ELEMENTS OF RINGS WITH EQUAL SPECTRAL IDEMPOTENTS

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Abstract

In this paper we define and study a generalized Drazin inverse x^{D} for ring elements x, and give a characterization of elements a, b for which $aa^{\mathrm{D}} = bb^{\mathrm{D}}$. We apply our results to the study of EP elements of a ring with involution.

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1. Introduction

This paper is motivated by a recent work of Castro et al. [2], which investigates the necessary and sufficient conditions for square complex matrices A, B to have the same eigenprojection at 0. This problem, under more restrictive conditions on A, B was first considered by Hartwig [7] more than 20 years ago.

The formulation of the problem for elements of rings requires the definition of an appropriate analogue of the eigenprojection, the so-called spectral idempotent, well known in the case of Banach algebras. We also define and investigate a generalized Drazin inverse for elements of rings that possess a spectral idempotent. The main result of this paper is a characterization of ring elements with equal spectral idempotents.

In rings with involution we can define the Moore–Penrose inverse and EP elements, that is, ring elements for which the Drazin and Moore–Penrose inverse exist and coincide. We give a new characterization of EP elements based on our main theorem.

2. Quasipolar elements in rings

In this paper 'ring' means an associative ring with unit $1 \neq 0$. Let $\mathcal{R}$ be a ring. The group of invertible elements is denoted by $\mathcal{R}^{-1}$.

For any element $a \in \mathcal{R}$ we define the *commutant* and the *double commutant* of a by

$$\begin{aligned} \operatorname{comm}(a) &= \{x \in \mathcal{R} : ax = xa\}, \\ \operatorname{comm}^2(a) &= \{x \in \mathcal{R} : xy = yx \text{ for all } y \in \operatorname{comm}(a)\}. \end{aligned}$$

The Jacobson *radical* of $\mathcal{R}$ is the two-sided ideal

$$\mathcal{R}^{\mathrm{rad}} = \{a \in \mathcal{R} : 1 + \mathcal{R}a \subset \mathcal{R}^{-1}\}.$$

DEFINITION 2.1. (Harte [5].) An element $a \in \mathcal{R}$ is *quasinilpotent* if, for every $x \in \operatorname{comm}(a)$, $1 + xa \in \mathcal{R}^{-1}$. The set of all quasinilpotent elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^{\mathrm{qnil}}$. The set of all nilpotent elements will be written as $\mathcal{R}^{\mathrm{nil}}$.

Clearly, $\mathcal{R}^{\mathrm{rad}} \subset \mathcal{R}^{\mathrm{qnil}}$. Further, $\mathcal{R}^{\mathrm{nil}} \subset \mathcal{R}^{\mathrm{qnil}}$ as

$$(1 + xa)^{-1} = \sum_{i=0}^{k-1} (-1)^i x^i a^i$$

if $a \in \mathcal{R}$ is nilpotent of index k and $x \in \operatorname{comm}(a)$ (see also [5, Theorems 3 and 4]). We note that in a ring, unlike in a Banach algebra, the sum of two commuting quasinilpotent elements need not be quasinilpotent. However, we have the following implication:

$$a \in \mathcal{R}^{-1} \text{ and } b \in \mathcal{R}^{\mathrm{qnil}} \cap \operatorname{comm}(a) \implies a + b \in \mathcal{R}^{-1}. \tag{2.1}$$

For a Banach algebra $\mathcal{R}$ it is well known [4, p. 251] that

$$a \in \mathcal{R}^{\mathrm{qnil}} \iff \lim_{n\to\infty} \|a^n\|^{1/n} = 0.$$

DEFINITION 2.2. An element $a \in \mathcal{R}$ is *quasipolar* if there exists $p \in \mathcal{R}$ such that

$$p^2 = p, \qquad p \in \operatorname{comm}^2(a), \qquad ap \in \mathcal{R}^{\mathrm{qnil}}, \qquad a + p \in \mathcal{R}^{-1}. \tag{2.2}$$

If a is quasipolar and $ap \in \mathcal{R}^{\mathrm{nil}}$ with the nilpotency index k, we say that a is *polar* of order k. Any idempotent p satisfying the above conditions is called a *spectral idempotent* of a. (The term 'quasipolar' comes from [5], and 'spectral' idempotent is borrowed from spectral theory in Banach algebras [4]. We shall see later that quasipolar elements are exactly the ones which are 'generalized Drazin invertible'—Theorem 4.2.)

PROPOSITION 2.3. *Any quasipolar element $a \in \mathcal{R}$ has a unique spectral idempotent denoted by a^π.*

PROOF. Suppose that p, q are spectral idempotents of a quasipolar element $a \in \mathcal{R}$. Then

$$\begin{aligned} 1-(1-p)q &= 1-(1-p)(a+p)^{-1}(a+p)q \\ &= 1-(1-p)(a+p)^{-1}aq = 1-b(aq). \end{aligned}$$

Since $p \in \mathrm{comm}^2(a)$, we have $b \in \mathrm{comm}(aq)$; $aq \in \mathcal{R}^{\mathrm{qnil}}$ implies $1-b(aq) \in \mathcal{R}^{-1}$. Then

$$1-(1-p)q = 1-(1-p)^2q^2 = (1-(1-p)q)(1+(1-p)q).$$

The invertibility of $1-(1-p)q$ implies that $(1-p)q=0$, that is, $q=pq$. Similarly we prove that $(1-q)p \in \mathcal{R}^{-1}$, and $p=qp=pq$. Then $p=q$. □

REMARK 2.4. From [8, Theorem 3.2] it follows that the condition $a+p \in \mathcal{R}^{-1}$ in (2.2) can be replaced by $1-p \in (\mathcal{R}a) \cap (a\mathcal{R})$.

The uniqueness of the spectral idempotent is used to prove the following result valid in rings with involution (see Section 5).

