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What teachers think about mathematical proof?

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Abstract: This paper presents a quantitative study, initial part of a larger work that also involves a qualitative component, which aims to study the conceptions of mathematics teachers in 5th to 9th grades (n=115) about mathematical proof. The results, that are based on the application of a questionnaire, show that teachers, despite their different academic backgrounds (all of them with a background in mathematics, but some performed courses with a strong pedagogical component and others with a predominant mathematical component), recognize the nature of proof and its importance in student learning, showing awareness of the need to adapt proof to students capabilities.

Résumé: Cet article présente une étude quantitative, première partie d'un plus grand travail qui implique également une component qualitative, qui vise à étudier les conceptions des enseignants de mathématiques de la 5ème à la 9ème année (n = 115) au sujet de la preuve mathématique. Les résultats, qui sont fondées sur l'application d'un questionnaire, montrent que les enseignants, en dépit de leurs différents formations académiques (chacun d'eux avec une formation en mathématiques, mais certains ont effectué cours avec une forte component pédagogique et les autres avec une component mathématique prédominante) , reconnaître la nature de la preuve et de son importance dans l'apprentissage des élèves, montrant prise de conscience de la nécessité d'adapter la preuve aux capacités des étudiants.

Introduction

The influence of knowledge, conceptions and beliefs of mathematics teachers in their professional practices has been widely documented by research (Ponte & Chapman, 2006; Thompson, 1992). This influence can result in opportunities for successful teaching practices, and, consequently, in rich student learning, or, on the contrary, in obstacles to the complex teaching-learning process. An example of this is the way the teacher outlooks and integrates mathematical proof in classroom activities. The mathematical proof is an important element in mathematics and mathematics teacher's activity, both in training and teaching practices. Recent curriculum changes in Portuguese mathematics programs (ME, 2007; MEC, 2013) have given more emphasis to mathematical proof. This is the context from which this work arises, that. We are interested in understanding how Portuguese Mathematics teachers look to mathematical proof and how they integrate it in their practices and also the goals they seek to achieve through it, having on the horizon, the current math programs.

In this study, proof is understood in a wide manner, and not strictly formal, covering reasoning and communication mathematical processes which allow to sign the veracity of certain mathematical statements by the force of the reason and not based on authority criteria, whatever it is. The broader research, combining quantitative with qualitative data, integrating survey of teachers (questionnaire and interview) and classroom observation, starts with a macro look at mathematics' teachers. It is that more general look that we present here, that is, we seek to an approach to conceptions of mathematics teachers in 5th to 9th grades by applying a structured questionnaire.

Background

To prove is an activity present in various fields of human action, whenever you want to ensure a certain statement, by its intrinsic value, and not by any other powers associated with who says or where says. This activity, although it is present in our everyday life, it is particularly relevant in contexts of genesis and communication of knowledge, such as the scientific production, in particular mathematics, and the case of the teaching-learning of mathematics in the classroom. In any of these contexts, prove is a condition of freedom and an affirmation of the "force of reason", being connected to the argumentation capacity (Boavida, 2005).

Proof, understood as the result of the proving action, assumes in various fields of science different forms that result from the nature of the knowledge that is concerned and especially from the practices followed by the communities in which that is developed. In mathematics, proof assumes proper contours, leading some authors (Dreyfus, 2000; Hanna, 2000, 2002; Knuth, 2002) to consider that proof is what distinguishes mathematics from other sciences.

Definitions for mathematical proof are usually related to the functions assigned to it. De Villiers (2003) considers six proof functions: i) verification; ii) explanation; iii) systematization; iv) discovery; v) intellectual challenge; and vi) communication. These proof functions are related to the context in which they are performed. While verification is more common in mathematician's activity, the explanation appears more linked to educational activity (CadwalladerOlsker, 2011). In turn, the communication function may either arise in the mathematician's or school mathematics activity. In this work, considering the educational context in which the research takes place, we have adopted a definition of mathematical proof as a process of argumentation, emphasizing it explicative and communicative functions, aimed learning of mathematical topics and transversal capabilities (Boavida, 2005).

This didactic value of mathematical proof has been stated by several authors and professional organizations (Hanna, 2000, 2002; NCTM, 2000). In this regard, the *Principles and Standards for School* Mathematics (NCTM, 2000) consider that must be provided to all students the opportunity to "recognize reasoning and proof as fundamental aspects of mathematics; make and investigate mathematical conjectures; develop and evaluate mathematical arguments and proofs; select and use various types of reasoning and methods of proof" (p. 56).

