The group inverse of a product*

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Abstract

In this paper, we characterize the existence and give an expression of the group inverse of a product of two regular elements by means of a ring unit.

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1 Introduction

In this paper, we consider elements on a general (associative) ring R with unity 1. We will follow the standard notation regarding generalized inverses. We say a is regular if $a \in aRa$. In this case, a particular solution to $axa = a$, called von Neumann inverse of a, is denoted by a^-. A reflexive inverse of a, denoted by a^+, is a common solution to $axa = a, x = xax$. A regular element a has a reflexive a^+, namely $a^-aa^=$, for any choice of von Neumann inverses $a^-, a^=$.

We say a is group invertible if there is a common solution to $axa = a, xax = x, ax = xa$. It is well known that such a solution is unique in case it exists. It is denoted by $a^\#$.

Our main goal is to characterize the group inverse of a product of regular elements, as well as to derive an expression of such a group inverse that does not rely on the knowledge of von Neumann regularity of the product.

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2 Main result

Let a, b be regular elements in R, with reflexive inverses a^+, b^+, respectively. Let also

$$w = (1 - bb^+)(1 - a^+a)$$

which we will assume to be regular in R.

Note that the regularity of w does *not* depend on the choices of a^+ and b^+. That is to say, if w is regular for a particular choice of a^+ and of b^+, then it must be regular for *all* choices of a^+ and b^+. This can be easily proved by noting that w being regular is equivalent to the regularity of the matrix $\begin{bmatrix} a & 0 \\ 1 & b \end{bmatrix}$, using [6], which it turn is equivalent to $(1 - bb^{=})(1 - a^{=}a)$ being regular, for any other choices of von Neumann inverses $a^{=}$ and $b^{=}$ of a and b.

Consider the matrix $M = \begin{bmatrix} ab & a \\ 0 & 1 \end{bmatrix} = AQ$ with $A = \begin{bmatrix} a & 0 \\ 1 & -b \end{bmatrix}, Q = \begin{bmatrix} b & 1 \\ 1 & 0 \end{bmatrix}$.

It is well known that $M^\#$ exists if and only if $(ab)^\#$ exists, using [1]. Furthermore, the (1,1) entry of $M^\#$ equals $(ab)^\#$. Also, $M^\#$ exists if and only if $U = AQ + I - AA^-$ is invertible, see [7], [5], in which case $(AQ)^\# = U^{-2}(AQ)$.

As $AQ + I - AA^- = A(Q - A^-) + I$ then $AQ + I - AA^-$ is invertible if and only if $(Q - A^-)A + I = QA + I - A^-A$ is invertible, using Jacobson's Lemma, which in turn means $(QA)^\#$ exists. Therefore, by considering the matrix $W = QA = \begin{bmatrix} ba + 1 & -b \\ a & 0 \end{bmatrix}$, then $(ab)^\#$ exists if and only if W is group invertible.

Using [4], the matrix W is group invertible if and only if

$$\begin{aligned} z &= (1 + ba)(1 - a^+a) + ba + (1 - ww^-)(1 - bb^+)(1 + ba) \\ &= 1 - a^+a + ba + (1 - ww^-)(1 - bb^+) \end{aligned}$$

is a unit.

We have, hence, the equivalence

$$(ab)^\# \text{ exists if and only if } 1 - a^+a + ba + (1 - ww^-)(1 - bb^+) \text{ is a unit.}$$

Using the expression presented in [4] does not give a tratable algorithm to actually compute $(ab)^\#$. We will, therefore, pursue a different strategy and compute the (1,1) entry of $M^\#$.

Recall that for $M = AQ$ and Q invertible, the group inverse of M exists if and only if $U = AQ + I - AA^-$ is invertible. For $A = \begin{bmatrix} a & 0 \\ 1 & -b \end{bmatrix}$, there exists A^- for which

$$AA^- = \begin{bmatrix} aa^+ & 0 \\ -(1 - ww^-)(1 - bb^+)a^+ & bb^+ + ww^-(1 - bb^+) \end{bmatrix},$$

using [6].