PROPOSITION 2.5. *Let $\mathcal{R}$ be a ring with involution. Then a is quasipolar if and only if a^* is quasipolar. In this case and $(a^*)^\pi = (a^\pi)^*$.*

PROOF. From $a+a^\pi \in \mathcal{R}^{-1}$ and $aa^\pi = a^\pi a \in \mathcal{R}^{\mathrm{qnil}}$ we obtain $a^*+(a^\pi)^* \in \mathcal{R}^{-1}$ and $a^*(a^*)^\pi = (a^*)^\pi a^* \in \mathcal{R}^{\mathrm{nil}}$ by applying the involution. □

For polar elements we can relax the condition that p double commutes with a:

PROPOSITION 2.6. *Let $a \in \mathcal{R}$, and let $p \in \mathcal{R}$ be such that*

$$p^2 = p, \qquad p \in \mathrm{comm}(a), \qquad ap \in \mathcal{R}^{\mathrm{nil}}, \qquad a+p \in \mathcal{R}^{-1}. \tag{2.3}$$

Then a is polar and $p = a^\pi$.

PROOF. Since $\mathcal{R}^{\mathrm{nil}} \subset \mathcal{R}^{\mathrm{qnil}}$, we only need to prove that $p \in \mathrm{comm}^2(a)$. For $ap \in \mathcal{R}^{\mathrm{nil}}$ there exists $k \in \mathbb{N}$ such that $(ap)^k = a^kp = 0$. Set $b = (a+p)^{-1}(1-p)$; then $ab=ba=1-p$. Let $x \in \mathrm{comm}(a)$. We have

$$\begin{aligned} (1-p)x-(1-p)x(1-p) &= (1-p)xp = (1-p)^kxp \\ &= b^ka^kxp = b^kxa^kp = 0, \end{aligned}$$

which implies $xp = pxp$. Similarly we show that $x(1-p)-(1-p)x(1-p)=0$, and $px = pxp$. This proves $px = xp$, and $p \in \mathrm{comm}^2(a)$. □

Observe that in general double commutativity of p with a is necessary.

3. Results on regular elements of rings

An element $a \in \mathcal{R}$ is *regular* (in the sense of von Neumann) if it has an *inner inverse* x, that is, if there exists $x \in \mathcal{R}$ such that $axa = a$. Any inner inverse of a will be denoted by a^-. The set of all regular elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^-$. Given $a \in \mathcal{R}$, we define the sets

$$a\mathcal{R} = \{ax : x \in \mathcal{R}\}, \qquad \mathcal{R}a = \{xa : x \in \mathcal{R}\},$$
$$a^0 = \{y \in \mathcal{R} : ay = 0\}, \qquad {}^0a = \{y \in \mathcal{R} : ya = 0\},$$

where $a\mathcal{R}$ and $\mathcal{R}a$ can be considered as finitely generated $\mathcal{R}$-modules; the same is true of a^0 and ${}^0a$ if $a \in \mathcal{R}^-$ (see Proposition 3.1 below). When considering a matrix A, these sets reflect, respectively, the column space of A, the row space of A, the kernel of A, and the kernel of A^{T}. However, we will work with these sets with no reference to rank, dimensional analysis or orthogonality. If $M \subset \mathcal{R}$, we can define

$$M\mathcal{R} = \{mx : m \in M,\ x \in \mathcal{R}\}, \qquad M^0 = \{x \in \mathcal{R} : Mx = \{0\}\};$$

similarly we define $\mathcal{R}M$ and ${}^0M$.

Some properties of these sets, established by Hartwig in [6, Proposition 6], will be needed in the following section. We include proofs for the sake of completeness.

PROPOSITION 3.1. *Given a, $b \in \mathcal{R}^-$ and A, $B \subset \mathcal{R}$, we have*

(i) $(1 - a^-a)\mathcal{R} = a^0$;

(ii) $a^0 = (\mathcal{R}a)^0$;

(iii) $\mathcal{R}a = {}^0(a^0) = {}^0((\mathcal{R}a)^0)$;

(iv) $A \subset B \implies {}^0A \supset {}^0B$.

PROOF. (i) As $a((1 - a^-a)y) = 0$, we have $(1 - a^-a)y \in a^0$. Conversely, if $ax = 0$, then $(1 - a^-a)x = x$ which implies $x \in (1 - a^-a)\mathcal{R}$.

(ii) Clearly, $a^0 \subset (\mathcal{R}a)^0$. The reverse inclusion is immediate when we take $x = 1$ in $(\mathcal{R}a)^0 = \{y \in \mathcal{R} : xay = 0 \text{ for all } x \in \mathcal{R}\}$.

(iii) Let $ya \in \mathcal{R}a$. Then $yax = 0$ for any $x \in a^0$, and $ya \in {}^0(a^0)$. Hence $\mathcal{R}a \subset {}^0(a^0)$.

Conversely let $y \in {}^0(a^0)$. Then $yx = 0$ for all $x \in a^0$. As $y = ya^-a + y(1 - a^-a)$ and $1 - a^-a \in a^0$ by (i) above, we have $y(1 - a^-a) = 0$, and $y = ya^-a \in \mathcal{R}a$. This proves ${}^0(a^0) \subset \mathcal{R}a$.

(iv) is obvious. $\square$

4. The g-Drazin inverse in rings

The original definition of the 'pseudoinverse' was given by Drazin [3] for elements of semigroups and polar elements of rings. It was generalized by Harte [5] to quasipolar elements, and studied by the first author in [8] in Banach algebras. In this section we survey the properties of the generalized Drazin inverse (called g-Drazin inverse) for quasipolar elements of rings; many of the results will appear in this setting for the first time.

Definition 4.1. An element $a \in \mathcal{R}$ is *generalized Drazin invertible* (or *g-Drazin invertible* for short) if there exists $b \in \mathcal{R}$ such that

$$b \in \mathrm{comm}^2(a), \qquad ab^2 = b, \qquad a^2b - a \in \mathcal{R}^{\mathrm{qnil}}. \tag{4.1}$$

Any element $b \in \mathcal{R}$ satisfying these conditions is a *g-Drazin inverse* of a. We denote the set of all g-Drazin invertible elements of $\mathcal{R}$ by $\mathcal{R}^{\mathrm{gD}}$. If a^2b-a in the above definition is nilpotent, then a is called *Drazin invertible* and b is called a *Drazin inverse* of a. The set of all Drazin invertible elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^{\mathrm{D}}$. The following result ensures that these concepts are well-defined.

