The inverse along a product and its applications

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Abstract

In this paper, we study the recently defined notion of the inverse along an element. An existence criterion for the inverse along a product is given in a ring. As applications, we present the equivalent conditions for the existence and expressions of the inverse along a matrix.

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1. Introduction

In this paper, R is an associative ring with unity 1. An element $a \in R$ is (von Neumann) regular if there exists $x \in R$ such that $axa = a$. Such x, an inner inverse of a, is denoted by a^-. We call b an outer inverse of a provided that $bab = b$. If b is both an inner and an outer inverse of a, then it is a reflexive inverse of a, and is denoted by a^+.

Given a semigroup S, S^1 denotes the monoid generated by S. Following Green [1], Green's preorders and relations in a semigroup are defined by

$a \leq_{\mathcal{L}} b \Leftrightarrow S^1 a \subset S^1 b \Leftrightarrow$ there exists $x \in S^1$ such that $a = xb$.

$a \leq_{\mathcal{R}} b \Leftrightarrow aS^1 \subset bS^1 \Leftrightarrow$ there exists $x \in S^1$ such that $a = bx$.

$a \leq_{\mathcal{H}} b \Leftrightarrow a \leq_{\mathcal{L}} b$ and $a \leq_{\mathcal{R}} b$.

$a \mathcal{L} b \Leftrightarrow S^1 a = S^1 b \Leftrightarrow$ there exist $x, y \in S^1$ such that $a = xb$ and $b = ya$.

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$a\mathcal{R}b \Leftrightarrow aS^1 = bS^1 \Leftrightarrow$ there exist $x, y \in S^1$ such that $a = bx$ and $b = ay$.
$a\mathcal{H}b \Leftrightarrow a\mathcal{L}b$ and $a\mathcal{R}b$.

Recently, Mary [4] introduced the notion of the inverse along an element that is based on Green's relation in a semigroup S. Given $a, d \in S$, an element $a \in S$ is invertible along d [4] if there exists b such that $dab = d = bad$ and $b \leq_{\mathcal{H}} d$. The element b above is unique if it exists, and is denoted by $a^{\|d}$. Recall that $a^{\|d}$ exists implies that d is regular. Later, Mary and Patrício [5] proved that a is invertible along d if and only if $d\mathcal{H}dad$, which gave a new existence criterion for the inverse along an element. Further, given a regular element d, they [5, 6] characterized the existence of $a^{\|d}$ by means of a unit and d^- in a ring. Moreover, the representation of $a^{\|d}$ is given. As applications, they [6] derived the equivalent conditions for the existence and the formula of the inverse along a regular lower triangular matrix. More results on the inverse along an element can be found in mathematical literature [3, 9].

Motivated by papers [5, 6], we investigate the inverse along a product pmq (m is regular) in a ring, extending the results in [5, 6]. As applications, the inverse along a regular matrix $\begin{bmatrix} d_1 & d_3 \\ d_2 & d_4 \end{bmatrix}$ is given under some conditions.

2. The inverse along a product *pmq*

In this section, we begin with some lemmas which play important roles in the sequel.

Lemma 2.1. *Given $a, b \in R$, then $1 + ab$ is invertible if and only if $1 + ba$ is invertible. Moreover, $(1 + ba)^{-1} = 1 - b(1 + ab)^{-1}a$.*

Lemma 2.1 is known as the Jacobson's Lemma (see e.g. [2]).

Lemma 2.2. ([8, Theorem 1]) *Let R be a ring and e an idempotent in R. Then $exe + 1 - e$ is invertible in R if and only if exe is invertible in eRe.*

The next theorem, a main result of this paper, gives an existence criterion of the inverse along a product pmq in a ring.

Theorem 2.3. *Let $p, a, q, m \in R$ with m regular. If $m \leq_{\mathcal{L}} pm$ and $m \leq_{\mathcal{R}} mq$, then the following conditions are equivalent*

(i) *a is invertible along pmq.*

(ii) *$u = mqap + 1 - mm^-$ is invertible.*

(iii) $v = qapm + 1 - m^- m$ *is invertible.*

In this case,

$$a^{\| pmq} = pu^{-1}mq = pmv^{-1}q.$$

PROOF. It follows from Lemma 2.1 that (ii)$\Leftrightarrow$(iii). Next, it is sufficient to prove (i)$\Leftrightarrow$(ii).

