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Fractional bioheat equation

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Abstract

In this work we develop a new mathematical model for the Pennes' bioheat equation assuming a fractional time derivative of single order. A numerical method for the solution of such equations is proposed, and, the suitability of the new model for modelling real physical problems is studied and discussed.

Key words: Time-fractional diffusion equation, Caputo derivative, bioheat equation, stability, convergence

1 Introduction

Pennes' [1] bioheat transfer equation, which describes the thermal distribution in human tissue, taking into account the influence of blood flow, (see Fig. 1) is given by,

$$\rho_t c_t \frac{\partial T(x,t)}{\partial t} = k \frac{\partial^2 T(x,t)}{\partial x^2} + W_b c_b \left(T_a - T\right) + q_m, \quad t > 0, \quad 0 < x < L, \tag{1}$$

where ρ_t, c_t are constants representing the density $\left[kg/m^3\right]$ and the specific heat $[J/\left(kg\,{}^\circ C\right)]$, respectively, and k is the tissue thermal conductivity$[J/(s.m\,{}^\circ C)]$; W_b is the mass flow rate

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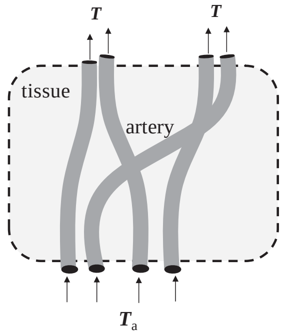

Figure 1: Heat transfer between blood vessels and tissue.

of blood per unit volume of tissue $\left[kg/\left(s.m^3\right)\right]$; c_b is the blood specific heat; q_m is the metabolic heat generation per unit volume $\left[J/(s.m^3)\right]$; T_a represents the temperature of arterial blood $[^\circ C]$; T is the temperature and the term $W_b c_b \left(T_a - T\right)$ represents the blood perfusion.

Although this model presents some limitations: the blood velocity field is not taken into account; assumes that thermal equilibration occurs in the capillaries; the blood leaving the tissue does not influence the temperature of the medium; it has been used by several researchers working in different research fields. The reason for that is that the model is simple, atractive and the new models proposed are complex presenting a large number of parameters that are difficult to obtain experimentally.

In order to overcome some of the model limitations, in this work we propose a new bioheat model, where the classical time derivation is substituted by a time-fractional derivative.

The bioheat equation presented before (Eq.1) is now written, using the time-fractional derivative instead of the the first-order time derivative, $\frac{\partial T(x,t)}{\partial t}$, generalizing in this way the original equation derived by Harry Pennes:

$$\frac{\partial^\alpha T(x,t)}{\partial t^\alpha} = A\frac{\partial}{\partial x}\left(k\left(x\right)\frac{\partial T(x,t)}{\partial x}\right) - BT(x,t) + C \quad 0 < t < T^*, \quad 0 < x < L, \qquad (2)$$

where $\frac{\partial^\alpha}{\partial t^\alpha}$ is the fractional Caputo derivative given by [2],

$$\frac{\partial^\alpha T(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma\left(1-\alpha\right)}\int_0^t \left(t-s\right)^{-\alpha}\frac{\partial T(x,s)}{\partial s}ds \qquad (3)$$

with $0 < \alpha < 1$, and $A = \frac{1}{\rho_t c_t \tau^{\alpha-1}}$, $B = \frac{W_b c_b}{\rho_t c_t \tau^{\alpha-1}}$, and $C = \frac{W_b c_b T_a + q_m}{\rho_t c_t \tau^{\alpha-1}}$. Note that $k\left(x\right)$ is a function of x, meaning that we can deal with possible anisotropy. Also, it is worth-mentioning the fact that we have added a new parameter $\tau\left[s\right]$ to the equation, so that it becomes dimensionally consistent.

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2 Numerical solution

For the numerical solution of Eq. 2, we consider a uniform space mesh on the interval $[0, L]$, defined by the gridpoints $x_i = i\Delta x$, $i = 0, \ldots, N$, where $\Delta x = \frac{L}{N}$, and we approximate the space derivative by a second order finite difference.

