Universidade do Minho
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Networks à la Hotelling

janeiro de 2014

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Tese de Doutoramento
Programa Doutoral em Matemática e Aplicações

Trabalho realizado sob a orientação do
Doutor Alberto Adrego Pinto
e da
Doutora Ana Jacinta Soares

janeiro de 2014

Acknowledgments

To Professor Alberto Adrego Pinto, my supervisor, for his guidance, availability, wise ideas and suggestions during all the process;

To Professor Ana Jacinta Soares, my co-supervisor, for the help, availability and suggestions;

To University of Minho, University of Aveiro, University of Porto and LIAAD-INESC TEC, for the conditions offered to this project and the warm host;

To ex-ku-malha, the reason why studying Mathematics became more interesting;

To Ana Vasconcelos, for the images, the English, the support, the patience and love;

To my parents, for all .

This thesis was supported by:

Fundação para a Ciência e Tecnologia FCT, through the PhD-grant SFRH/BD/33762/2009.

Fundação para a Ciência e a Tecnologia (FCT) through Project "Dynamics and Applications", with reference PTDC/MAT/121107/2010.

LIAAD-INESC TEC through 'Strategic Project - LA 14 - 2013-2014' with reference PEst-C/EEI/LA0014/2013

Abstract

The theme of this PhD Thesis is mainly related to the areas of Game Theory and Industrial Organization. This work develops concretely two problems related with the Hotelling model of spatial competition. The first one consists in the introduction of incomplete information on the production costs of the two firms in the Hotelling model. Under explicit conditions on the production costs, we determine the Bayesian-Nash equilibrium prices for every probability distribution of the production costs. The second problem addresses an extension of the Hotelling model from the line to a network. In this problem, we establish conditions, depending on the production cost of the firms and in the network structure, that guarantee the existence of a Nash price equilibrium for all kind of networks. Furthermore, the explicit formula of the equilibrium prices is determined. Using an approach similar to the one used in the first problem, the case of incomplete information on the production costs of the firms in the network was also studied. Both problems analyse the two classical variations of the Hotelling model: linear transportation costs and quadratic transportation costs. Under linear transportation costs, we also analysed the case when the transportation costs can vary according to the firms.

Resumo

O tema desta tese de doutoramento insere-se, principalmente, nas áreas de Teoria de Jogos e Organização Industrial. Neste trabalho desenvolveram-se concretamente dois problemas relacionados com o modelo de Hotelling de competição espacial. O primeiro consiste na introdução de informação incompleta nos custos de produção das duas firmas no modelo de Hotelling. Com condições explícitas sobre os custos de produção, foram determinados os equilíbrios Bayesianos de Nash em preços para qualquer distribuição de probabilidade dos custos de produção. O segundo problema aborda uma extensão do modelo de Hotelling na linha para uma rede (network). No âmbito deste problema foram estabelecidas condições, dependendo dos custos de produção de cada empresa e da estrutura da rede, que garantem a existência de um equiíbrio de Nash em preços para todos o tipos de redes. Para além da garantia de existência, a fórmula explícita dos preços em equilíbrio é determinada. Usando uma abordagem semelhante à usada no primeiro problema foi ainda estudado o caso de informação incompleta nos custos de produção das firmas na rede. Em ambos os problemas foram analisadas as duas variações clássicas do modelo de Hotelling: custos de transporte lineares e custos de transporte quadráticos. Para custos de transporte lineares foi ainda analisado o caso em que os custos de transporte podem variar com a firma.

Contents

Introduction

Since the seminal work of Hotelling [25], the model of spatial competition has been seen by many researchers as an attractive framework for analyzing oligopoly markets (see [9, 24, 27, 30, 31, 32, 33, 37, 38]).

In his model, Hotelling present a city represented by a line segment where a uniformly distributed continuum of consumers have to buy a homogeneous commodity. Consumers have to support linear transportation costs when buying the commodity in one of the two firms of the city. The firms compete in a two-staged location-price game, where simultaneously choose their location and afterwards set their prices in order to maximize their profits. Hotelling concluded that firms would agglomerate at the center of the line, an observation referred as the "Principle of Minimum Differentiation". In 1979, D'Aspremont et al. [2] show that the "Principle of Minimum Differentiation" is invalid, since there was no price equilibrium solution for all possible locations of the firms, in particular when they are not far enough from each other. Moreover, in the same article, D'Aspremont et al. introduce a modification in the Hotelling model, considering quadratic transportation costs instead of linear. The introduction of this feature removed the discontinuities verified in the profit and demand functions, which was a problem in Hotelling model and they show that, under quadratic transportation costs, a price equilibrium exists for all locations and a location equilibrium exists and involves maximum product differentiation, i.e. the firms opt to locate at the extremes of the line.

Hotelling and D'Aspremont et al. consider that the production costs of both firms are equal to zero. Ziss [41] introduce a modification in the model of D'Aspremont et al. by allowing for different production costs between the two firms and examines the effect of heterogeneous production technologies on the location problem. Ziss shows that a price equilibrium exists for all locations and concludes that when the difference between the production costs is small, a price and location equilibrium exists in which the firms prefer to locate in different extremes of the line. However, if the difference between the production costs is sufficiently large, a location equilibrium does not exist.

Using linear transportation costs, Boyer et al. [5] study the case where the firms choose sequentially their location and then compete in delivered prices (see [26]) assuming that the first mover has perfect information, while the second mover does not know if the opponent firm has a low or high production cost. Using quadratic transportation costs, a similar model but under mill pricing setting was studied by Boyer et al. [6] and by Biscaia and Sarmento [4] in the case where firms simultaneously choose their locations. However, Boyer et al. [6] and Biscaia and Sarmento [4] consider that the uncertainty on the productions costs exists only during the first subgame in location strategies. Then the production costs are revealed to the firms before the firms have to choose their optimal price strategies and so the second subgame has complete information.

In the first part of this work (Chapter 1) we study the Hotelling model with incomplete information on the production costs of both firms. We do not study the Hotelling models in which the location choice by the firms plays a major rule, but models of price competition under spatial nature and we study the linear and quadratic cases separately. With linear transportation costs, we assume that the location of firms is fixed at the extremes of the line, avoiding the problem of non existence of equilibrium pointed by D'Aspremont et al. [2] and so we do not study the first subgame in location strategies. However, with quadratic transportation costs we consider all possible locations for the firms in the line. With linear transportation costs, we consider a more general model, where the transportation cost depends on the firm.