(i)$\Rightarrow$(ii) Suppose that a is invertible along pmq. From $m \leq_{\mathcal{L}} pm$ and $m \leq_{\mathcal{R}} mq$, then there exist p' and q' such that $p'pm = m = mqq'$. In view of [5, Theorem 2.2], we know that a is invertible along pmq if and only if $pmq\mathcal{H}pmqapmq$. There are $x, y \in R$ such that

$$pmq = xpmqapmq = pmqapmqy. \tag{1}$$

Multiplying the above equation (1) by p' on the left yields

$$mq = mqapmqy.$$

Multiplying the above equation (1) by q' on the right yields

$$pm = xpmqapm.$$

Hence,

$$mqapmm^-(mqyq'm^-mm^-) = mm^- = (mm^-p'xpmm^-)mqapmm^-.$$

The equalities above show that $mqapmm^-$ is invertible in mm^-Rmm^-. By Lemma 2.2, $mqapmm^- + 1 - mm^-$ is invertible in R. Again, Lemma 2.1 ensures that $u = mqap + 1 - mm^-$ is invertible.

(ii)$\Rightarrow$(i) Suppose that u, therefore v are invertible. From $um = mv = mqapm$, it follows that $pmq = pu^{-1}mqapmq = pmqapmv^{-1}q$ and $pu^{-1}mq = pmv^{-1}q$. Pose $b = pu^{-1}mq = pmv^{-1}q$, then $b \leq_{\mathcal{H}} pmq$ since $pu^{-1}mq = pu^{-1}p'pmq = pmqq'v^{-1}q$.

Hence, a is invertible along pmq. Moreover,

$$a^{\| pmq} = pu^{-1}mq = pmv^{-1}q.$$

The proof is completed. □

If p is left invertible and q is right invertible, then $m\mathcal{L}pm$ and $m\mathcal{R}mq$. As a special result of Theorem 2.3, we have the following corollary.

Corollary 2.4. *Let $p, a, q, m \in R$ with m regular. If p is left invertible and q is right invertible, then the following conditions are equivalent*

(i) *a is invertible along pmq.*

(ii) *$u = mqap + 1 - mm^-$ is invertible.*

(iii) *$v = qapm + 1 - m^-m$ is invertible.*

In this case,

$$a^{\|pmq} = pu^{-1}mq = pmv^{-1}q.$$

Taking $p = q = 1$, we get

Corollary 2.5. ([5, Theorem 3.2] and [6, Theorem 1.3]) *Let m be a regular element of a ring R. Then the following are equivalent*

(i) *a is invertible along m.*

(ii) *$u = ma + 1 - mm^-$ is invertible.*

(iii) *$v = am + 1 - m^-m$ is invertible.*

In this case,

$$a^{\|m} = u^{-1}m = mv^{-1}.$$

3. Applications to the inverse along a matrix

Mary, Patrício [6] gave some equivalent conditions for the existence of the inverse along a regular lower triangular matrix $\begin{bmatrix} d_1 & 0 \\ d_2 & d_4 \end{bmatrix}$ over a Dedekind-finite ring. It would be interesting to find the related existence criteria and formula of the inverse along a regular matrix $D = \begin{bmatrix} d_1 & d_3 \\ d_2 & d_4 \end{bmatrix}$, in the general case.

By $R_{2\times 2}$ we denote the ring of 2×2 matrices over R. Let $D = \begin{bmatrix} d_1 & d_3 \\ d_2 & 0 \end{bmatrix} \in R_{2\times 2}$ and $D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} d_2 & 0 \\ d_1 & d_3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} =: PMQ$. Given a lower triangular matrix $M = \begin{bmatrix} d_2 & 0 \\ d_1 & d_3 \end{bmatrix}$ with d_2 and d_3 regular, Patrício and Puystjens [7] proved that M is regular if and only if $w = (1 - d_3d_3^+)d_1(1 - d_2^+d_2)$ is regular. In this case,

$$MM^- = \begin{bmatrix} d_2d_2^+ & 0 \\ (1 - ww^-)(1 - d_3d_3^+)d_1d_2^+ & d_3d_3^+ + ww^-(1 - d_3d_3^+) \end{bmatrix}.$$

Next, we consider the inverse along a regular matrix, whose $(2,2)$ entry is zero.