The matrix U then becomes

$$U = \begin{bmatrix} ab + 1 - aa^+ & a \\ (1 - ww^-)(1 - bb^+)a^+ & 2 - bb^+ - ww^-(1 - bb^+) \end{bmatrix}.$$

Multiplication on the right by $K = \begin{bmatrix} 1 & 0 \\ a^+ - b & 1 \end{bmatrix}$ gives

$$G = \begin{bmatrix} 1 & a \\ \alpha & 2 - bb^+ - ww^-(1 - bb^+) \end{bmatrix},$$

where

$$\begin{aligned} \alpha &= (1 - ww^-)(1 - bb^+)a^+ + \left(2 - bb^+ - ww^-(1 - bb^+)\right)(a^+ - b) \\ &= a^+ - b + 2(1 - ww^-)(1 - bb^+)a^+, \end{aligned}$$

as $(1 - bb^+)b = 0$.

We are left with showing when is G invertible. We do so using the Schur complement on the (1,1) entry. This Schur complement equals

$$\begin{aligned} G/I &= (2 - bb^+ - ww^-(1 - bb^+)) - ((1 - ww^-)(1 - bb^+)a^+ + \\ &\quad + \left(2 - bb^+ - ww^-(1 - bb^+)\right)(a^+ - b))a \\ &= \left(2 - bb^+ - ww^-(1 - bb^+)\right)(1 - a^+a + ba) - (1 - ww^-)(1 - bb^+)a^+a \\ &= (1 + (1 - ww^-)(1 - bb^+))(1 - a^+a + ba) - (1 - ww^-)(1 - bb^+)a^+a \\ &= 1 - a^+a + ba + (1 - ww^-)(1 - bb^+)(1 - 2a^+a) \\ &= 1 - a^+a + ba + (1 - ww^-)(1 - bb^+)(1 - a^+a) + \\ &\quad +(1 - ww^-)(1 - bb^+)a^+a \\ &= 1 - a^+a + ba + (1 - ww^-)(1 - bb^+) \end{aligned}$$

This gives, and as previously shown,

$(ab)^\#$ exists if and only if $z = 1 - a^+a + ba + (1 - ww^-)(1 - bb^+)$ is a unit.

As a side note, we construct another unit associated with z, namely we may show that $z = 1 - a^+a + ba + (1 - ww^-)(1 - bb^+)$ is a unit if and only if $z' = 1 - aa^+ + ab - a(1 - ww^-)(1 - bb^+)a^+$ is a unit. This follows by the sequence of identities $(1 - ww^-)(1 - bb^+) = (1-ww^-)(1-bb^+)(1-a^+a+a^+a) = (1-ww^-)(1-bb^+)a^+a$ together with Jacobson's Lemma.

We remark that given a reflexive inverse w^+ of w, the element $\tilde{w} = (1-a^+a)w^+(1-bb^+)$ is an idempotent reflexive inverse of w. As such z and z' simplify to $1-a^+a+ba+1-bb^+-w\tilde{w}$ and $1 + ab - abb^+a^+ - aw\tilde{w}a^+$, respectively.

We know, using [5, Corollary 3.3(4)], that $(AQ)^\#$ exists if and only if U is invertible, in which case $(AQ)^\# = U^{-2}(AQ)$. The matrices U and G are equivalent, and we are able to relate their inverses by means of the matrix K. Indeed, since $G = UK$, then $U^{-1} = KG^{-1}$. Firstly, we need to compute the inverse of G, for which we will use the following known result:

Lemma 2.1.

$$\begin{bmatrix} 1 & y \\ x & z \end{bmatrix}^{-1} = \begin{bmatrix} 1 + y\zeta^{-1}x & -y\zeta^{-1} \\ -\zeta^{-1}x & \zeta^{-1} \end{bmatrix},$$

where $\zeta = z - yx$ *is the Schur complement.*

Our purpose is to derive an expression for $(ab)^{\#}$, which equals the (1,1) entry of $M^{\#}$.