Theorem 4.2. *An element $a \in \mathcal{R}$ is g-Drazin invertible if and only if a is quasipolar. In this case $a \in \mathcal{R}$ has a unique g-Drazin inverse a^{D} given by the equation*

$$b = (a + a^\pi)^{-1}(1 - a^\pi) = (1 - a^\pi)(a + a^\pi)^{-1}. \tag{4.2}$$

Proof. Suppose first that a is quasipolar with the spectral idempotent p, and set $b = (a+p)^{-1}(1-p)$. Then $b \in \mathrm{comm}^2(a)$. Further,

$$ab^2 = a(1-p)(a+p)^{-2} = (a+p)(1-p)(a+p)^{-2} = (1-p)(a+p)^{-1} = b,$$

and

$$\begin{aligned} a^2b - a &= a^2(1-p)(a+p)^{-1} - a \\ &= a(a+p)(a+p)^{-1}(1-p) - a \\ &= -ap \in \mathcal{R}^{\mathrm{qnil}}. \end{aligned}$$

Conversely assume that a is g-Drazin invertible with a g-Drazin inverse b, and set $p = 1 - ab$. Then $p \in \mathrm{comm}^2(a)$, and

$$(1-p)^2 = a^2b^2 = a(ab^2) = ab = 1 - p,$$

which implies $p^2 = p$. Finally, to prove that $a + p \in \mathcal{R}^{-1}$, we observe that

$$(a+p)(b+p) = ab + ap + bp + p = 1 - p + ap + p = 1 + ap \in \mathcal{R}^{-1} \tag{4.3}$$

as $bp = b(1-ab) = b - ab^2 = 0$. From $(a+p)b = ab+pb = 1-p+pb = 1-p$ it follows that $b = (a+p)^{-1}(1-p)$. The uniqueness of the spectral idempotent of a proves the uniqueness of the g-Drazin inverse b. □

The preceding theorem together with Proposition 2.5 implies the following result valid in rings with involution (see Section 5).

PROPOSITION 4.3. *Let $\mathcal{R}$ be a ring with involution. Then a is g-Drazin invertible if and only if a^* is g-Drazin invertible. In this case $(a^*)^{\mathrm{D}} = (a^{\mathrm{D}})^*$.*

DEFINITION 4.4. The g-Drazin index $i(a)$ of a quasipolar element $a \in \mathcal{R}$ is defined by

$$i(a) = \begin{cases} 0 & \text{if } a \in \mathcal{R}^{-1}, \\ k & \text{if } a^2b - a \text{ is nilpotent of index } k \in \mathbb{N}, \\ \infty & \text{otherwise.} \end{cases} \tag{4.4}$$

If $i(a) \leq 1$, we call a *group invertible*; the Drazin inverse of a is then called the *group inverse*, and is denoted by $a^{\mathrm{D}} = a^{\#}$. The set of all group invertible elements will be denoted by $\mathcal{R}^{\#}$.

We observe that the g-Drazin index of $a \in \mathcal{R}$ is finite if and only if a is polar. The sets $\mathcal{R}^{\mathrm{gD}}$, $\mathcal{R}^{\mathrm{D}}$ and $\mathcal{R}^{\#}$ coincides with the set of all quasipolar, polar and simply polar elements of $\mathcal{R}$, respectively. Note that $\mathcal{R}^{\mathrm{gD}} \supset \mathcal{R}^{\mathrm{D}} \supset \mathcal{R}^{\#} \supset \mathcal{R}^{-1}$. We make the following useful observation.

PROPOSITION 4.5. *An element $a \in \mathcal{R}$ is Drazin invertible if and only if there exists $k \in \mathbb{N}$ such that a^k is group invertible.*

In addition to (4.2) we have the following useful relations between the spectral idempotent and the g-Drazin inverse established in the proof of Theorem 4.2:

$$a^\pi = 1 - a^{\mathrm{D}}a = 1 - aa^{\mathrm{D}}, \qquad a^\pi a^{\mathrm{D}} = a^{\mathrm{D}}a^\pi = 0. \tag{4.5}$$

By (4.3) we also have that $a^{\mathrm{D}} + a^\pi \in \mathcal{R}^{-1}$. This leads to the following.

PROPOSITION 4.6. *If $a \in \mathcal{R}^{\mathrm{gD}}$, then $a^{\mathrm{D}} \in \mathcal{R}^{\#}$, and $(a^{\mathrm{D}})^\pi = a^\pi$. In addition, $a^{\mathrm{D}} \in \mathcal{R}^-$.*

PROOF. All we need to prove is that a^{D} is regular for any $a \in \mathcal{R}^{\mathrm{gD}}$. Write $b = a^{\mathrm{D}}$. Then $bb^{\mathrm{D}}b = b(1 - b^\pi) = b - bb^\pi = b - a^{\mathrm{D}}a^\pi = b$. □

Equation (4.2) can be improved as follows.

PROPOSITION 4.7. *Let $a \in \mathcal{R}^{\mathrm{gD}}$. If $x \in \mathcal{R}^{-1} \cap \mathrm{comm}(a)$, then $a + xa^\pi \in \mathcal{R}^{-1}$ and*

$$a^{\mathrm{D}} = (a + xa^\pi)^{-1}(1 - a^\pi). \tag{4.6}$$

PROOF. Let $x \in \mathcal{R}^{-1} \cap \mathrm{comm}(a)$. Then x commutes with a^π, and $aa^\pi + x \in \mathcal{R}^{-1}$ according to (2.1). Hence

$$\begin{aligned} a + xa^\pi &= (a + xa^\pi)a^\pi + (a + xa^\pi)(1 - a^\pi) \\ &= (aa^\pi + x)a^\pi + (a + a^\pi)(1 - a^\pi), \end{aligned}$$

which shows that

$$(a + xa^\pi)^{-1} = (aa^\pi + x)^{-1}a^\pi + (a + a^\pi)^{-1}(1 - a^\pi).$$

The result follows from the equation

$$(a + xa^\pi)a^{\mathrm{D}} = aa^{\mathrm{D}} + xa^\pi a^{\mathrm{D}} = 1 - a^\pi$$

obtained from (4.5). □

REMARK 4.8. In rings the double commutativity of b with a in Definition 4.1 is necessary to guarantee the uniqueness of the g-Drazin inverse. In [8, Lemma 2.4] it is erroneously claimed that the uniqueness of the g-Drazin inverse follows from $b \in \mathrm{comm}(a)$. However, commutativity is sufficient when $\mathcal{R}$ is a Banach algebra or $a^2b - a$ is nilpotent rather than quasinilpotent.