Theorem 3.1. *Let* $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, $D = \begin{bmatrix} d_1 & d_3 \\ d_2 & 0 \end{bmatrix} \in R_{2\times 2}$ *with* d_2 *and* d_3 *regular. If* $c^{\|d_2}$ *exists, then* $A^{\|D}$ *exists if and only if* $\xi = \beta - \alpha c^{\|d_2} a$ *is invertible.*

In this case, $A^{\|D} = \begin{bmatrix} \xi^{-1}(d_1 - \alpha c^{\|d_2}) & \xi^{-1}d_3 \\ c^{\|d_2}[1 - a\xi^{-1}(d_1 - \alpha c^{\|d_2})] & -c^{\|d_2} a\xi^{-1}d_3 \end{bmatrix}$, *where*

$$\begin{aligned} \alpha &= d_1c + d_3d - (1 - ww^-)(1 - d_3d_3^+)d_1d_2^+, \\ \beta &= d_1a + d_3b + (1 - ww^-)(1 - d_3d_3^+), \\ \xi &= \beta - \alpha c^{\|d_2} a. \end{aligned}$$

Proof. We have $MAP = \begin{bmatrix} d_2c & d_2a \\ d_1c + d_3d & d_1a + d_3b \end{bmatrix}$. Hence,

$$U = MAP + I - MM^- = \begin{bmatrix} u & d_2a \\ \alpha & \beta \end{bmatrix}, \text{ where}$$

$$\begin{aligned} u &= d_2c + 1 - d_2d_2^+, \\ \alpha &= d_1c + d_3d - (1 - ww^-)(1 - d_3d_3^+)d_1d_2^+, \\ \beta &= d_1a + d_3b + (1 - ww^-)(1 - d_3d_3^+). \end{aligned}$$

Since $c^{\|d_2}$ exists, it follows that $u = d_2c + 1 - d_2d_2^+$ is invertible and $c^{\|d_2} = u^{-1}d_2$. Using Schur complements we get the factorization

$$U = \begin{bmatrix} 1 & 0 \\ \alpha u^{-1} & 1 \end{bmatrix} \begin{bmatrix} u & 0 \\ 0 & \xi \end{bmatrix} \begin{bmatrix} 1 & c^{\|d_2}a \\ 0 & 1 \end{bmatrix},$$

where $\xi = \beta - \alpha c^{\|d_2} a$. Hence, U is invertible if and only if ξ is invertible.

Note that $U^{-1} = \begin{bmatrix} 1 & -c^{\|d_2}a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u^{-1} & 0 \\ 0 & \xi^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\alpha u^{-1} & 1 \end{bmatrix}$. Then

$$A^{\|D} = PU^{-1}M = \begin{bmatrix} \xi^{-1}(d_1 - \alpha c^{\|d_2}) & \xi^{-1}d_3 \\ c^{\|d_2}[1 - a\xi^{-1}(d_1 - \alpha c^{\|d_2})] & -c^{\|d_2} a\xi^{-1}d_3 \end{bmatrix}.$$

The proof is completed. □

Remark 3.2. In the above Theorem, if c is not invertible along d_2, $A^{\| D}$ may exist. Next, we give an example to illustrate it.

Take $D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and A to be the 2×2 identity matrix over any field. Since 0 is not invertible along 1 then the $(1, 2)$ entry of A is not invertible the $(2, 1)$ entry of D, and yet A is invertible along D since they are both invertible.

Now, suppose that d_4 in the matrix D is regular and set $e = 1 - d_4 d_4^+$, $f = 1 - d_4^+ d_4$ and $s = d_1 - d_3 d_4^+ d_2$. We have the following decomposition

$$D = \begin{bmatrix} d_1 & d_3 \\ d_2 & d_4 \end{bmatrix} = \begin{bmatrix} 1 & d_3 d_4^+ \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s & d_3 f \\ e d_2 & d_4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ d_4^+ d_2 & 1 \end{bmatrix} =: PMQ.$$

We next discuss the inverse of $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ along a regular matrix D, under certain conditions.