Our main goal is to study the price formation in the second subgame with incomplete information on the production costs of both firms. The incomplete information consists on each firm knowing its production cost but being uncertain about the competitor's cost as usual in oligopoly theory (see [11, 12, 13, 14, 15, 16, 17, 21, 28, 29]). We show that the first and second moments of the probability distribution in the production costs are the only relevant information for the price formation and all the other relevant economic quantities.

We introduce the definition of local optimum price strategy that is characterized by a local optimum property and by a duopoly property. We say that a price strategy for both firms is a local optimum price strategy if (i) any small deviation of a price of a firm provokes a decrease in its own ex-ante profit (local optimum property); and (ii) both firms have non-empty market for every pair of production costs (duopoly property). We observe that a Nash price equilibrium satisfying the duopoly property has to be a local optimum price strategy.

First, we introduce a bounded costs condition that defines a bound for the production costs in terms only of the exogenous variables that are the transportation cost and the road length of the segment line (and, in the quadratic case, the localization of both firms). We prove that the second subgame has a local optimum price strategy with the duopoly property if and only if the condition holds and that the local optimum price strategy for the firms is unique. Then, we introduce a mild additional bounded costs condition and we prove that under these two conditions, the local optimum price strategy is a Bayesian-Nash price strategy. Furthermore, we compute explicitly the formula for the local optimum price strategy that is simple and leaves clear the influence of the relevant economic exogenous quantities in the price formation. In particular, we observe that the local optimum price strategy does not depend on the distributions of the production costs of the firms, except on their first moments. We note that the novelty and elegance of the proof consists in computing explicitly the expected prices of the optimal strategies before computing the optimal strategies. Our techniques allowed the results to be universal in the incomplete information scenario because they apply to all probability distributions in the production costs.

We explicitly compare the ex-ante and ex-post profits, consumer surplus and welfare. We prove that, under specific bounded costs conditions, the ex-post profit of a firm is smaller than its ex-ante profit if and only if the production cost of the other firm is greater than its expected cost. We do a comparative analysis of profits, consumer surplus and welfare with complete and incomplete information.

Other models have been developed where the line in the Hotelling model is replaced by other topologies as for example in the Salop Model [37], where the line is replaced by the circle, or in the Spokes model [9]. In the second part of this work (Chapter 2) we introduce the Hotelling town model, extending the Hotelling model to a network, where the firms are located at the neighbourhood of the nodes and the consumers are distributed along the edges (roads), that can have different sizes. This part of the work is related with the area of network games (see [8, 20, 19, 23]). However, these studies locate firms and consumers at nodes, following the modeling methodology common in social network analysis. In particular, the edges in these only serve the purpose of connecting two nodes. The networks presented here are fundamentally different because consumers are assumed uniformly distributed along the edges of the network.

Again, we study the linear and quadratic cases separately. Moreover, in the linear case, we consider that transportation costs that consumers have to support can be different for each firm of the network.

We extend the definition of local optimum price strategy to the Hotelling network and, similarly as in Chapter 1, we introduce (weak) bounded conditions on production costs and road lengths that depend on the maximum and minimum values of the production costs, on the road lengths in the network and on the transportation costs. Under the (weak) bounded conditions, we prove that the price competition game has a local optimum price strategy. Under other (strong) bounded conditions that depend also on the maximum node degree of the network, we prove that the local optimum price strategy is a Nash price equilibrium strategy. We give an explicit series expansion formula for the Nash price equilibrium that shows explicitly how the Nash price equilibrium of a firm depends on the production costs, road market sizes and firms locations. Furthermore, the influence of a firm in the Nash price equilibrium of other firm decreases exponentially with the distance between the firms.

Assuming that the firms could not know the entire network, we introduce the idea of space bounded information (see Subsections 2.1.4 and 2.2.3), that defines how deep a firm can see in the network from its location in terms of the production costs, node degrees and road sizes and we show how a firm can estimate its own local optimum price.

With linear transportation costs, we study the location game and we prove that, if the firms are located at the neighbourhood of the nodes of degree greater than 2, the local optimal localization strategy is achieved when the firms are at the vertices of the network (see Subsection 2.1.3).

In Section 2.3, considering that the firms are located at the vertices of the network, we extend the Hotelling model with linear transportation cost allowing that the firms in the network can charge different transportation costs. In Section 2.4 we deal with the problem of uncertainty on the Hotelling network and we find the Bayesian Nash equilibrium strategy in prices.

Finally, in the conclusions we discuss the results and we present some possible directions of future works.

Chapter 1

Hotelling model

This chapter contains a general presentation of the classical Hotelling model where the firms have different production costs and introduces the price competition in the Hotelling model with uncertainty in the production costs of both firms. We consider the two usual approaches of the Hotelling model, and we study separately the scenarios of linear and quadratic transportation costs.

1.1 Linear transportation costs

In this section, we consider that the firms have associated different *transportation costs* t_A and t_B and we study the Hotelling model [25] with uncertainty in the production costs of both firms with linear transportation costs. For the linear Hotelling model with firms located at the boundaries of the segment line, we study the price competition in a scenario of incomplete information in the production costs of both firms.

We introduce the bounded uncertain costs $BUC1$ condition that defines a bound for the costs in terms of the transportation cost and the road length of the line. Under the bounded costs $BUC1$ condition we compute the unique local optimum price strategy for the firms with the property that the market shares of both firms are not empty for any outcome of production costs. We introduce a mild additional bounded uncertain costs $BUC2$ and, under the $BUC1$ and $BUC2$ conditions, we prove that the local optimum price strategy is a Bayesian-Nash price strategy. Finally, we do a complete analysis of profits, consumer surplus and welfare under complete and incomplete information.

In the last subsection we present the results of the section where the linear transportation costs are equal to both firms, $t_A = t_B = t$, as originally presented by Hotelling.

1.1.1 Hotelling model under complete information

The buyers of a commodity will be supposed uniformly distributed along a line with length l. In the two ends of the line there are two firms A and B, located at positions 0 and l respectively, selling the same commodity with unitary *production costs* c_A and c_B. No customer has any preference for either seller except on the ground of price plus *transportation cost* t_A or t_B.

Denote A's *price* by p_A and B's *price* by p_B. The point of division $x = x(p_A, p_B) \in]0, l[$ between the regions served by the two entrepreneurs is determined by the condition that at this place it is a matter of indifference whether one buys from A or from B (see Figure 1.1).