PROPOSITION 4.9. *Let $a \in \mathcal{R}$, and let $b \in \mathcal{R}$ be such that*

$$b \in \mathrm{comm}(a), \qquad ab^2 = b, \qquad a^2b - a \in \mathcal{R}^{\mathrm{nil}}. \tag{4.7}$$

Then a is polar, and $a^{\mathrm{D}} = b$.

PROOF. Let $p = 1 - ab$. Then it can be easily verified that $p \in \mathrm{comm}(a)$, $p^2 = p$, $ap \in \mathcal{R}^{\mathrm{nil}}$, and $(a + p)(b + p) = 1 + ap \in \mathcal{R}^{-1}$ which implies $a + p \in \mathcal{R}^{-1}$. Thus p satisfies the conditions of Proposition 2.6, and $p = a^\pi \in \mathrm{comm}^2(a)$. Hence a is polar and $b = (a + p)^{-1}(1 - p) \in \mathrm{comm}^2(a)$. This proves $b = a^{\mathrm{D}}$. □

REMARK 4.10. Drazin [3] defined a *pseudo-inverse* of $a \in \mathcal{R}$ as an element $a' \in \mathcal{R}$ satisfying $aa' = a'a$, $a(a')^2 = a'$ and $a^{m+1}a' = a^m$ for some positive integer m. (For $m = 0$ we get $a \in \mathcal{R}^{-1}$ and $a' = a^{-1}$.) It can be verified that these conditions on a' are equivalent to (4.7). Hence the Drazin original definition applies only to polar elements, in which case $a' = a^{\mathrm{D}}$.

5. The Moore–Penrose inverse

An involution $x \mapsto x^*$ in a ring $\mathcal{R}$ is an anti-isomorphism of degree 2, that is,

$$(a^*)^* = a, \quad (a + b)^* = a^* + b^*, \quad (ab)^* = b^*a^*.$$

We say that a is *Moore–Penrose invertible* if the equations

$$bab = b, \quad aba = a, \quad (ab)^* = ab, \quad (ba)^* = ba \tag{5.1}$$

have a common solution; such solution is unique if it exists (see [11]), and is usually denoted by $a^\dagger$. The set of all Moore–Penrose invertible elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^\dagger$.

The next well known lemma (see [11, p .407]) asserts that two one-sided invertibility conditions imply the Moore–Penrose invertibility.

LEMMA 5.1. *Let $a \in \mathcal{R}$. Then $a \in \mathcal{R}^\dagger$ if and only if there exist $x, y \in \mathcal{R}$ such that $axa = a = aya$, $(ax)^* = ax$ and $(ya)^* = ya$. In this case $a^\dagger = yax$.*

DEFINITION 5.2. An element $a \in \mathcal{R}$ is **-cancellable* if

$$a^*ax = 0 \implies ax = 0 \quad \text{and} \quad xaa^* = 0 \implies xa = 0. \tag{5.2}$$

A ring $\mathcal{R}$ is **-reducing* if all elements are *-cancellable. This is equivalent to $a^*a = 0 \implies a = 0$ for all a. A **-regular* ring is a *-reducing regular ring.

Applying the involution to (5.2), we observe that a is *-cancellable if and only if a^* is *-cancellable. It is often useful to observe that

$$a \text{ is *-cancellable} \implies a^*a \text{ and } aa^* \text{ are *-cancellable.} \tag{5.3}$$

Generalized inverses in *-regular rings, including the Moore–Penrose inverse, were studied by Hartwig in [6]. The local *-cancellation property was used by Puystjens and Robinson in [12] to study the Moore–Penrose inverse of a morphism in a category with involution. The condition $\|x^*x\| = \|x\|^2$ guarantees that any C^*-algebra (called a Hilbert algebra in [4, Section 8.8]) is a *-reducing ring.

THEOREM 5.3. *Let $a \in \mathcal{R}$. Then $a \in \mathcal{R}^\dagger$ if and only if a is *-cancellable and a^*a is group invertible. Then also aa^* is group invertible and*

$$a^\dagger = (a^*a)^\# a^* = a^*(aa^*)^\#. \tag{5.4}$$

PROOF. Suppose that $a \in \mathcal{R}^\dagger$ and $a^*ax = 0$. Then

$$ax = aa^\dagger ax = (aa^\dagger)^* ax = (a^\dagger)^* a^*ax = 0.$$

Similarly we prove that $xaa^* = 0 \implies xa = 0$. Hence a is *-cancellable. The Moore–Penrose invertibility of a^*a is obtained by verifying that $(a^*a)^\dagger = a^\dagger(a^\dagger)^*$. Since a^*a is symmetric, $(a^*a)^\# = (a^*a)^\dagger$.

Suppose that a is *-cancellable and a^*a is group invertible, and write $x = (a^*a)^\# a^*$. The conditions $xax = x$, $(ax)^* = ax$ and $(xa)^* = xa$ can be verified by a direct calculation. By the group invertibility, $a^*a(a^*a)^\pi = 0$, and $a(a^*a)^\pi = 0$ by *-cancellation. This gives

$$a - axa = a(1 - (a^*a)^\# a^*a) = a(a^*a)^\pi = 0.$$

Hence $x = a^\dagger$ and the first equation in (5.4) is proved.

We observe that $a \in \mathcal{R}^\dagger$ if and only of $a^* \in \mathcal{R}^\dagger$. Applying the preceding result to a^* in place of a, we get the rest of the theorem. □

The following is the main result on the existence of Moore–Penrose inverse in rings with involution. Many of the equivalences were observed earlier for matrices; we note that the *-cancellability holds automatically in the *-regular ring of complex matrices of the same order. The equivalence of conditions (i) and (ix) was proved by Puystjens and Robinson [12, Lemma 3] in categories with involution.

