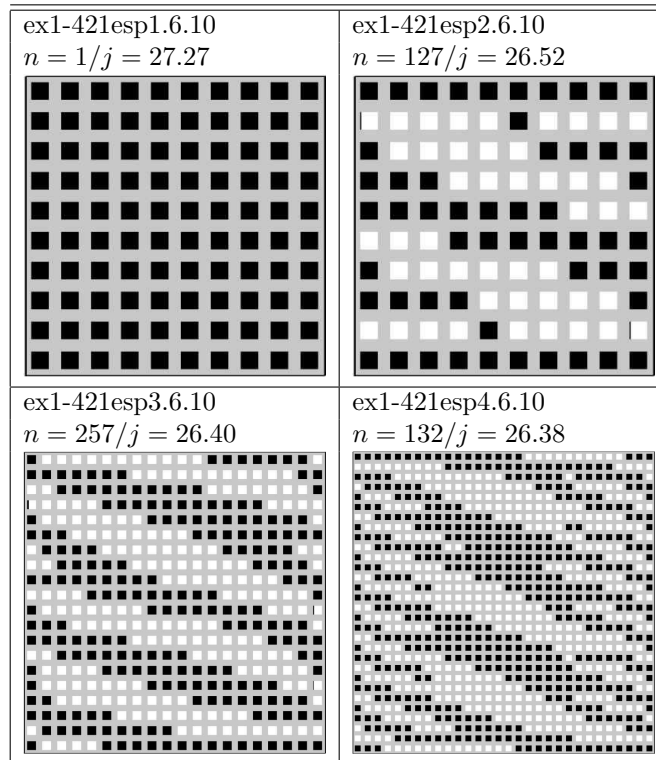


TABLE 6. Example 1-421: $\bar{\sigma}_{11} = 4., \bar{\sigma}_{22} = 2., \bar{\sigma}_{12} = 1.$ — numerical solution with $k = 9.01$ and rank-1 microstructure without rotation; in cases *esp1* and *esp2* we considered 100 elements, in *esp3* we considered 400 elements and in *esp4* 900 elements. The initial approximation of τ for *esp1* 0.96 (the analytical solution); for the other cases, the initial approximation was 0.5

Once again, near the critical value of k the difficulty in orienting the structure leads to a nonhomogeneous solution whose pattern is shown in Table 6. The obtained pattern shows a reinforcement of the direction of the larger principal stress.

According to the examples presented, one sees that in the cases where $\bar{\sigma}_{11} > \bar{\sigma}_{22}$ there are some values of k where the numerical solutions for rank-1 microstructure without cell rotation (suffix 10) are not constant throughout the domain. That is, this occurs when the orientation of the layers is perpendicular to the direction of application of the dominated applied force. When we detail some of these cases with finer meshes it is possible to see that there is a tendency for an alignment of the strong material with the larger principal stress. Near the boundary there is a mixture of both materials, which is the way to transmit the applied forces to the solid, while keeping the total work done to a minimum.

5.2. Example 2. In this example, we consider a cantilever having a narrow rectangular cross section of unit width, with height h and length L , bent by a force having a resultant P applied on $x_1 = L$. All the other edges are free from loading (Figure ??).

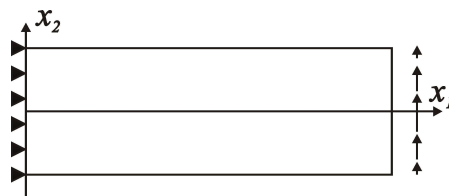


FIGURE 3. Example 2 — geometry and boundary conditions

Denoting by $I = \frac{bh^3}{12}$ the moment of inertia with respect to an axis perpendicular to the plane x_1x_2 , we have

$$\sigma_{11}(u^\tau) = -\frac{P}{I}(L - x_1)x_2, \sigma_{22}(u^\tau) = 0, \sigma_{12}(u^\tau) = \frac{P}{2I}\left(x_2^2 - \left(\frac{h}{2}\right)^2\right),$$

which enables us to write

$$\chi_\tau = k(\rho^+ - \rho^-) - \frac{E^+ - E^-}{E^+ E^-} \left(((1 - \nu^2) + E^+ E^- \frac{\nu^2}{I_2^2}) \frac{P^2}{I^2} x_1^2 x_2^2 + (1 + \nu) \frac{P^2}{2I^2} \left(x_2^2 - \left(\frac{h}{2}\right)^2 \right)^2 \right).$$

Considering $h = 1, L = 4$ and $P = 1$, we have that

$$\bar{\tau}(x) = \begin{cases} 0 & \text{if } 306 \leq k \\ \dots & \text{if } 0 < k < 306 \\ 1 & \text{if } k = 0 \end{cases}$$

TABLE 7. Example 2 — part of the analytical solution for rank-1 microstructure without rotation

The corresponding numerical examples are shown in Tables 8 and 9. One observes that as the material cost k becomes lower the stiffest (more expensive) material is progressively added from the regions where the internal stresses are higher to the lower stressed ones, as expected.