The (1,1) entry of $M^{\#}$ is obtained by multiplying the first row of U^{-2} by the first column of AQ, which is $\begin{bmatrix} ab \\ 0 \end{bmatrix}$. So, in fact we just need the (1,1) entry of U^{-2}, which is then multiplied on the right by ab to give $(ab)^{\#}$.

We recall that $G = UK$, where $K = \begin{bmatrix} 1 & 0 \\ a^+ - b & 1 \end{bmatrix}$, which gives $U^{-1} = KG^{-1}$ and $U^{-2} = KG^{-1}KG^{-1}$. Pre-multiplication with K does not afect the first row, and so we just need the (1,1) element of $G^{-1}KG^{-1}$. Calculations show that

$$U^{-2} = (KG^{-1})^{-2} = \begin{bmatrix} (1 + az^{-1}\alpha)^2 - az^{-1}(a^+ - b)(1 + az^{-1}\alpha) + az^{-2}\alpha & ? \\ ? & ? \end{bmatrix}$$

We will need the simplification

$$b - zb = a^+ab - bab, \tag{1}$$

from where we obtain

$$\begin{aligned} \alpha ab &= (a^+ - b)ab = b - zb \\ (1 + az^{-1}\alpha)ab &= az^{-1}b. \end{aligned}$$

Indeed, $\alpha ab = aa^+ab - bab + 2(1 - ww^-)(1 - bb^+)a^+ab$ whose last summand can be expressed as $2(1 - ww^-)(1 - bb^+)a^+ab = -2(1 - ww^-)(1 - bb^+)(1 - a^+a - 1)b = -2(1 - ww^-)w + 2(1 - ww^-)(1 - bb^+)b = 0$, and therefore $\alpha ab = a^+ab - bab$.

Therefore,

$$(AQ)^{\#} = \begin{bmatrix} (ab)^{\#} & ? \\ 0 & ? \end{bmatrix} = \begin{bmatrix} \left((1 + az^{-1}\alpha)^2 - az^{-1}(a^+ - b)(1 + az^{-1}\alpha) + az^{-2}\alpha\right) ab & ? \\ 0 & ? \end{bmatrix}$$

from which we obtain the general formula

$$\begin{aligned} (ab)^{\#} &= \left((1 + az^{-1}\alpha)^2 - az^{-1}(a^+ - b)(1 + az^{-1}\alpha) + az^{-2}\alpha\right) ab \\ &= ab + 2(az^{-1}b - ab) + az^{-1}\alpha(az^{-1}b - ab) - \\ &\quad az^{-1}(a^+ - b)az^{-1}b + az^{-1}(z^{-1}b - b) \\ &= \left(az^{-1}\alpha az^{-1}b - az^{-1}(a^+ - b)az^{-1}b\right) + az^{-2}b \\ &= 2az^{-1}(1 - ww^-)(1 - bb^+)z^{-1}b + az^{-2}b \\ &= 2az^{-1}b - 2\left(az^{-1}b\right)^2 + az^{-2}b. \end{aligned}$$

From $1 - a^+a = z^{-1} - z^{-1}a^+a$ we obtain, by post-multiplying by b,

$$b - a^+ab = z^{-1}b - z^{-1}a^+ab \tag{2}$$

which implies

$$az^{-1}b = az^{-1}a^+ab. \tag{3}$$

Now, from (1) we have $z^{-1}b = b + z^{-1}a^+ab - z^{-1}bab$ which implies, using (2), that

$$z^{-1}bab = a^+ab \tag{4}$$

which in turns delivers

$$ab = az^{-1}bab. \tag{5}$$

Using (4) and (5), together with $(ab)^\# = 2az^{-1}b - 2\left(az^{-1}b\right)^2 + az^{-2}b$, we write the idempotent $(ab)^\#(ab)$ as

$$\begin{aligned} (ab)^\# ab &= 2az^{-1}b - 2\left(az^{-1}b\right)^2 + az^{-2}b \\ &= 2az^{-1}bab - 2az^{-1}bab + az^{-1}z^{-1}bab \\ &= az^{-1}a^+ab \end{aligned}$$