THEOREM 5.4. *For $a \in \mathcal{R}$ the following conditions are equivalent*:

(i) $a \in \mathcal{R}^\dagger$;

(ii) $a^* \in \mathcal{R}^\dagger$;

(iii) *a is *-cancellable and $a^*a \in \mathcal{R}^\dagger$*;

(iv) *a is *-cancellable and $aa^* \in \mathcal{R}^\dagger$*;

(v) *a is *-cancellable and $a^*a \in \mathcal{R}^{\mathrm{D}}$*;

(vi) *a is *-cancellable and $aa^* \in \mathcal{R}^{\mathrm{D}}$*;

(vii) *a is *-cancellable and $a^*a \in \mathcal{R}^\#$*;

(viii) *a is *-cancellable and $aa^* \in \mathcal{R}^\#$*;

(ix) *a is *-cancellable and both aa^* and a^*a are regular*;

(x) $a \in aa^*\mathcal{R} \cap \mathcal{R}a^*a$;

(xi) *a is *-cancellable and a^*aa^* is regular.*

PROOF. First we prove the implications

$$\text{(i)} \Longrightarrow \text{(iii)} \Longrightarrow \text{(v)} \Longrightarrow \text{(vii)} \Longrightarrow \text{(i)}. \tag{5.5}$$

(i) $\Longrightarrow$ (iii) Follows from Theorem 5.3 and its proof.

(iii) $\Longrightarrow$ (v) A Moore–Penrose invertible symmetric element is Drazin (in fact group) invertible.

(v) $\Longrightarrow$ (vii) Since a is *-cancellable, then so is $x = a^*a$ by (5.3). Hence x is Drazin invertible, symmetric and *-cancellable. We have $(x^\pi)^* = (x^*)^\pi = x^\pi$ by Proposition 2.5. Let $k \in \mathbb{N}$ be such that $x^k x^\pi = 0$. From the symmetry of x and its *-cancellability we deduce that $xx^\pi = 0$. Hence $x = a^*a \in \mathcal{R}^\#$.

(vii) $\implies$ (i) This follows from Theorem 5.3.

Since $a \in \mathcal{R}^\dagger \iff a^* \in \mathcal{R}^\dagger$, (5.5) gives immediately

$$\text{(ii)} \implies \text{(iv)} \implies \text{(vi)} \implies \text{(viii)} \implies \text{(ii)},$$

and the equivalence of (i)–(viii) is established.

(viii) $\implies$ (ix) As we showed, (viii) is equivalent to (vii), and together they yield (ix) (group invertibility implies regularity).

(ix) $\implies$ (x) From $aa^*xaa^* = aa^*$ we get $aa^*xa = a$, and $a^*aya^*a = a^*a$ implies $aya^*a = a$ by the *-cancellability of a. Hence, $a \in aa^*\mathcal{R} \cap \mathcal{R}a^*a$.

(x) $\implies$ (i) If $a = aa^*u = va^*a$ are consistent, then $a^*u = (aa^*u)^*u = u^*aa^*u = u^*a$. Similarly, $va^* = av^*$. Further, $au^*a = aa^*u = a$ and $av^*a = va^*a = a$. Then $a \in \mathcal{R}^\dagger$ by Lemma 5.1 with $x = v^*$ and $y = u^*$.

(i) $\implies$ (xi) We note that $a^*aa^*((a^\dagger)^* a^\dagger (a^\dagger)^*)a^*aa^* = a^*aa^\dagger aa^* = a^*aa^*$.

(xi) $\implies$ (x) If $a^*aa^*ca^*aa^* = a^*aa^*$, then, by using the *-cancellability of a twice, we get $aa^*xa^*a = a$, which implies $a \in aa^*\mathcal{R} \cap \mathcal{R}a^*a$. $\square$

From the equivalence of (i) and (vi) (or (i) and (vii)) in the preceding theorem we recover [9, Theorem 2.4] in C^*-algebras and [13, Lemma 2] in *-reducing rings.

6. Elements with equal spectral idempotents

In this section we give a characterization of elements of $\mathcal{R}$ with equal spectral idempotents. In view of (4.5) we observe that

$$a^\pi = b^\pi \iff aa^{\mathrm{D}} = bb^{\mathrm{D}}.$$

This problem was studied by Hartwig [7] for matrices over a ring in the special case when $ba^{l+1} = a^l$ and $ab^{k+1} = b^k$. Our investigation is motivated by a recent study of Castro et al. [2] for the case of complex matrices.

Theorem 6.1. *Let $a \in \mathcal{R}^{\mathrm{gD}}$ and $b \in \mathcal{R}$. The following conditions are equivalent:*

(i) $b \in \mathcal{R}^{\mathrm{gD}}$ *and* $a^\pi = b^\pi$;

(ii) $a^\pi \in \mathrm{comm}^2(b)$, $ba^\pi \in \mathcal{R}^{\mathrm{qnil}}$ *and* $b + a^\pi \in \mathcal{R}^{-1}$;

(iii) $a^\pi \in \mathrm{comm}^2(b)$, $ba^\pi \in \mathcal{R}^{\mathrm{qnil}}$ *and* $a^{\mathrm{D}}b + a^\pi \in \mathcal{R}^{-1}$;

(iv) $b \in \mathcal{R}^{\mathrm{gD}}$, $a^{\mathrm{D}}b + a^\pi \in \mathcal{R}^{-1}$ *and* $b^{\mathrm{D}} = (a^{\mathrm{D}}b + a^\pi)^{-1}a^{\mathrm{D}}$;

(v) $b \in \mathcal{R}^{\mathrm{gD}}$ *and* $b^{\mathrm{D}} - a^{\mathrm{D}} = a^{\mathrm{D}}(a - b)b^{\mathrm{D}}$;

(vi) $b \in \mathcal{R}^{\mathrm{gD}}$, $a^\pi \in \mathrm{comm}(b)$ *and* $1 - (b^\pi - a^\pi)^2 \in \mathcal{R}^{-1}$;

(vii) $b^{\mathrm{D}}\mathcal{R} \subset a^{\mathrm{D}}\mathcal{R}$ *and* $(b^{\mathrm{D}})^0 \subset (a^{\mathrm{D}})^0$.

PROOF. The equivalence of (i) and (ii) is Definition 2.2.