Theorem 3.3. *Let* $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, $D = \begin{bmatrix} d_1 & d_3 \\ d_2 & d_4 \end{bmatrix} \in R_{2\times 2}$ *with* d_4 *regular. With the notations above, if* $d_3 f = 0$ *and* $a^{\| s}$ *exists, then* $A^{\| D}$ *exists if and only if* $\xi = \beta - \alpha a^{\| s}(a d_3 d_4^+ + c)$ *is invertible.*

In this case, $A^{\| D} = \begin{bmatrix} x_1 d_2 + x_2 s & x_1 d_4 \\ \xi^{-1}(d_2 - \alpha a^{\| s}) & \xi^{-1} d_4 \end{bmatrix}$, *where*

$$\begin{aligned} u &= sa + 1 - ss^+, \\ t &= ed_2(1 - s^+ s), \\ \alpha &= d_2 a + d_4 b - (1 - tt^-) e d_2 s^+, \\ \beta &= (d_2 a + d_4 b) d_3 d_4^+ + d_2 c + d_4 d + (1 - tt^-) e, \\ \xi &= \beta - \alpha a^{\| s}(a d_3 d_4^+ + c), \\ x_1 &= [(1 - a^{\| s} a) d_3 d_4^+ - a^{\| s} c] \xi^{-1}, \\ x_2 &= u^{-1} - x_1 \alpha u^{-1}. \end{aligned}$$

Proof. If $d_3 f = 0$, then $M = \begin{bmatrix} s & 0 \\ e d_2 & d_4 \end{bmatrix}$. Note that the regularity of D is equivalent to the regularity of M. Hence, it follows from [7, Theorem 1] that

$$I - MM^- = \begin{bmatrix} 1 - ss^+ & 0 \\ -(1 - tt^-) e d_2 s^+ & (1 - tt^-) e \end{bmatrix},$$

where $t = ed_2(1 - s^+ s)$.

Note that $MQAP = \begin{bmatrix} sa & s(ad_3d_4^+ + c) \\ d_2a + d_4b & (d_2a + d_4b)d_3d_4^+ + d_2c + d_4d \end{bmatrix}$. We have

$$U = MQAP + I - MM^- = \begin{bmatrix} u & s(ad_3d_4^+ + c) \\ \alpha & \beta \end{bmatrix},$$

where

$$\begin{aligned} u &= sa + 1 - ss^+, \\ \alpha &= d_2a + d_4b - (1 - tt^-)ed_2s^+, \\ \beta &= (d_2a + d_4b)d_3d_4^+ + d_2c + d_4d + (1 - tt^-)e. \end{aligned}$$

In this case,

$$U^{-1} = \begin{bmatrix} 1 & -a^{\|s}(ad_3d_4^+ + c) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u^{-1} & 0 \\ 0 & \xi^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\alpha u^{-1} & 1 \end{bmatrix},$$

where $\xi = \beta - \alpha a^{\|s}(ad_3d_4^+ + c)$.

By calculations, $A^{\|D} = PU^{-1}MQ = \begin{bmatrix} x_1d_2 + x_2s & x_1d_4 \\ \xi^{-1}(d_2 - \alpha a^{\|s}) & \xi^{-1}d_4 \end{bmatrix}$, where

$$\begin{aligned} x_1 &= [(1 - a^{\|s}a)d_3d_4^+ - a^{\|s}c]\xi^{-1}, \\ x_2 &= u^{-1} - x_1\alpha u^{-1}. \end{aligned}$$

The proof is completed. □

Remark 3.4. In Theorem 3.3, $A^{\|D}$ may exist and yet $d_3f \neq 0$ and $a^{\|s}$ exists.

Indeed, suppose that $R = \mathbb{Z}/6\mathbb{Z}$ and let $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, $D = \begin{bmatrix} d_1 & d_3 \\ d_2 & d_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \in R_{2\times 2}$. Note that A is invertible along D, using Corollary 2.5. From $d_4^+ = 2$, we have $f = 1 - d_4^+d_4 = 3$ and $d_3f = 3 \neq 0$. Note that $sa + 1 - ss^- = 5$ is invertible, from which a is invertible along s.