In this sense, Portuguese programs of Mathematics (5th to 9th grades) suggest student work with proof, first informally and then with a progressive degree of formalization. In 5th and 6th grades it is suggested in the program, for example, that "to the sum of the amplitudes of internal and external angles of a triangle resort to informal proof" (ME, 2007, p. 38). For students of 7th to 9th grades, mathematics program advocates that students should "understand the notion of demonstration and be able to do deductive reasoning" (ME, 2007, p. 51). The current mathematics programs of basic education (MEC, 2013) also gave greater prominence to proof, although in order to give it greater level of formalization.

In this context of curricular recommendation, and because of their potential to influence practices, it seemed pertinent to study what teachers think about mathematical proof and how they conceive proof in student's activity and in their own professional activity.

Methodology

In this paper, we focus on conceptions of mathematics teachers of 5th to 6th grades (2nd cycle) (n=43) and 7th to 9th grades (3rd cycle) (n=72) on the mathematical proof, adopting a quantitative approach in the treatment of information resulting from the application a questionnaire (Gall, Gall & Borg, 2003). In the sample selection, we sent questionnaires to elementary schools of the northern of Portugal, where the majority of the schools stand. There two districts were chosen, one inland and the other from the coast. Doing so, we pretend to cover a diversity of schools. The sampling method was by convenience (Hill & Hill, 2012), since the questionnaires were distributed in various schools by some teachers who had contact with the Project team.

This methodological approach arises from the fact that, initially, we want to meet in a broad way teachers' conceptions about mathematical proof, with no intention to generalize to all the mathematics teachers. The sample is formed by all questionnaires received. Among participant teachers (n=115), the female gender is prevalent (85), the age is 41 years old (ranging between 30 and 62 and the average age is 44) and 15 is the mode of years of service (ranging from 5 to 35 years).

The questionnaire consists of five parts: the first part includes four questions about age, gender, school year and years of service; the second part consists of 14 closed questions about proof in mathematics; the third part includes 11 closed questions about the proof in the student activity (5th to 9th grades); the fourth part has nine closed questions about proof in teacher activity; and the fifth and final part includes eight questions about proof in mathematics curricula.

In data analysis, responses were organized and processed using the SPSS software. Analysis was guided by the following dimensions: (i) proof in mathematics; (ii) proof in elementary school student's activity; and (iii) proof in teacher's activity. In these dimensions, the answers to the items of the questions relate to the selection of a frequency option, according to the scale: Strongly Disagree (SD); Disagree (D); Neither Agree Nor Disagree (NAND); Agree (A) and From the answers we have determined average and standard deviations for all options after the coded options SD, D, NAND, A and TA with the values 1, 2, 3, 4 and 5, respectively.

From the numerical values, we applied the Student's t-test for independent samples, considering the group of the 2nd cycle teachers (5^{th} to 6^{th} grades) and the group of teachers of the 3rd cycle (7^{th} to 9^{th} grades). In the statistical analysis performed adopted the level of significance of 0.05.

Results

To understand what teachers think about mathematical proof, we organize information from their responses to a questionnaire according to the dimensions already mentioned.

Proof in Mathematics

The underlying formality to logical argument of proofing the veracity of a mathematical statement leads, in general, teachers to distinguish this method from experimental ones that are used in other areas of knowledge. In addition to the deductive method, teachers recognize other proof methods. Whichever proof method to which they use to prove the veracity of a mathematical result, to most teachers this activity is evidenced by being the basis for mathematical knowledge construction (Table 1).

	2^{nd} cycle		3^{rd} cycle	
	$\bar{x}$	s	$\bar{x}$	s
The proof in maths has different nature of the proof in other sciences.	3,5	1,07	3,7	1,05
The deductive method is the only method that proves mathematical results.	2,1	0,84	1,9	0,88
The proof is essential for the construction of mathematical knowledge.	3,8	0,99	3,9	0,83

Table 1. Nature of mathematical proof.

Comparing the average of the two groups of teachers (see Table 1), we find that there are no statistically significant differences between these groups. Teachers' conceptions about proof are associated with the functions they give to it.

For all teachers, proof performs several functions, including the verification and explanation of a mathematical statement's veracity. Already the discovery/ invention function of new mathematical results gathers indecision among teachers, which may be due to the absence of this proof function in the school context. With respect to the function of systematizing mathematical statement, observation of Table 2 show there are considerable differences between the means of two groups of teachers (Table 2).

	2nd cycle		3rd cycle	
	$\bar{x}$	s	$\bar{x}$	s
The proof has the function of verification the mathematical statement.	4,1	0,92	4,1	0,99
The proof has the function of explanation the mathematical statement.	4,1	0,83	3,8	0,96
The proof has the function of discovery / invention of new results.	3,1	1,01	2,9	1,16
The proof has the function of systematization a mathematical statement.	3,6	0,85	3,1	1,09

Table 2. Functions of mathematical proof.