(ii) $\Longleftrightarrow$ (iii) We show that under the assumption $a^\pi \in \mathrm{comm}^2(b)$ and $a^\pi b \in \mathcal{R}^{\mathrm{qnil}}$,

$$b + a^\pi \in \mathcal{R}^{-1} \iff a^{\mathrm{D}}b + a^\pi \in \mathcal{R}^{-1}. \tag{6.1}$$

Observe that

$$(a^{\mathrm{D}} + a^\pi)((1 - a^\pi)b + a^\pi) = a^{\mathrm{D}}b + a^\pi. \tag{6.2}$$

Since $a^{\mathrm{D}} + a^\pi \in \mathcal{R}^{-1}$, from (6.2) we obtain

$$(b + a^\pi) - a^\pi b \in \mathcal{R}^{-1} \iff a^{\mathrm{D}}b + a^\pi \in \mathcal{R}^{-1}.$$

As $a^\pi b \in \mathcal{R}^{\mathrm{qnil}}$, (6.1) will follow when we show that $a^\pi b$ commutes with $b + a^\pi$ (obvious) and $a^{\mathrm{D}}b + a^\pi$ (not so obvious):

$$\begin{aligned} a^\pi b(a^{\mathrm{D}}b + a^\pi) &= a^\pi b a^{\mathrm{D}} b + a^\pi b a^\pi = b a^\pi a^{\mathrm{D}} b + a^\pi b = a^\pi b, \\ (a^{\mathrm{D}}b + a^\pi)a^\pi b &= a^{\mathrm{D}} b a^\pi b + a^\pi b = a^{\mathrm{D}} a^\pi b^2 + a^\pi b = a^\pi b. \end{aligned}$$

This proves the equivalence of (ii) and (iii).

(iii) $\Longrightarrow$ (iv) Let (iii) hold. From the equivalence of (i) and (iii) we conclude that $a^\pi = b^\pi$. Then

$$(a^{\mathrm{D}}b + a^\pi)b^{\mathrm{D}} = a^{\mathrm{D}}bb^{\mathrm{D}} + a^\pi b^{\mathrm{D}} = a^{\mathrm{D}}(1 - a^\pi) + b^\pi b^{\mathrm{D}} = a^{\mathrm{D}}$$

in view of (4.5), and (iv) follows.

(iv) $\Longrightarrow$ (v) If $b^{\mathrm{D}} = (a^{\mathrm{D}}b + a^\pi)^{-1}a^{\mathrm{D}}$, then $a^{\mathrm{D}} = (a^{\mathrm{D}}b + a^\pi)b^{\mathrm{D}}$, and

$$b^{\mathrm{D}} - a^{\mathrm{D}} = (1 - a^{\mathrm{D}}b - a^\pi)b^{\mathrm{D}} = (a^{\mathrm{D}}a - a^{\mathrm{D}}b)b^{\mathrm{D}} = a^{\mathrm{D}}(a - b)b^{\mathrm{D}}.$$

(v) $\Longrightarrow$ (i) From $b^{\mathrm{D}} - a^{\mathrm{D}} = a^{\mathrm{D}}(a - b)b^{\mathrm{D}}$ we get $b^{\mathrm{D}} = a^{\mathrm{D}}(b^\pi + ab^{\mathrm{D}})$. Multiplying this expression on the right by $b^{\mathrm{D}}b^2$, after a short calculation we get $bb^{\mathrm{D}} = a^{\mathrm{D}}ab^{\mathrm{D}}b$. Writing $aa^{\mathrm{D}} = 1 - a^\pi$ and $bb^{\mathrm{D}} = 1 - b^\pi$, we get $a^\pi = a^\pi b^\pi$.

Similarly, multiplying $a^{\mathrm{D}} = (a^\pi + a^{\mathrm{D}}b)b^{\mathrm{D}}$ on the left by a^2a^{D}, we get $aa^{\mathrm{D}} = aa^{\mathrm{D}}bb^{\mathrm{D}}$, and $b^\pi = a^\pi b^\pi$. Hence $a^\pi = b^\pi$.

(i) $\Longrightarrow$ (vi) is clear.

(vi) $\Longrightarrow$ (i) From $ba^\pi = a^\pi b$ it follows that $b^\pi a^\pi = a^\pi b^\pi$ since $b^\pi \in \mathrm{comm}^2(b)$. Then $1 - (b^\pi - a^\pi)^2 = (1 - a^\pi + b^\pi)(1 - b^\pi + a^\pi)$, and $1 - a^\pi + b^\pi$, $1 - b^\pi + a^\pi \in \mathcal{R}^{-1}$. Further, $a^\pi(1 - a^\pi + b^\pi) = a^\pi b^\pi = b^\pi(1 - b^\pi + a^\pi)$. Hence

$$\begin{aligned} a^\pi &= (1 - a^\pi + b^\pi)^{-1}a^\pi b^\pi = (1 - a^\pi + b^\pi)^{-1}(1 - a^\pi + b^\pi)a^\pi b^\pi = a^\pi b^\pi, \\ b^\pi &= (1 - b^\pi + a^\pi)^{-1}b^\pi a^\pi = (1 - b^\pi + a^\pi)^{-1}(1 - b^\pi + a^\pi)b^\pi a^\pi = a^\pi b^\pi. \end{aligned}$$

(vii) $\Longrightarrow$ (i) From $(b^{\mathrm{D}})^0 \subset (a^{\mathrm{D}})^0$ it follows that $\mathcal{R}a^{\mathrm{D}} \subset \mathcal{R}a^{\mathrm{D}}$. Indeed, a^{D}, b^{D} are regular (with inner inverses $a^{\mathrm{D}}a$, $b^{\mathrm{D}}b$, respectively). By Proposition 3.1,

$$(\mathcal{R}b^{\mathrm{D}})^0 = (b^{\mathrm{D}})^0 \subset (a^{\mathrm{D}})^0 = (\mathcal{R}a^{\mathrm{D}})^0,$$

and

$$\mathcal{R}b^{\mathrm{D}} = {}^0((\mathcal{R}b^{\mathrm{D}})^0) \supset {}^0((\mathcal{R}a^{\mathrm{D}})^0) = \mathcal{R}a^{\mathrm{D}}.$$