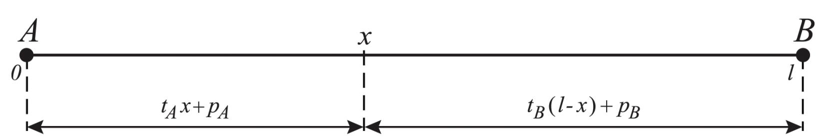


Figure 1.1: Hotelling's linear city with different transportation costs

The point x is the location of the *indifferent consumer* to buy from firm A or firm B, if

$$p_A + t_A\, x = p_B + t_B\, (l - x)$$

Solving for x, we obtain

$$x = \frac{p_B - p_A + t_B\, l}{t_A + t_B}.$$

Both firms have a non-empty market share if and only if $x \in]0, l[$. Hence, both firms have a non-empty market share if and only if the prices satisfy

$$-t_B\, l < p_B - p_A < t_A\, l \tag{1.1}$$

We note that

$$|p_A - p_B| < \min\{t_A, t_B\}\, l$$

implies inequality (1.1). Assuming inequality (1.1), both firms A and B have a non-empty demand (x and $l-x$) and the *profits* of the two firms are defined respectively by

$$\pi_A = (p_A - c_A)\, x = (p_A - c_A)\, \frac{p_B - p_A + t_B\, l}{t_A + t_B}; \tag{1.2}$$

and

$$\pi_B = (p_B - c_B)\,(l - x) = (p_B - c_B)\, \frac{p_A - p_B + t_A\, l}{t_A + t_B}. \tag{1.3}$$

Two of the fundamental economic quantities in oligopoly theory are the consumer surplus CS and the welfare W. The consumer surplus is the gain of the consumers community for given price strategies of both firms. The welfare is the gain of the state that includes the gains of the consumers community and the gains of the firms for given price strategies of both firms.

Let us denote by v_T the total amount that consumers are willing to pay for the commodity. The total amount $v(y)$ that a consumer located at y pays for the commodity is given by

$$v(y) = \begin{cases} p_A + t_A\, y & \text{if } 0 < y < x; \\ p_B + t_B\,(l - y) & \text{if } x < y < l. \end{cases}$$

The *consumer surplus* CS is the difference between the total amount that a consumer is willing to pay v_T and the total amount that the consumer pays $v(y)$

$$CS = \int_0^l v_T - v(y)\mathrm{d}y. \tag{1.4}$$

The *welfare* W is given by adding the profits of firms A and B with the consumer surplus

$$W = CS + \pi_A + \pi_B. \tag{1.5}$$

Definition 1.1.1. *A price strategy* $(\underline{p}_A, \underline{p}_B)$ *for both firms is a* local optimum price strategy *if (i) for every small deviation of the price* $\underline{p}_A$ *the profit* π_A *of firm A decreases, and for every small deviation of the price* $\underline{p}_B$ *the profit* π_B *of firm B decreases (local optimum property); and (ii) the indifferent consumer exists, i.e.* $0 < \underline{x} < l$ *(duopoly property).*

Let us compute the local optimum price strategy $(\underline{p}_A, \underline{p}_B)$. Differentiating π_A with respect to p_A and π_B with respect to p_B and equalizing to zero, we obtain the first order conditions (FOC). The FOC imply that

$$\underline{p}_A = \frac{1}{3}\left(2\, c_A + c_B + (t_A + 2\, t_B)\, l\right) \tag{1.6}$$

and

$$\underline{p}_B = \frac{1}{3}\left(c_A + 2\, c_B + (2\, t_A + t_B)\, l\right). \tag{1.7}$$

We note that the first order conditions refer to jointly optimizing the profit function (1.2) with respect to the price p_A and the profit function (1.3) with respect to the price p_B.

Since the profit functions (1.2) and (1.3) are concave, the second-order conditions for this maximization problem are satisfied and so the prices (1.6) and (1.7) are indeed maxima for the functions (1.2) and (1.3), respectively. The corresponding equilibrium profits are given by

$$\underline{\pi}_A = \frac{(c_B - c_A + (t_A + 2\, t_B)\, l)^2}{9\,(t_A + t_B)} \tag{1.8}$$

and

$$\underline{\pi}_B = \frac{(c_A - c_B + (2\,t_A + t_B)\,l)^2}{9\,(t_A + t_B)}. \tag{1.9}$$

Furthermore, the indifferent consumer location corresponding to the maximizers $\underline{p}_A$ and $\underline{p}_B$ of the profit functions π_A and π_B is

$$\underline{x} = \frac{c_B - c_A + (t_A + 2\,t_B)\,l}{3\,(t_A + t_B)}.$$

Finally, for the pair of prices $(\underline{p}_A, \underline{p}_B)$ to be a local optimum price strategy, we need assumption (1.1) to be satisfied with respect to these pair of prices. We observe that assumption (1.1) is satisfied with respect to the pair of prices $(\underline{p}_A, \underline{p}_B)$ if and only if the following condition with respect to the production costs is satisfied.

Definition 1.1.2. *The Hotelling model satisfies the* bounded costs (BC) *condition, if*

$$-(t_A + 2\,t_B)\,l < c_B - c_A < (2\,t_A + t_B)\,l.$$

We note that

$$|c_A - c_B| < 3\,\min\{t_A, t_B\}\,l$$

implies the BC condition.

We note that under the BC condition the prices are higher than the production costs $\underline{p}_A > c_A$ and $\underline{p}_B > c_B$. Hence, there is a local optimum price strategy if and only if the BC condition holds. Furthermore, under the BC condition, the pair of prices $(\underline{p}_A, \underline{p}_B)$ is the local optimum price strategy.

We note that, if a Nash price equilibrium satisfies the duopoly property then it is a local optimum price strategy. However, a local optimum price strategy is only a local strategic maximum. Hence, the local optimum price strategy to be a Nash equilibrium must also be global strategic maximum. We are going to show that this is the case.

Following D'Aspremont et al. [2], we note that the profits of the two firms, valued at local optimum price strategy are globally optimal if they are at least as great as the payoffs that firms would earn by undercutting the rivals' price and supplying the whole market.