In Theorem 3.3, if d_4 is invertible, then $e = f = 0$. Hence, we have the following corollary.

Corollary 3.5. *Let $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, $D = \begin{bmatrix} d_1 & d_3 \\ d_2 & d_4 \end{bmatrix} \in R_{2\times 2}$ with d_4 invertible. If $a^{\|s}$ exists, then $A^{\|D}$ exists if and only if $\xi = \beta - \alpha a^{\|s}(ad_3d_4^{-1} + c)$ is invertible.*

In this case, $A^{\|D} = \begin{bmatrix} x_1d_2 + x_2s & x_1d_4 \\ \xi^{-1}(d_2 - \alpha a^{\|s}) & \xi^{-1}d_4 \end{bmatrix}$, *where*

$$\begin{aligned} s &= d_1 - d_3d_4^{-1}d_2, \\ u &= sa + 1 - ss^{+}, \\ \alpha &= d_2a + d_4b, \\ \beta &= \alpha d_3d_4^{-1} + d_2c + d_4d, \\ \xi &= \beta - \alpha a^{\|s}(ad_3d_4^{-1} + c), \\ x_1 &= [(1 - a^{\|s}a)d_3d_4^{-1} - a^{\|s}c]\xi^{-1}, \\ x_2 &= u^{-1} - x_1\alpha u^{-1}. \end{aligned}$$

In Theorem 3.3, take $d_3 = 0$, then $s = d_1$. We can get the formula and equivalence for the existence of the inverse along a regular lower triangular matrix obtained in [6].

Corollary 3.6. ([6, Theorem 3.1]) *Let* $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, $D = \begin{bmatrix} d_1 & 0 \\ d_2 & d_4 \end{bmatrix} \in R_{2\times 2}$ *with* d_4 *regular. With the notations above, if* $a^{\|d_1}$ *exists, then* $A^{\|D}$ *exists if and only if* $\xi = \beta - \alpha a^{\|d_1}c$ *is invertible.*

In this case, $A^{\|D} = \begin{bmatrix} a^{\|d_1} & -a^{\|d_1}c\xi^{-1}d_4 \\ \xi^{-1}(d_2 - \alpha a^{\|d_1}) & \xi^{-1}d_4 \end{bmatrix}$, *where*

$$\begin{aligned} u &= d_1a + 1 - d_1d_1^{+}, \\ t &= ed_2(1 - d_1^{+}d_1), \\ \alpha &= d_2a + d_4b - (1 - tt^{-})ed_2d_1^{+}, \\ \beta &= d_2c + d_4d + (1 - tt^{-})e, \\ \xi &= \beta - \alpha a^{\|d_1}c. \end{aligned}$$

By taking $ed_2 = 0$ in Theorem 3.3, we get the following corollary.

Corollary 3.7. *Let* $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, $D = \begin{bmatrix} d_1 & d_3 \\ d_2 & d_4 \end{bmatrix} \in R_{2\times 2}$ *with* d_4 *regular. With the notations above, if* $ed_2 = d_3f = 0$ *and* $a^{\|s}$ *exists, then* $A^{\|D}$ *exists if and only if* $\xi = \beta - \alpha a^{\|s}(ad_3d_4^{+} + c)$ *is invertible.*

In this case, $A^{\|D} = \begin{bmatrix} x_1d_2 + x_2s & x_1d_4 \\ \xi^{-1}(d_2 - \alpha a^{\|s}) & \xi^{-1}d_4 \end{bmatrix}$, *where*

$$\begin{aligned} u &= sa + 1 - ss^{+}, \\ \alpha &= d_2a + d_4b, \\ \beta &= \alpha d_3d_4^{+} + d_2c + d_4d + e, \\ \xi &= \beta - \alpha a^{\|s}(ad_3d_4^{+} + c), \\ x_1 &= [(1 - a^{\|s}a)d_3d_4^{+} - a^{\|s}c]\xi^{-1}, \\ x_2 &= u^{-1} - x_1\alpha u^{-1}. \end{aligned}$$

Question 3.8. Given a regular matrix D, can we give further equivalent conditions such that $A^{\|D}$ exists without additional conditions ?

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