Comparing the average of the two groups, the application of T-Test determines statistically significant differences in the item, "The proof has the function of systematization of a mathematical statement" ($p=0.007$), which is more emphasized by 5th/6th grades teachers than by 7th to 9th grades teachers.

The proof in elementary school student's activity

The consideration of mathematical proof in school context leads us to investigate the role that students play in this activity. Teachers of both cycles agree with the involvement of students in the proof of mathematical results, which, in their perspective, leads students to understand the nature of this activity. In this involvement, they disagree that the proof should be reserved for the best students (Table 3).

	2nd cycle		3rd cycle	
	$\bar{x}$	s	$\bar{x}$	s
Students must participate in the proof of mathematical results.	3,8	0,85	3,8	0,86
Proofs should be made only by the best students.	2,1	0,92	2,4	1,08
Proving leads students to understand the nature of mathematical activity.	3,7	0,85	3,9	0,83
Students should use the mathematical results without proving them.	2,7	0,85	3,0	0,93

Table 3. The student's activity in mathematical proof.

The comparison of the means of teachers' answers of each school cycles reflects that there are no statistically significant differences between the groups (Table 3). Concerning the use by the students of mathematical results without being proved, teachers expressed indecision. The involvement of students on proving mathematical results gathers the agreement of teachers of both cycles in the development of students understanding of mathematical concepts and capabilities to reason logically and to communicate mathematically (Table 4).

	2nd cycle		3rd cycle	
	$\bar{x}$	s	$\bar{x}$	s
Proving increases understanding of mathematical concepts by students.	3,9	0,87	3,6	0,88
Proving develops student’s ability to reason logically.	3,9	0,83	4,1	0,69
Proving develops student’s mathematical communication capability.	3,9	0,92	3,9	0,76

Table 4. The mathematical proof in student learning.

Comparing the average of the two groups of teachers (see Table 4), we observe that there are no statistically significant differences between these groups.

The proof in the teacher's activity

The systematization of knowledge and the abstract nature involved in the proof of mathematical results are factors that increase the complexity of this activity (de Villiers, 1990). The complexity inherent of the activity of proving mathematical results seems to be the reason why teachers did not express their agreement to the difficulty of integrating the proof in their teaching strategies, and developing this activity in their lessons and engage students in activities that lead to conjecture and proving mathematical results. Despite this hesitation, teachers tend to disagree that proof does need not be offered to students of elementary school (Table 5).

	2nd cycle		3rd cycle	
	$\bar{x}$	s	$\bar{x}$	s
I have difficulty integrating the proof in my classroom.	2,9	1,07	3,2	1,15
I often prove the mathematical results in my classroom.	3,2	0,92	3,2	0,95
In my classroom, I challenge students to formulate and prove conjectures.	3,3	0,91	3,3	0,89
I believe that it is not necessary teach elementary school students to prove.	2,5	1,10	2,4	1,06

Table 5. Proof in the teaching practice of the mathematics teacher.

The comparison of means of two groups of teachers (see Table 5) does not highlight significant differences between these groups.

Final considerations

Mathematics teachers recognize the specificity of mathematical proof distinguishing the nature of this activity of experimental methods. In addition to the deductive method, they recognize other proof methods of mathematical results. This might be connected to the conceptual framework of how they organize mathematical knowledge.

Teachers from both cycles of teaching identify multiple functions of proof, such as verification and explanation of a mathematical statement. The systematization function is prevalent in teachers of the 2nd cycle (5th to 6th grades), which is understandable considering the student’s school grade. Teachers in both cycles agree with the participation of students, and not just the top ones, on the proof of mathematical results, because it favors the development of mathematical concepts and reasoning and mathematical communication, with special emphasis to argumentation capacity, as advocated by Boavida (2005). This result shows that teachers recognize the didactical recommendations concerning the mathematical proof (Hanna, 2000, 2002; NCTM, 2000). Nevertheless, mathematics teachers have difficulties integrating proof situations in their classes, which can be derived from the fact that this is an activity that is conceptually demanding or this is a practice that needs teacher training, as pointed by Boavida (2005). Despite the difficulties, teachers consider that it is necessary to involve students on the proof of mathematical results. In summary, teachers of the two cycles, although with different backgrounds (all of them with a background in mathematics, but ones performed courses with a strong pedagogical component and the others with a predominant mathematical component), reveal similar conceptions about the mathematical proof. Considering not just the difficulties but also the desire that teachers expressed in integrating mathematical proof in their teaching strategies it makes sense to promote formation dynamics, such as training activities, to encourage this integration.

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