Firm A may gain the whole market, undercutting its rival by setting

$$p_A^M = \underline{p}_B - t_A\, l - \epsilon, \text{ with } \epsilon > 0.$$

In this case the profit amounts to

$$\pi_A^M = \left(\underline{p}_B - t_A\, l - \epsilon - c_A\right)\, l = \frac{1}{3}\left(2\, c_B - 2\, c_A + (t_B - t_A)\, l\right)\, l - \epsilon\, l.$$

A similar argument is valid for store B. Undercutting this rival, setting

$$p_B^M = \underline{p}_A - t_B\, l - \epsilon,$$

it would earn

$$\pi_B^M = \left(\underline{p}_A - t_B\, l - \epsilon - c_B\right)\, l = \frac{1}{3}\left(2\, c_A - 2\, c_B + (t_A - t_B)\, l\right)\, l - \epsilon\, l.$$

The conditions for such undercutting not to be profitable are $\underline{\pi}_A \geq \pi_A^M$ and $\underline{\pi}_B \geq \pi_B^M$. Hence, since $\epsilon > 0$, proving that

$$\frac{(c_B - c_A + (t_A + 2\, t_B)\, l)^2}{9\,(t_A + t_B)} \geq \frac{1}{3}\left(2\, c_B - 2\, c_A + (t_B - t_A)\, l\right)\, l \qquad (1.10)$$

is sufficient to prove that $\underline{\pi}_A \geq \pi_A^M$. Similarly, proving that

$$\frac{(c_A - c_B + (t_B + 2\, t_A)\, l)^2}{9\,(t_A + t_B)} \geq \frac{1}{3}\left(2\, c_A - 2\, c_B + (t_A - t_B)\, l\right)\, l \qquad (1.11)$$

is sufficient to prove that $\underline{\pi}_B \geq \pi_B^M$.

However, conditions (1.10) and (1.11) are satisfied because they are equivalent to

$$(c_A - c_B + (2\, t_A + t_B)\, l)^2 \geq 0$$

and

$$(c_B - c_A + (t_A + 2\,t_B)\,l)^2 \geq 0.$$

Therefore, if $(\underline{p}_A, \underline{p}_B)$ is a local optimum price strategy then $(\underline{p}_A, \underline{p}_B)$ is a Nash price equilibrium.

By equation (1.4), the consumer surplus $\underline{CS}$ with respect to the local optimum price strategy $(\underline{p}_A, \underline{p}_B)$ is given by

$$\begin{aligned}\underline{CS} &= \int_0^l v_T - v(x)\mathrm{d}x \\ &= v_T\, l - \frac{5\,t_B + 4\,t_A}{6}\, l^2 - \frac{c_A + 2\,c_B}{3}\, l + \frac{(c_B - c_A + (t_A + 2\,t_B)\,l)^2}{18(t_A + t_B)}.\end{aligned} \tag{1.12}$$

By equation (1.5), the welfare $\underline{W}$ is given by

$$\begin{aligned}\underline{W} &= v_T\, l - \frac{t_A + t_B}{18}\, l^2 - \frac{c_A + 2\,c_B}{3}\, l + \\ &+ \frac{2\,(c_A - c_B)(t_A - 4\,t_B)\,l - 5\,t_A\,t_B\,l^2 + 5(c_A - c_B)^2}{18\,(t_A + t_B)}.\end{aligned}$$

1.1.2 Incomplete information on the production costs

The incomplete information consists in each firm to know its production cost but to be uncertain about the competitor's cost. In this subsection, we introduce a simple notation that is fundamental for the elegance and understanding of the results presented in this section.

Let the triples (I_A, Ω_A, q_A) and (I_B, Ω_B, q_B) represent (finite, countable or uncountable) sets of types I_A and I_B with σ-algebras Ω_A and Ω_B and probability measures q_A and q_B, over I_A and I_B, respectively.

We define the expected values $E_A(f)$, $E_B(f)$ and $E(f)$ with respect to the probability measures q_A and q_B as follows:

$$E_A(f) = \int_{I_A} f(z,w)\,\mathrm{d}q_A(z); \quad E_B(f) = \int_{I_B} f(z,w)\,\mathrm{d}q_B(w)$$

and

$$E(f) = \int_{I_A}\int_{I_B} f(z,w)\,\mathrm{d}q_B(w)\mathrm{d}q_A(z).$$

Let $c_A : I_A \to \mathbb{R}_0^+$ and $c_B : I_B \to \mathbb{R}_0^+$ be measurable functions where $c_A^z = c_A(z)$ denotes the production cost of firm A when the type of firm A is $z \in I_A$ and $c_B^w = c_B(w)$ denotes the production cost of firm B when the type of firm B is $w \in I_B$. Furthermore, we assume that the expected values of c_A and c_B are finite

$$E(c_A) = E_A(c_A) = \int_{I_A} c_A^z\,\mathrm{d}q_A(z) < \infty;$$
$$E(c_B) = E_B(c_B) = \int_{I_B} c_B^w\,\mathrm{d}q_B(w) < \infty.$$

We assume that $\mathrm{d}q_A(z)$ denotes the probability of the *belief* of the firm B on the production costs of the firm A to be c_A^z. Similarly, we assume that $\mathrm{d}q_B(w)$ denotes the probability of the belief of the firm A on the production costs of the firm B to be c_B^w.

The simplicity of the following cost deviation formulas is crucial to express the main results of this section in a clear and understandable way. The *cost deviations* of firm A and firm B

$$\Delta_A : I_A \to \mathbb{R}_0^+ \quad \text{and} \quad \Delta_B : I_B \to \mathbb{R}_0^+$$

are given respectively by $\Delta_A(z) = c_A^z - E(c_A)$ and $\Delta_B(w) = c_B^w - E(c_B)$. The *cost deviation* between the firms

$$\Delta_C : I_A \times I_B \to \mathbb{R}_0^+$$

is given by $\Delta_C(z,w) = c_A^z - c_B^w$. Since the meaning is clear, we will use through the section the following simplified notation:

$$\Delta_A = \Delta_A(z); \quad \Delta_B = \Delta_B(w) \text{ and } \Delta_C = \Delta_C(z,w).$$

The *expected cost deviation* Δ_E between the firms is given by $\Delta_E = E(c_A) - E(c_B)$. Hence,

$$\Delta_C - \Delta_E = \Delta_A - \Delta_B.$$

Let V_A and V_B be the variances of the production costs c_A and c_B, respectively. We observe that

$$E(\Delta_C) = \Delta_E; \quad E(\Delta_A^2) = E_A(\Delta_A^2) = V_A; \quad E(\Delta_B^2) = E_B(\Delta_B^2) = V_B. \quad (1.13)$$

Furthermore,

$$E_A(\Delta_C^2) = \Delta_B^2 + V_A + \Delta_E\,(\Delta_E - 2\,\Delta_B); \quad (1.14)$$

$$E_B(\Delta_C^2) = \Delta_A^2 + V_B + \Delta_E\,(\Delta_E + 2\,\Delta_A); \quad (1.15)$$

$$E(\Delta_C^2) = \Delta_E^2 + V_A + V_B. \quad (1.16)$$

1.1.3 Local optimal price strategy under incomplete information

In this section, we introduce incomplete information in the classical Hotelling game and we find the local optimal price strategy. We introduce the bounded uncertain costs condition that allows us to find the local optimum price strategy.