For the discretization of the fractional derivative we also assume uniform meshes, with a time step $\Delta t = Time/R$ (R is the number of divisions of the grid) and time gridpoints t^l, $l = 0, 1, ..., R$, and, we use the backward finite difference formula provided by Diethelm [2] ($\mathcal{O}\left((\triangle t)^{2-\alpha_j}\right)$). Denoting $T(x_i, t_l)$ by T_i^l, and $k\left(x_i \pm \frac{\Delta x}{2}\right)$ by $k_{i\pm\frac{1}{2}}$ and neglecting the $\mathcal{O}\left((h)^2\right)$ and $\mathcal{O}\left((\triangle t)^{2-\alpha_j}\right)$ terms, the finite difference scheme is then given by,

$$\begin{aligned}\frac{(\triangle t)^{-\alpha}}{\Gamma(2-\alpha)} \sum_{m=0}^{l} a_{m,l}^{(\alpha)} \left(T_i^{l-m} - T_i^0\right) &= A\frac{k_{i+\frac{1}{2}}T_{i+1}^l - \left(k_{i+\frac{1}{2}} + k_{i-\frac{1}{2}}\right)T_i^l + k_{i-\frac{1}{2}}T_{i-1}^l}{(\Delta x)^2} \\ &+ f\left(x_i, t_l, T_i^l\right) \quad i = 2, ..., N-2, \; l = 1, ..., R\end{aligned} \tag{4}$$

with $f\left(x_i, t_l, T_i^l\right) = -BT_i^l + C$.

For consistency with the order of the spatial discretization at grid points $i = 2, ..., N-2$, we also assume a second a order approximation for the Neumann boundary conditions. For that, a second order forward and backward finite difference scheme was used.

Stability and Convergence

In this section we provide two useful theorems for the stability and convergence of the numerical mehtod proposed.

Theorem (stability) 1. *Let $0 < \varepsilon \leq \Delta t$, the scheme given by Eq. 4 is unconditionally stable with respect to the initial conditions.*

Theorem (convergence) 1. *Let $0 < \varepsilon \leq \Delta t$, if the solution of (2) is of class C^2 with respect to t and of class C^4 with respect to x, then there exists a constant C_0 independent of Δx and Δt such that,*

$$\left\|\mathbf{e}^l\right\|_{\varepsilon} \leq C_0\left((\Delta t)^{2-\alpha} + (\Delta x)^2\right), \quad l = 0, 1, \ldots. \tag{5}$$

where $\mathbf{e}^l = \left[e_1^l, e_2^l, ..., e_{N-1}^l\right]$, $\quad l = 1, 2, ...,$ is vector of the errors at time step l, with $e_i^l = T(x_i, t_l) - T_i^l$ $l = 1, 2, ...,$ $i = 1, ..., N-1$ the error at each point of the mesh.

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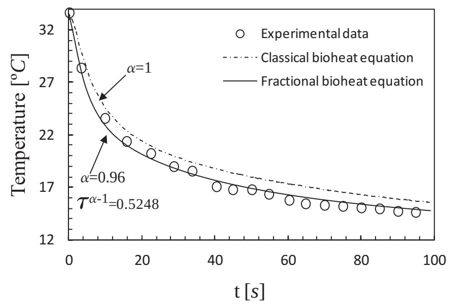

Figure 2: Fitting experimental data.

3 Modelling

In order to test the fractional bioheat model, we used the experimental data provided by Barcroft and Edholme [3] for the temperature variation inside a human arm. One of their experiments consisted of measuring the temperature decrease of the subcutaneous tissue ($1\,cm$ below the skin surface) when the forearm is submersed in a $12°C$ water bath.

In Fig. 2, we show that the proposed fractional bioheat equation can be used to improve the accuracy of the classical numerical predictions, when compared with classical models.

4 Conclusions

In this work we have derived a numerical method for the solution of the fractional bioheat equation of single order with a variable diffusion coefficient. We observed that the new bioheat model can be used to better predict subdifusion processes.

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