Using (3), this equals $az^{-1}b$ and therefore $az^{-1}b$ is an idempotent, the unit of the group generated by ab. This simplifies the expression of $(ab)^\#$ to

$$(ab)^\# = az^{-2}b.$$

It comes with no surprise that the expression of $(ab)^\#$ is of the form aXb, for a suitable X.

We have, from the above, our main result:

Theorem 2.2. *Let a, b be regular elements in R with reflexive inverses a^+ and b^+, respectively. Assume, also, that $w = (1 - bb^+)(1 - a^+a)$ is regular. Then $(ab)^\#$ exists if and only if $z = 1 - a^+a + ba + (1 - ww^-)(1 - bb^+)$ is a unit. In this case,*

$$(ab)^\# = az^{-2}b.$$

3 Special cases

On ag where g is a unit

We consider a special instance of a product, where one element is a unit. Precisely, let $a, g \in R$ with g a unit. In [5, Corollary 3.2], the existence of the group inverse of ag was related to the existence of a unit, and an expression of $(ag)^\#$ was given. By Theorem 2.2, we know that

$(ag)^\#$ exists if and only if $z = 1 - a^+a + ga + (1 - ww^-)(1 - gg^+)$ is a unit. But $g^+ = g^{-1}$ so that $(ag)^\#$ exists if and only if $z = 1 - a^+a + ga$ is a unit, which is equivalent to the criterion of [5, Corollary 3.2] by Jacobson's Lemma. However, Theorem 2.2 gives us

$$(ag)^\# = a(1 - a^+a + ga)^{-2}g$$

which is different from [5, Corollary 3.2]. On the other hand, we can also consider the case of a group inverse of ga. In this case the w in the previous theorems is 0. $(ga)^\#$ exists if and only if $ag - 1 + aa^+$ is a unit, and

$$(ga)^\# = g(ag - 1 + aa^+)^{-2}a.$$

As the existence of $(ag)^\#$ is equivalent with the existence of $(ga)^\#$ (and using $-g$ instead of g in the second case), we get:

Corollary 3.1. *The following conditions are equivalent:*

1. *ag is group invertible;*
2. *ga is group invertible;*
3. *$z = 1 - a^+a + ga$ is a unit;*
4. *$\eta = 1 - aa^+ + ag$ is a unit;*

In which case:

$$(ag)^\# = az^{-2}g \text{ and } (ga)^\# = g\eta^{-2}a.$$

The units in the previous corollary are strongly related to the existence of the inverse of g along a, see [3].

As a special case, when $g = 1$, we recover the classical result ([5, Corollary 3.3.]):

Corollary 3.2. *The following conditions are equivalent:*

1. *a is group invertible;*
2. *$z = 1 - a^+a + a$ is a unit;*
3. *$\eta = 1 - aa^+ + a$ is a unit;*

In which case:

$$(a)^\# = az^{-2} \text{ and } (a)^\# = \eta^{-2}a.$$

On the sum

We now apply the results of the previous section to the sum of ring elements to obtain a known characterization of the group inverse of a sum.

Let a, b be ring elements such that $a+b$ is regular. Consider $A = \begin{bmatrix} a & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ b & 0 \end{bmatrix}$, for which $AB = \begin{bmatrix} a+b & 0 \\ 0 & 0 \end{bmatrix}$. This is a key factorization that allows us to characterize the group inverse of $a+b$ by the group inverse of AB.

Using the results on the previous section, $(AB)^{\#}$ exists if and only if $H = I - A^{+}A + BA - (I - WW^{-})(I - BB^{+})$ is invertible. We will now undertake the computation of this matrix, for particular choices of inner and reflexive inverses.