A *price strategy* (p_A, p_B) is given by a pair of functions $p_A : I_A \to \mathbb{R}_0^+$ and $p_B : I_B \to \mathbb{R}_0^+$ where $p_A^z = p_A(z)$ denotes the price of firm A when the type of firm A is $z \in I_A$ and $p_B^w = p_B(w)$ denotes the price of firm B when the type of firm B is $w \in I_B$. We note that $E(p_A) = E_A(p_A)$ and $E(p_B) = E_B(p_B)$.

The *indifferent consumer* $x : I_A \times I_B \to (0, l)$ is given by

$$x^{z,w} = \frac{p_B^w - p_A^z + t_B\, l}{t_A + t_B}. \tag{1.17}$$

The ex-post profit of the firms is the effective profit of the firms given a realization of the production costs for both firm. Hence, it is the main economic information for both firms. However, the incomplete information prevents the firms to have access to their ex-post profits except after the firms have already decided their price strategies. The *ex-post profits* $\pi_A^{EP} : I_A \times I_B \to \mathbb{R}_0^+$ and $\pi_B^{EP} : I_A \times I_B \to \mathbb{R}_0^+$ are given by

$$\pi_A^{EP}(z, w) = \pi_A(z, w) = (p_A^z - c_A^z)\, x^{z,w}$$

and

$$\pi_B^{EP}(z, w) = \pi_B(z, w) = (p_B^w - c_B^w)\,(l - x^{z,w}).$$

The ex-ante profit of the firms is the expected profit of the firm that knows its production cost but are uncertain about the production cost of the competitor firm. The *ex-ante profits* $\pi_A^{EA} : I_A \to \mathbb{R}_0^+$ and $\pi_B^{EA} : I_B \to \mathbb{R}_0^+$ are given by

$$\pi_A^{EA}(z) = E_B(\pi_A^{EP}) \quad \text{and} \quad \pi_B^{EA}(w) = E_A(\pi_B^{EP}). \tag{1.18}$$

We note that, the *expected profit* $E(\pi_A^{EP})$ of firm A is equal to $E_A(\pi_A^{EA})$ and the *expected profit* $E(\pi_B^{EP})$ of firm B is equal to $E_B(\pi_B^{EA})$.

The incomplete information forces the firms to have to choose their price strategies using their knowledge of their ex-ante profits, to which they have access, instead of the ex-post profits, to which they do not have access except after the price strategies are decided.

Definition 1.1.3. *A price strategy* $(\underline{p}_A, \underline{p}_B)$ *for both firms is a* local optimum price strategy *if (i) for every* $z \in I_A$ *and for every small deviation of the price* $\underline{p}_A^z$ *the ex-ante profit* $\pi_A^{EA}(z)$ *of firm* A *decreases, and for every* $w \in I_B$ *and for every small deviation of the price* $\underline{p}_B^w$ *the ex-ante profit* $\pi_B^{EA}(w)$ *of firm* B

decreases (local optimum property); and (ii) for every $z \in I_A$ and $w \in I_B$ the indifferent consumer exists, i.e. $0 < \underline{x}^{z,w} < l$ (duopoly property).

We introduce the $BUC1$ condition that has the crucial economical information that can be extracted from the exogenous variables. The $BUC1$ condition allow us to know if there is, or not, a local optimum price strategy in the presence of uncertainty for the production costs of both firms.

Definition 1.1.4. *The Hotelling model satisfies the* bounded uncertain costs ($BUC1$) *condition, if*

$$-2\,(t_A + 2\,t_B)\,l < \Delta_E - 3\,\Delta_C < 2\,(2\,t_A + t_B)\,l,$$

for all $z \in I_A$ and for all $w \in I_B$.

We note that

$$|3\,\Delta_C - \Delta_E| < 6\,\min\{t_A, t_B\}\,l$$

implies BUC condition.

For $i \in \{A, B\}$, we define

$$c_i^m = \min_{z \in I_i}\{c_i^z\} \quad \text{and} \quad c_i^M = \max_{z \in I_i}\{c_i^z\}.$$

Let

$$\overline{\Delta} = \max_{i,j \in \{A,B\}}\{c_i^M - c_j^m\}$$

Thus, the bounded uncertain costs and location $BUC1$ is implied by the following stronger $SBUC1$ condition.

Definition 1.1.5. *The Hotelling model satisfies the* strong bounded uncertain costs ($SBUC1$) *condition, if*

$$\overline{\Delta} < 3\,\min\{t_A, t_B\}\,l.$$

The following theorem is a key economical result in oligopoly theory. First, it tells us about the existence, or not, of a local optimum price strategy only by accessing a simple inequality in the exogenous variables and so available to both firms. Secondly, it gives us explicit and simple formulas that allow the firms to know the relevance of the exogenous variables in their price strategies and corresponding profits.

Theorem 1.1.1. *There is a local optimum price strategy* $(\underline{p}_A, \underline{p}_B)$ *if and only if the* $BUC1$ *condition holds. Under the* $BUC1$ *condition, the expected prices of the local optimum price strategy are given by*

$$E(\underline{p}_A) = \frac{t_A + 2\,t_B}{3}\, l + E(c_A) - \frac{\Delta_E}{3}; \tag{1.19}$$

$$E(\underline{p}_B) = \frac{2\,t_A + t_B}{3}\, l + E(c_B) + \frac{\Delta_E}{3}. \tag{1.20}$$

Furthermore, the local optimum price strategy $(\underline{p}_A, \underline{p}_B)$ *is unique and it is given by*

$$\underline{p}_A^z = E(\underline{p}_A) + \frac{\Delta_A}{2}; \quad \underline{p}_B^w = E(\underline{p}_B) + \frac{\Delta_B}{2}. \tag{1.21}$$

We observe that the difference between the expected prices of both firms has a very useful and clear economical interpretation in terms of the localization and expected cost deviations.

$$E(\underline{p}_A) - E(\underline{p}_B) = \frac{t_B - t_A}{3} + \frac{\Delta_E}{3}$$

Furthermore, for different production costs, the differences between the optimal prices of a firm are proportional to the differences of the production costs

$$\underline{p}_A^{z_1} - \underline{p}_A^{z_2} = \frac{c_A^{z_1} - c_A^{z_2}}{2}.$$

and

$$\underline{p}_B^{w_1} - \underline{p}_B^{w_2} = \frac{c_B^{w_1} - c_B^{w_2}}{2}.$$

for all $z_1, z_2 \in I_A$ and $w_1, w_2 \in I_B$. Hence, half of the production costs value is incorporated in the price.