The inclusions $\mathcal{R}b^{\mathrm{D}} \supset \mathcal{R}a^{\mathrm{D}}$ and $b^{\mathrm{D}}\mathcal{R} \subset a^{\mathrm{D}}\mathcal{R}$ imply the consistency of the equations

$$a^{\mathrm{D}} = ybb^{\mathrm{D}}, \qquad aa^{\mathrm{D}}x = b^{\mathrm{D}}, \tag{6.3}$$

since $a^{\mathrm{D}}\mathcal{R} = aa^{\mathrm{D}}\mathcal{R}$ and $\mathcal{R}b^{\mathrm{D}} = \mathcal{R}bb^{\mathrm{D}}$. Equation (6.3) is equivalent to

$$(1 - aa^{\mathrm{D}})b^{\mathrm{D}} = 0 = a^{\mathrm{D}}(1 - bb^{\mathrm{D}}), \tag{6.4}$$

which in turn implies

$$a^{\mathrm{D}} = a^{\mathrm{D}}bb^{\mathrm{D}} \quad \text{and} \quad b^{\mathrm{D}} = aa^{\mathrm{D}}b^{\mathrm{D}}.$$

Then $aa^{\mathrm{D}} = aa^{\mathrm{D}}bb^{\mathrm{D}}$ and $bb^{\mathrm{D}} = aa^{\mathrm{D}}bb^{\mathrm{D}}$. Thus $aa^{\mathrm{D}} = bb^{\mathrm{D}}$, and $a^\pi = b^\pi$.

(i) $\Longrightarrow$ (vii) As $aa^{\mathrm{D}} = bb^{\mathrm{D}}$, then $b^{\mathrm{D}}\mathcal{R} = bb^{\mathrm{D}}\mathcal{R} = aa^{\mathrm{D}}\mathcal{R} = a^{\mathrm{D}}\mathcal{R}$. Similarly, $\mathcal{R}b^{\mathrm{D}} = \mathcal{R}a^{\mathrm{D}}$, which implies $(\mathcal{R}b^{\mathrm{D}})^0 = (\mathcal{R}a^{\mathrm{D}})^0$, or $(a^{\mathrm{D}})^0 = (b^{\mathrm{D}})^0$ according to Proposition 3.1. □

Specializing the equivalence of conditions (i)–(v) in the preceding theorem to complex matrices, we recover [2, Theorem 2.1]. Condition (vi) appears to be new. Hartwig [7, Corollary 2] proved that if $ba^{l+1} = a^l$ and $ab^{k+1} = b^k$, then $aa^{\mathrm{D}} = bb^{\mathrm{D}}$ if and only if a^{k+l} and b^{k+l} commute.

Remark 6.2. The condition $1 - (b^\pi - a^\pi)^2 \in \mathcal{R}^{-1}$ in (vi) is equivalent to the simultaneous validity of $1 - a^\pi + b^\pi \in \mathcal{R}^{-1}$ and $1 - b^\pi + a^\pi \in \mathcal{R}^{-1}$. We show that it cannot be replaced by $1 - a^\pi + b^\pi \in \mathcal{R}^{-1}$ (or $1 - b^\pi + a^\pi \in \mathcal{R}^{-1}$) alone. Let $\mathcal{R}$ be the ring of all real 3×3 matrices, and set

$$a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then $a^{\mathrm{D}} = a$, $b^{\mathrm{D}} = b$ and

$$a^\pi = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad b^\pi = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We note that $ba^\pi = a^\pi b$, and

$$1 - a^\pi + b^\pi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \in \mathcal{R}^{-1},$$

while $a^\pi \neq b^\pi$.

7. EP elements in rings with involution

Complex matrices and Hilbert space operators A with the property that the ranges of A and A^* coincide are known as EP or range-hermitian operators. For a discussion of EP matrices see [1, Chapter 4]. A detailed study of EP elements in involutory rings was undertaken by Hartwig [6]. The concept has been studied recently in the setting of C^*-algebras [10].

Definition 7.1. An element a of a ring $\mathcal{R}$ with involution is said to be EP if $a \in \mathcal{R}^{\mathrm{gD}} \cap \mathcal{R}^\dagger$ and $a^{\mathrm{D}} = a^\dagger$. An element a is *generalized* EP (or gEP for short) if there exists $k \in \mathbb{N}$ such that a^k is EP.

We recall the following well known characterization of EP elements (see, for instance, [6, 10]):

$$a \text{ is EP} \iff aa^\dagger = a^\dagger a.$$

In [2], the authors gave characterization of complex EP matrices based on properties of matrices with the same eigenprojection at 0. This section is motivated by these results. The key to the characterization of EP elements is the following proposition involving equality of spectral idempotents of various elements given without proof in [10, Corollary 2.2] in the setting of C^*-algebras.

Theorem 7.2. *For $a \in \mathcal{R}$ the following conditions are equivalent*:

(i) a *is* EP;

(ii) $a \in \mathcal{R}^{\#}$ *and* $a^\pi = (a^*)^\pi$;

(iii) $a \in \mathcal{R}^{\mathrm{gD}} \cap \mathcal{R}^\dagger$ *and* $a^\pi = (a^*a)^\pi$;

(iv) $a \in \mathcal{R}^{\mathrm{gD}} \cap \mathcal{R}^\dagger$ *and* $a^\pi = (aa^*)^\pi$;

(v) $a \in \mathcal{R}^\dagger$ *and* $(a^*a)^\pi = (aa^*)^\pi$.

Proof. (i) $\Longrightarrow$ (ii). Assume that a is EP. The group invertibility of a follows from the equation $aa^\pi = a(1 - a^{\mathrm{D}}a) = a(1 - a^\dagger a) = a - aa^\dagger a = 0$. Further, $(a^*)^\pi = (a^\pi)^* = (1 - a^\dagger a)^* = 1 - a^\dagger a = a^\pi$.

(ii) $\iff$ (iii). If (ii) holds, then $a^* \in \mathcal{R}^{\#}$, $a^\pi a^* a = a^* a a^\pi = 0$, and

$$a^*a + a^\pi = (a^* + a^\pi)(a + a^\pi) \in \mathcal{R}^{-1}$$

by properties of spectral idempotents of a^* and a. From the definition of a spectral idempotent we conclude that $(a^*a)^\pi = a^\pi$. A direct check reveals that $a^{\#}$ satisfies the definition of $a^\dagger$; hence $a \in \mathcal{R}^\dagger$.

Conversely, if (iii) holds, then $a^*a \in \mathcal{R}^{\#}$ by Theorem 5.3, and consequently $a^*a(a^*a)^\pi = 0$. By the *-cancellation for a, $aa^\pi = a(a^*a)^\pi = 0$, which shows that $a \in \mathcal{R}^{\#}$. Since a^π is symmetric, (ii) holds.