We will take $A^{+} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $B^{+} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, which will deliver $A^{+}A = \begin{bmatrix} 0 & 0 \\ a & 1 \end{bmatrix}$ and $BB^{+} = \begin{bmatrix} 1 & 0 \\ b & 0 \end{bmatrix}$.

Also, $BA = \begin{bmatrix} a & 1 \\ ba & b \end{bmatrix}$ and $W = (I - BB^{+})(I - A^{+}A) = \begin{bmatrix} 0 & 0 \\ -a-b & 0 \end{bmatrix}$. Since $a+b$ is regular, then W is regular and we may take $W^{-} = \begin{bmatrix} 0 & (-a-b)^{-} \\ 0 & 0 \end{bmatrix}$. The associated idempotents take the form $WW^{-} = \begin{bmatrix} 0 & 0 \\ 0 & (a+b)(a+b)^{-} \end{bmatrix}$ and $I - WW^{-} = \begin{bmatrix} 1 & 0 \\ 0 & 1-(a+b)(a+b)^{-} \end{bmatrix}$.

We therefore obtain

$$(I - WW^{-})(I - BB^{+}) = \begin{bmatrix} 0 & 0 \\ -(1-(a+b)(a+b)^{-})b & 1-(a+b)(a+b)^{+} \end{bmatrix}.$$

The invertible matrix takes the form

$$\begin{bmatrix} 1+a & 1 \\ -a+ba+1(1-(a+b)(a+b)^{+})b & b-1+(a+b)(a+b)^{+} \end{bmatrix},$$

whose Schur complement equals $(a+b)+1-(a+b)^{+}(a+b)$ and which has to be a ring unit. This is coherent with the known result.

The example of a trace product

Let a, b be two elements of the ring R such that $Rab = Rb$ and $abR = aR$ (one says that ab is a trace product). Then it is known, by a theorem of Clifford ([2, Proposition 2.3.7]), that this is equivalent with the existence of an idempotent $e \in R$ such that $Ra = Re$ and $bR = eR$. In particular, a and b are von Neumann regular and we can find a^{+} and b^{+} such that $a^{+}a = bb^{+} = e$. For this particular choice, $w = (1-bb^{+})(1-a^{+}a) = (1-e)$ is idempotent, and the element $z = 1 - a^{+}a + ba + (1 - ww^{-})(1 - bb^{+})$ reduces to $z = 1 - a^{+}a + ba$. As the invertibility of z implies the invertibility of all the elements of the form $1-a^+a+ba$, we get by Theorem 2.2:

Corollary 3.3. *Let ab be a trace product. Then a and b are regular, and for any choice of a^+, ab is group invertible if and only if $z=1-a^+a+ba$ is a unit. In this case,*

$$(ab)^{\#}=a(1-a^+a+ba)^{-2}b.$$

Once again, we recognize the criterion of invertibility of b along a.

4 Final remarks

1. In this paper, we are primarly interested in the group invertibility (on the group inverse) of the product ab. Since the group inverse of ab equals $az^{-2}b$, what can be said about ba?

   If z^{-1} is an inner inverse, and since ab is group invertible, then $1+ab-(ab)(ab)^{\#}$ is a unit, that is $1+ab-az^{-1}b$ is a unit. By Jacobson lemma, $1+ba-baz^{-1}$ is a unit, and if z^{-1} is an inner inverse of ba, then ba is actually group invertible.