The ex-post profit of the firms is the effective profit of the firms given a realization of the production costs for both firms. Hence it is the main economic information for both firms. By equation (1.1.10), the ex-post profit of firm A is

$$\underline{\pi}_A^{EP}(z,w) = \frac{(2\,(t_A + 2\,t_B)l - 3\,\Delta_A - 2\,\Delta_E)\,(2\,(t_A + 2\,t_B)l + \Delta_E - 3\,\Delta_C)}{36\,(t_A + t_B)}$$

and the ex-post profit of firm B is

$$\underline{\pi}_B^{EP}(z,w) = \frac{(2\,(2\,t_A + t_B)l - 3\,\Delta_B + 2\,\Delta_E)\,(2\,(2\,t_A + t_B)l - \Delta_E + 3\,\Delta_C)}{36\,(t_A + t_B)}.$$

The ex-ante profit of a firm is the expected profit of the firm that knows its production cost but is uncertain about the production costs of the competitor firm. Since the ex-post profit of firm A, $\underline{\pi}_A^{EP}(z,w)$, is given by

$$\frac{(2\,(t_A + 2\,t_B)l - 3\,\Delta_A - 2\,\Delta_E)\,(2\,(t_A + 2\,t_B)l + \Delta_E + 3\,(c_B^w - c_A^z))}{36\,(t_A + t_B)},$$

the ex-ante profit of firm A, $\underline{\pi}_A^{EA}(z)$, is

$$\frac{(2\,(t_A + 2\,t_B)l - 3\,\Delta_A - 2\,\Delta_E)\,(2\,(t_A + 2\,t_B)l + \Delta_E + 3\,(E(c_B) - c_A^z))}{36\,(t_A + t_B)}.$$

Hence,

$$\underline{\pi}_A^{EA}(z) = \frac{(2\,(t_A + 2\,t_B)\,l - 3\,\Delta_A - 2\,\Delta_E)^2}{36\,(t_A + t_B)}. \tag{1.22}$$

Similarly, the ex-ante profit of firm B is

$$\underline{\pi}_B^{EA}(w) = \frac{(2\,(2\,t_A + t_B)\,l - 3\,\Delta_B + 2\,\Delta_E)^2}{36\,(t_A + t_B)}. \tag{1.23}$$

Let α_A and α_B be given by

$$\alpha_A = \max\{E(c_B) - c_B^w : w \in I_B\} \text{ and } \alpha_B = \max\{E(c_A) - c_A^z : z \in I_A\}.$$

The following corollary gives us the information of the market size of both firms by giving the explicit localization of the indifferent consumer with respect to the local optimum price strategy.

Corollary 1.1.1. *Under the $BUC1$ condition, the indifferent consumer $x^{z,w}$ is given by*

$$\underline{x}^{z,w} = \frac{t_A + 2\,t_B}{3\,(t_A + t_B)}\,l + \frac{\Delta_E - 3\,\Delta_C}{6\,(t_A + t_B)}. \tag{1.24}$$

The pair of prices $(\underline{p}_A, \underline{p}_B)$ satisfies

$$\underline{p}_A^z - c_A^z \geq \alpha_A/2; \qquad \underline{p}_B^w - c_B^w \geq \alpha_B/2. \tag{1.25}$$

Proof of Theorem 1.1.1 and Corollary 1.1.1.
Under incomplete information, each firm seeks to maximize its ex-ante profit. From (1.18), the ex-ante profit for firm A is given by

$$\begin{aligned} \pi_A^{EA}(c_A^z) &= \int_{I_B} (p_A^z - c_A^z)\left(\frac{p_B^w - p_A^z + t_B\,l}{t_A + t_B}\right) \mathrm{d}q_B(w) \\ &= (p_A^z - c_A^z)\left(\frac{E(p_B) - p_A^z + t_B\,l}{t_A + t_B}\right). \end{aligned}$$

From the first order condition FOC applied to the ex-ante profit of firm A we obtain

$$p_A^z = \frac{c_A^z + E(p_B) + t_B\,l}{2}. \tag{1.26}$$

Similarly,

$$\pi_B^{EA}(c_B^w) = (p_B^w - c_B^w)\left(\frac{E(p_A) - p_B^w + t_A\,l}{t_A + t_B}\right), \tag{1.27}$$

and, by the FOC, we obtain

$$p_B^w = \frac{c_B^w + E(p_A) + t_A\, l}{2}. \tag{1.28}$$

Then, from (1.26) and (1.28),

$$\begin{aligned} E(p_A) &= \frac{E(c_A) + E(p_B) + t_B\, l}{2}; \\ E(p_B) &= \frac{E(c_B) + E(p_A) + t_A\, l}{2}. \end{aligned}$$

Solving the system of two equations, we obtain that

$$\begin{aligned} E(p_A) &= \frac{t_A + 2\, t_B}{3}\, l + \frac{2\, E(c_A) + E(c_B)}{3}; \\ E(p_B) &= \frac{2\, t_A + t_B}{3}\, l + \frac{E(c_A) + 2\, E(c_B)}{3}. \end{aligned}$$

Hence, equalities (1.19) and (1.20) are satisfied. Replacing (1.20) in (1.26) and replacing (1.19) in (1.28) we obtain that

$$\begin{aligned} p_A^z &= \frac{c_A^z}{2} + \frac{t_A + 2\, t_B}{3}\, l + \frac{E(c_A) + 2\, E(c_B)}{6}; \\ p_B^w &= \frac{c_B^w}{2} + \frac{2\, t_A + t_B}{3}\, l + \frac{2\, E(c_A) + E(c_B)}{6}. \end{aligned}$$

Hence, equation (1.21) is satisfied.

Replacing in equation (1.17) the values of $\underline{p}_A$ and $\underline{p}_B$ given by the equation (1.21) we obtain that the indifferent consumer $x^{z,w}$ is given by

$$x^{z,w} = \frac{t_A + 2\, t_B}{3\,(t_A + t_B)}\, l + \frac{3\,(c_B^w - c_A^z) + E(c_A) - E(c_B)}{6\,(t_A + t_B)}$$

Hence, equation (1.24) is satisfied. Therefore, $(\underline{p}_A, \underline{p}_B)$ satisfies property (ii) if and only if the $BUC1$ condition holds.