(ii) $\iff$ (iv). This is the equivalence (ii) $\iff$ (iii) with a^* in place of a.

(iii) and (iv) together obviously imply (v).

(v) $\implies$ (i). If $a \in \mathcal{R}^\dagger$, then a is *-cancellable, and a^*a and aa^* are group invertible by Theorem 5.3. According to (5.4) we have

$$a^\dagger a = (a^*a)^{\#} a^* a = 1 - (a^*a)^\pi = 1 - (aa^*)^\pi = aa^*(aa^*)^{\#} = aa^\dagger,$$

and a is EP. □

Part (ii) of the preceding proposition states that an element is EP if and only if a is group invertible and the elements a and a^* have the same spectral idempotent. When we apply our main Theorem 6.1 to this situation, a number of conditions will coalesce. In particular, we have the following result.

THEOREM 7.3. *An element $a \in \mathcal{R}$ is* EP *if and only if a is group invertible and one of the following equivalent conditions holds*:

(a) $a^{\#}a$ *is symmetric*;

(b) $(a^{\#})^* = aa^{\#}(a^{\#})^*$;

(c) $(a^{\#})^* = (a^{\#})^* a^{\#} a$;

(d) $a^{\#}(a^\pi)^* = a^\pi (a^{\#})^*$.

PROOF. First assume that $a \in \mathcal{R}^{\#}$.

(a) $\implies$ (b). From $(a^{\#})^2 a = a^{\#}$ we obtain $a^{\#}(a^{\#}a)^* = a^{\#}$ by the symmetry of $a^{\#}a$. Then $a^{\#}a^*(a^{\#})^* = a^{\#}$; applying involution, we get (b).

(b) $\iff$ (c). Condition (c) is obtained from (b) with a^* in place of a by applying involution.

(b) $\implies$ (d). We have

$$a^\pi (a^{\#})^* = a^\pi a a^{\#} (a^{\#})^* = a^\pi (1 - a^\pi)(a^{\#})^* = 0.$$

Hence $a^\pi (a^{\#})^* = 0 = a^{\#}(a^\pi)^*$.

Assume that $a \in \mathcal{R}^{\#}$ and (d) holds. From (d) we get

$$(1 - a^{\#}a)(a^{\#})^* = a^{\#}(1 - a^*(a^{\#})^*)$$

and

$$(a^*)^\# - a^\# = a^\#(a - a^*)(a^*)^\#.$$

By Theorem 6.1 (vi) applied to $b = a^*$ we get $(a^*)^\pi = a^\pi$; hence a is EP by Theorem 7.2 (ii).

Conversely, if a is EP, then according to Theorem 7.2 (ii) a is group invertible, and $a^\pi = (a^*)^\pi = (a^\pi)^*$,that is, a^π is symmetric; then $a^\# a$ is also symmetric. □

In the following theorem we obtain a particularly simple and elegant characterization of EP elements in a ring with involution.

THEOREM 7.4. *An element $a \in \mathcal{R}$ is* EP *if and only if a is g-Drazin invertible and one of the following equivalent conditions holds*:

(a) $a^* a^\pi = 0$;

(b) $a^\pi a^* = 0$;

(c) $a^* = a^* a^{\mathrm{D}} a$;

(d) $a^* = a^{\mathrm{D}} a a^*$.

PROOF. Assume that $a \in \mathcal{R}^{\mathrm{gD}}$; then also $a^* \in \mathcal{R}^{\mathrm{gD}}$.

Under this assumption, the equivalence of (a) and (c) follows from the equation $a^* - a^* a^{\mathrm{D}} a = a^*(1 - a^{\mathrm{D}} a) = a^* a^\pi$. Applying (a) to a^* in place of a and taking involution, we see that (a) is equivalent to (b); similarly, (c) is equivalent to (d).

Suppose that $a \in \mathcal{R}^{\mathrm{gD}}$ and (c) holds. We show that $a^{\mathrm{D}} a$ is symmetric:

$$(a^{\mathrm{D}} a)^* = a^*(a^{\mathrm{D}})^* = a^{\mathrm{D}} a a^*(a^{\mathrm{D}})^* = (a^{\mathrm{D}} a)(a^{\mathrm{D}} a)^*.$$

Since $(a^{\mathrm{D}} a)(a^{\mathrm{D}} a)^*$ is symmetric, so is $a^{\mathrm{D}} a$. From $a^* = a^{\mathrm{D}} a a^*$ we get $a^\pi a^* = 0$, which implies $a a^\pi = 0$. Then $a \in \mathcal{R}^\#$, and a is EP by Theorem 7.3 (i).

Conversely assume that a is EP. Then $a \in \mathcal{R}^\#$ and a^π is symmetric by Theorem 7.2 (ii). Hence $a^* a^\pi = (a^\pi a)^* = 0$, and (a) holds. □

For matrices we recover [2, Theorem 5.2 (ii)]—without the redundant condition that $a^{\mathrm{D}} a$ is symmetric.

As a final result of this paper we obtain the following characterization of gEP elements of $\mathcal{R}$ (see Definition 7.1) which follows from Theorems 7.3 and 7.4.

THEOREM 7.5. *An element $a \in \mathcal{R}$ is* gEP *if and only if $a \in \mathcal{R}^{\mathrm{D}}$ and one of the following equivalent conditions holds*:

(a) a^π *is symmetric*;

(b) $a^{\mathrm{D}}a$ *is symmetric*;

(c) $a^k \in \mathcal{R}^\#$ *and* $(a^{\mathrm{D}})^k(a^\pi)^* = 0$ *for some* k;

(d) $a^k \in \mathcal{R}^\#$ *and* $(a^\pi)^*(a^{\mathrm{D}})^k = 0$ *for some* k;

(e) $a^k \in \mathcal{R}^\#$ *and* $(a^{\mathrm{D}})^k(a^\pi)^*$ *is symmetric for some* k;

(f) $a^k(a^\pi)^* = 0$ *for some* $k \in \mathbb{N}$;

(g) $(a^\pi)^*a^k = 0$ *for some* $k \in \mathbb{N}$;

(h) $a^k = (a^{\mathrm{D}}a)^*a^k$ *for some* $k \in \mathbb{N}$;

(i) $a^k = a^k(a^{\mathrm{D}}a)^*$ *for some* $k \in \mathbb{N}$.

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