   Let us compute $z(baz^{-1}ba-ba)$.

$$\begin{aligned} z(baz^{-1}ba-ba) &= \left(1-a^+a+ba+(1-ww^-)(1-bb^+)\right)(baz^{-1}ba-ba) \\ &= \left(1-a^+a+ba\right)(baz^{-1}ba-ba) \text{ as } (1-bb^+)b=0 \\ &= (baz^{-1}ba-ba)+(-a^++b)\left(abaz^{-1}ba-aba\right) \end{aligned}$$

   But equation (5) gives $ab=az^{-1}bab$ and since we have shown that $az^{-1}b=ab(ab)^{\#}$, then it commutes with ab and $ab=abaz^{-1}b$. This gives that the second term in the above sum is 0, and $z(baz^{-1}ba-ba)=(baz^{-1}ba-ba)$, or equivalently, $(1-z)(baz^{-1}ba-ba)=0$. Multiplying by z^{-1} on the left gives $(baz^{-1}ba-ba)=z^{-1}baz^{-1}ba-z^{-1}ba$ or

$$(z^{-1}-1)(baz^{-1}ba-ba) = 0. \tag{6}$$

   Note that $1-z^{-1}$ is a unit if and only if $1-z$ is a unit. Indeed, setting $\chi=1-z$ then $1=z^{-1}-z^{-1}\chi=z^{-1}-\chi z^{-1}$ which imply $z^{-1}-1=z^{-1}\chi=\chi z^{-1}$ and $\chi=1-z$ is a unit. Conversely, if $\chi=1-z$ is a unit then $z^{-1}-1=z^{-1}\chi$ we obtain the desired conclusion.

   Suppose $1-z$ is a unit. Then z^{-1} is a inner inverse of ba (and $z^{-1}ba$ is idempotent). Of course, if $z^{-1}ba$ is idempotent then z^{-1} is always an inner inverse of ba ($ba=zz^{-1}ba=zz^{-1}baz^{-1}ba=baz^{-1}ba$.)

2. Consider $(G_z, .)$ the group generated by z, and $(G_{ab}, .)$ the group generated by ab. Then we can prove that $\phi : (G_z, .) \to (G_{ab}, .)$ defined by $\phi(z^k) = az^{k-1}b$ is an isomorphism, or put in an other form: $az^n b = (ab)^{n+1}$ for all $n \in \mathbb{Z}$.

   This can be proved by induction on $\mathbb{Z}$. First, we have proved in our paper that $z(b - a^+ab) = b - a^+ab$ and $z^{-1}bab = a^+ab$. By induction on $\mathbb{Z}$, this imply that $z^n(b - a^+ab) = b - a^+ab$ for all $n \in \mathbb{Z}$ hence $az^n b = az^n a^+ab = az^{n-1}bab$ for all $n \in \mathbb{Z}$. Now our equation is true for $n = 0$. Assume it is true for $n \in \mathbb{N}$. Then $az^{n+1}b = az^n bab = (ab)^{n+1}$. Suppose now it is true for $-n \in \mathbb{N}$, that is $az^{-n}b = (ab)^{1-n}$. Then we prove that $az^{-n-1}b$ is the group inverse of $(ab)^n$. We have $az^{-n-1}b(ab)^n = az^{-n-1}b(ab)(ab)^{n-1} = az^n b(ab)^{n-1} = (ab)^0$. As symmetrically, $(ab)^n az^{-n-1}b = (ab)^0$, then $az^{-n-1}b$ is the group inverse of $(ab)^n$.

3. We can compute other units by duality, using the opposite ring $(R, \times)$, $x \times y = yx$. Precisely, ab is group invertible in $(R, .)$ if and only if $b \times a$ is group invertible in $(R, \times)$, and by our theorem this happens if and only if $1 - b^+ \times b + a \times b + (1 - w \times w^-) \times (1 - a \times a^+)$ (if a^+ and b^+ are inverses in $(R, .)$ then they are also inverses in $(R, \times)$), with $w = (1 - a \times a^+) \times (1 - b^+ \times b)$, or equivalently if and only if $t = 1 - bb^+ + ba + (1 - a^+a)(1 - w^-w)$ is a unit with $w = (1 - bb^+)(1 - a^+a)$ (classical one). Note that this unit is *a priori* different from the other ones in the paper. If we continue the duality principle, we end with $b \times t^{-2} \times a$ is the group inverse of $b \times a$, that is $at^{-2}b$ is the group inverse of ab, so that this unit works equivalently as our z.

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