Since the ex-ante profit functions (1.26) and (1.27) are concave, the second-order conditions for this maximization problem are satisfied and so the prices $\underline{p}_A^z$ and $\underline{p}_B^w$ are indeed maxima for the functions (1.26) and (1.27), respectively. Therefore, the pair $(\underline{p}_A^z, \underline{p}_B^w)$ satisfies property (i) and so $(\underline{p}_A^z, \underline{p}_B^w)$ is a local optimum price strategy.

Let us prove that $\underline{p}_A^z$ and $\underline{p}_B^w$ satisfy inequalities (1.25). By equation (1.21),

$$\underline{p}_A^z - c_A^z = \frac{t_A + 2\,t_B}{3}\,l - \frac{c_A^z}{2} + \frac{E(c_A) + 2\,E(c_B)}{6};$$
$$\underline{p}_B^w - c_B^w = \frac{2\,t_A + t_B}{3}\,l - \frac{c_B^w}{2} + \frac{2\,E(c_A) + E(c_B)}{6}.$$

By the $BUC1$ condition, for every $w \in I_B$, we obtain

$$\begin{aligned} &6\,(\underline{p}_A^z - c_A^z) - 2\,(t_A + 2\,t_B)\,l = -3\,c_A^z + E(c_A) + 2\,E(c_B) \\ &= 3\,(E(c_B) - c_B^w) - 3\,(c_A^z - c_B^w) + E(c_A) - E(c_B) \\ &> 3\,(E(c_B) - c_B^w) - 2\,(t_A + 2\,t_B)\,l. \end{aligned}$$

Similarly, by the $BUC1$ condition, for every $z \in I_A$, we obtain

$$\begin{aligned} &6\,(\underline{p}_B^w - c_B^w) - 2\,(2\,t_A + t_B)\,l = -3\,c_B^w + 2\,E(c_A) + E(c_B) \\ &= 3\,(E(c_A) - c_A^z) - 3\,(c_B^w - c_A^z) - E(c_A) + E(c_B) \\ &> 3\,(E(c_A) - c_A^z) - 2\,(2\,t_A + t_B)\,l. \end{aligned}$$

Hence, inequalities (1.25) are satisfied. □

1.1.4 Bayesian Nash equilibrium

We note that, if a Bayesian-Nash price equilibrium satisfies the duopoly property then it is a local optimum price strategy. However, a local optimum price strategy is only a local strategic maximum. Hence, the local optimum price strategy to be a Bayesian-Nash equilibrium must also be global strategic maximum. In this subsection, we are going to show that this is the case.

Following D'Aspremont et al. [2], we note that the profits of the two firms, valued at local optimum price strategy are globally optimal if they are at least as great as the payoffs that firms would earn by undercutting the rivals's price and supplying the whole market for all admissible subsets of types I_A and I_B.

Definition 1.1.6. *A price strategy* $(\underline{p}_A, \underline{p}_B)$ *for both firms is a* Bayesian-Nash*, if for every* $z \in I_A$ *and for every deviation of the price* $\underline{p}_A^z$ *the ex-ante profit* $\pi_A^{EA}(z)$ *of firm* A *decreases, and for every* $w \in I_B$ *and for every deviation of the price* $\underline{p}_B^w$ *the ex-ante profit* $\pi_B^{EA}(w)$ *of firm* B *decreases.*

Let $(\underline{p}_A, \underline{p}_B)$ be the local optimum price strategy. Given the type w_0 of firm B, firm A may gain the whole market, undercutting its rival by setting

$$p_A^M(w_0) = \underline{p}_B^{w_0} - t_A\, l - \epsilon, \text{ with } \epsilon > 0$$

Hence, by $BUC1$ condition $p_A^M(w_0) \leq p_A^z$ for all $z \in I_A$. We observe that if firm A chooses the price $p_A^M(w_0)$ then, by equalities (1.17) and (1.21), the whole market belongs to Firm A for all types w of firm B with $c^w \geq c^{w_0}$. Let

$$x(w; w_0) = \min\left\{ l, \frac{p_B^w - p_A^M(w_0) + t_B\, l}{t_A + t_B} \right\}.$$

Thus, the *expected profit* with respect to the price $p_A^M(w_0)$ for firm A is

$$\pi_A^{EA,M}(w_0) \quad = \quad \int_{I_B} \left(p_A^M(w_0) - c_A^z \right) \, x(w; w_0)\, \mathrm{d}q_B(w).$$

Let $w_M \in I_B$ such that $c_B^{w_M} = c_B^M$. Since $c^{w_M} \geq c^{w_0}$ for every $w_0 \in I_B$, we obtain

$$\pi_A^{EA,M}(w_0) \leq \left(p_A^M(w_0) - c_A^z \right) \, l \leq \left(p_A^M(w_M) - c_A^z \right) l \qquad (1.29)$$

Given the type z_0 of firm A, firm B may gain the whole market, undercutting its rival by setting

$$p_B^M(z_0) = \underline{p}_A^{z_0} - t_B\, l - \epsilon, \text{ with } \epsilon > 0.$$

Hence, by $BUC1$ condition $p_B^M(z_0) \leq p_B^w$ for all $w \in I_B$. We observe that if firm B chooses the price $p_B^M(z_0)$ then, by equalities (1.17) and (1.21), the whole market belongs to Firm B for all types z of firm A with $c^z \geq c^{z_0}$. Let

$$x(z; z_0) = \max\left\{0, \frac{p_B^M(z_0) - p_A^z + t_B\, l}{t_A + t_B}\right\}$$

Thus, the *expected profit* with respect to the price $p_B^M(z_0)$ of firm B is

$$\pi_B^{EA,M}(z_0) \;=\; \int_{I_A} \left(p_B^M(z_0) - c_B^w\right)\, (l - x(z; z_0))\, \mathrm{d}q_A(z).$$

Let $z_M \in I_A$ such that $c_A^{z_M} = c_A^M$. Since $c^{z_M} \geq c^{z_0}$ for every $z_0 \in I_A$, we obtain

$$\pi_B^{EA,M}(z_0) \leq \left(p_B^M(z_0) - c_B^w\right)\, l \leq \left(p_B^M(z_M) - c_B^w\right) l. \qquad (1.30)$$

Remark 1.1.1. *Under the $BUC1$ condition, the strategic equilibrium $(\underline{p}_A, \underline{p}_B)$ is the unique pure Bayesian-Nash equilibrium with the duopoly property if for every $z \in I_A$ and every $w \in I_B$,*

$$\pi_A^{EA,M}(w) \leq \underline{\pi}_A^{EA}(z) \qquad \textit{and} \qquad \pi_B^{EA,M}(z) \leq \underline{\pi}_B^{EA}(w). \qquad (1.31)$$

Let

$$X_{i,j} = 3\, c_j^M + 2\, E(c_i) + E(c_j) - 6\, c_i^m + 2\,(t_j - t_i)\, l$$

and

$$Y_{i,j} = 2\,(t_i + 2\, t_j)\, l + E(c_i) + 2\, E(c_j) - 3\, c_i^M.$$

Definition 1.1.7. *The Hotelling model satisfies the* bounded uncertain costs

$(BUC2)$ *condition, if*

$$6\,(t_A + t_B)\,X_{A,B}\,l \le Y_{A,B}^2 \tag{1.32}$$

and

$$6\,(t_A + t_B)\,X_{B,A}\,l \le Y_{B,A}^2. \tag{1.33}$$

Thus, the bounded uncertain costs condition $BUC2$ is implied by the following stronger $SBUC2$ condition.