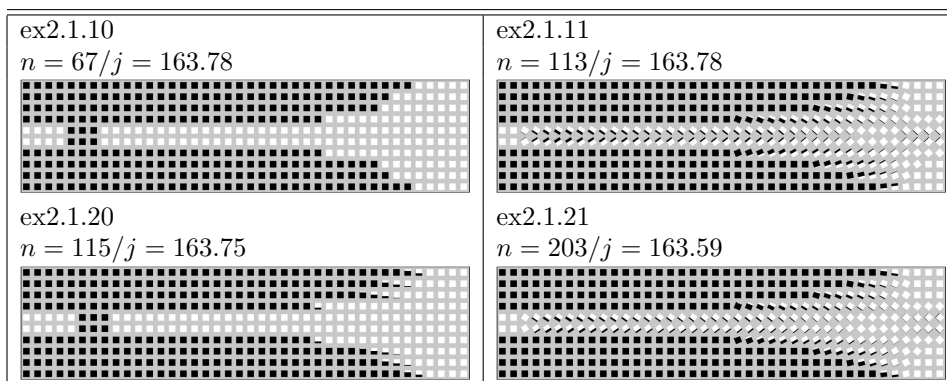
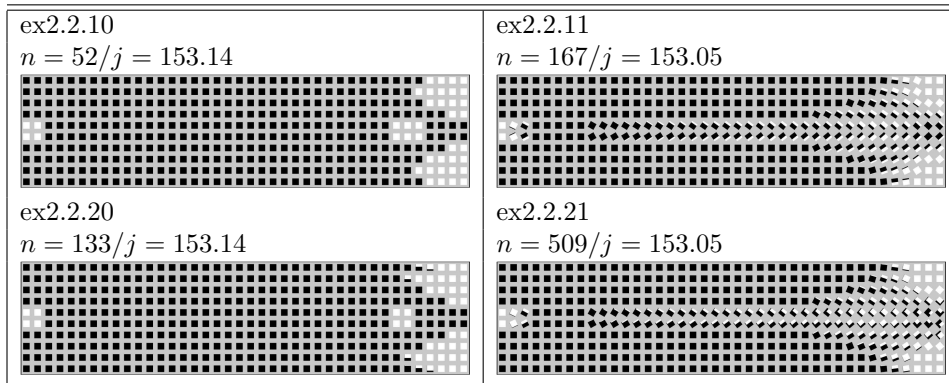
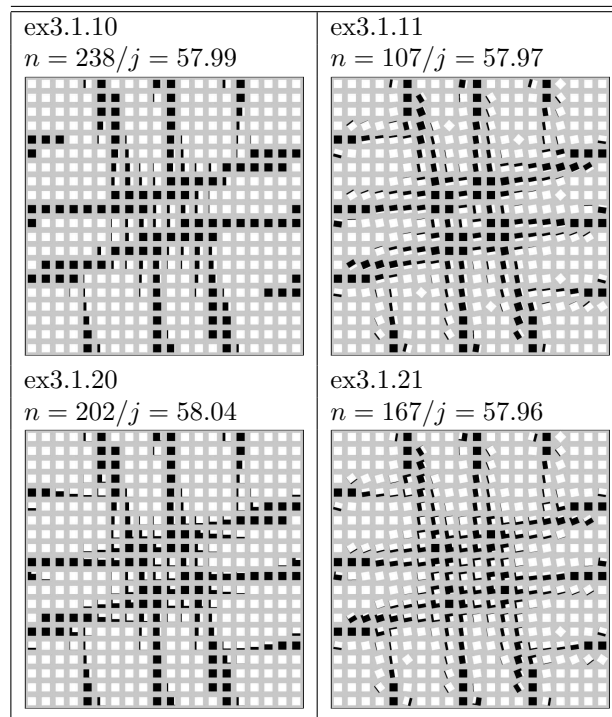


TABLE 8. Example 2 — numerical solution with $k = 4$

TABLE 9. Example 2 — numerical solution with $k = 2.5$ TABLE 10. Example 3 — numerical solution with $k = 20$

5.3. **Example 3.** As a last example, we consider a square plate simply-supported on the bottom side, where there are tip forces along the sides, which make with them an angle of thirty degrees (Figure ??).

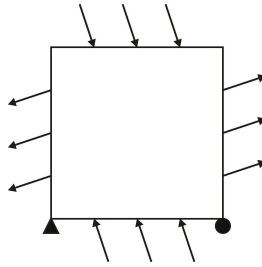


FIGURE 4. Example 3 — geometry and boundary conditions

The confidence the previous examples gave to us and the results obtained for the case in which the applied loads act perpendicularly to the border (not shown), enable us to apply the method to this more complex example, even if in this case there is not an explicit analytical solution.

The numerical examples are shown in Table 10.

As expected, one observes the adaptation and the orientation of the microstructure to the applied loads and if one considers that each finite element is in the situation of the whole problem of example 1, it is possible to verify that the obtained solution, in each finite element, is in agreement with the analytical one, shown in the first example.

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