Let

$$t_m = \min\{t_A, t_B\} \qquad \text{and} \qquad t_M = \max\{t_A, t_B\}.$$

Definition 1.1.8. *The Hotelling model satisfies the* strong bounded uncertain costs $(SBUC2)$ *condition, if*

$$9\,\overline{\Delta} < \left(\frac{3\,t_m^2 - 2\,t_M + 2\,t_m}{t_M}\right) l$$

We observe that the $SBUC2$ condition implies $SBUC1$ condition and so implies the $BUC1$ condition.

Theorem 1.1.2. *If the Hotelling model satisfies the $BUC1$ and $BUC2$ conditions the local optimum price strategy $(\underline{p}_A, \underline{p}_B)$ is a Bayesian-Nash equilibrium.*

Corollary 1.1.2. *If the Hotelling model satisfies $SBUC2$ condition the local optimum price strategy $(\underline{p}_A, \underline{p}_B)$ is a Bayesian-Nash equilibrium.*

Proof. By equalities (1.22) and (1.23), we obtain that $\underline{\pi}_A^{EA}(z_M) \le \underline{\pi}_A^{EA}(z)$ and $\underline{\pi}_B^{EA}(w_M) \le \underline{\pi}_B^{EA}(w)$ for all $z \in I_A$ and for all $w \in I_B$. Hence, putting conditions (1.29), (1.30) and (1.31) together, we obtain the following sufficient condition for the local optimum price strategy $(\underline{p}_A, \underline{p}_B)$ to be a Bayesian-Nash equilibrium:

$$(p_A^M(w_M) - c_A^m)\,l \le \underline{\pi}_A^{EA}(z_M) \qquad \text{and} \qquad (p_B^M(z_M) - c_B^m)\,l \le \underline{\pi}_B^{EA}(w_M). \tag{1.34}$$

By equalities (1.22) and (1.23) we obtain that

$$\underline{\pi}_A^{EA}(z_M) = \frac{(2\,(t_A + 2\,t_B)\,l + E(c_A) + 2\,E(c_B) - 3\,c_A^M)^2}{36\,(t_A + t_B)} = \frac{Y_{A,B}^2}{36\,(t_A + t_B)}$$

and

$$\underline{\pi}_B^{EA}(w_M) = \frac{(2\,(2\,t_A + t_B)\,l + 2\,E(c_A) + E(c_B) - 3\,c_B^M)^2}{36\,(t_A + t_B)} = \frac{Y_{B,A}^2}{36\,(t_A + t_B)}.$$

Also, from (1.1.10), we know that

$$\begin{aligned}
p_A^M(w_M) - c_A^m &= \underline{p}_B^{w_M} - t_A\,l - \epsilon - c_A^m \\
&= \frac{1}{6}\,(3\,c_B^M + 2\,E(c_A) + E(c_B) - 6\,c_A^m + 2\,(t_B - t_A)\,l) - \epsilon \\
&= \frac{1}{6}\,X_{A,B} - \epsilon.
\end{aligned}$$

and

$$\begin{aligned}
p_B^M(z_M) - c_B^m &= \underline{p}_A^{z_M} - t_B\,l - \epsilon - c_B^m \\
&= \frac{1}{6}\,(3\,c_A^M + E(c_A) + 2\,E(c_B) - 6\,c_B^m + 2\,(t_B - t_A)\,l) - \epsilon \\
&= \frac{1}{6}\,X_{B,A} - \epsilon.
\end{aligned}$$

Hence, condition (1.34) holds if inequalities (1.32) and (1.33) are satisfied. □

1.1.5 Comparative profit analysis

From now on, we assume that the $BUC1$ condition holds and that the price strategy $(\underline{p}_A, \underline{p}_B)$ is the local optimum price strategy determined in Theorem 1.1.1.

We observe that the difference between the ex-post profits of both firms has a very useful and clear economical interpretation in terms of the expected cost deviations.

Let $X = \Delta_B\,(2\,t_A + t_B) - \Delta_A\,(t_A + 2\,t_B)$. The difference $\underline{\pi}_A^{EP}(z,w) - \underline{\pi}_B^{EP}(z,w)$ is given by

$$\frac{t_B - t_A}{3}\,l^2 + \frac{2\,X + (\Delta_E - 3\,\Delta_C)\,(4\,(t_A + t_B) - \Delta_A - \Delta_B)}{12\,(t_A + t_B)}.$$

Furthermore, for different production costs, the differences between the ex-post profit of firm A, $\underline{\pi}_A^{EP}(z_1,w) - \underline{\pi}_A^{EP}(z_2,w)$, is given by

$$\frac{(c_A^{z_2} - c_A^{z_1})\,(4\,(t_A + 2\,t_B)\,l - \Delta_E + 3\,(c_B^w + E(c_A) - c_A^{z_1} - c_A^{z_2}))}{12\,(t_A + t_B)}.$$

The difference between the ex-post profit of firm B, $\underline{\pi}_B^{EP}(z,w_1) - \underline{\pi}_B^{EP}(z,w_2)$, is given by

$$\frac{(c_B^{w_2} - c_B^{w_1})\,(4\,(2\,t_A + t_B)\,l + \Delta_E + 3\,(c_A^z + E(c_B) - c_B^{w_1} - c_B^{w_2}))}{12\,(t_A + t_B)}$$

for all $z, z_1, z_2 \in I_A$ and $w, w_1, w_2 \in I_B$.

We observe that the difference between the ex-ante profits of both firms has a very useful and clear economical interpretation in terms of the expected cost deviations.

The difference $\underline{\pi}_A^{EA}(z) - \underline{\pi}_B^{EA}(w)$ is given by

$$\frac{t_B - t_A}{3}\,l^2 + \frac{(\Delta_A + \Delta_B)\,(3\,(\Delta_A - \Delta_B) + 4\,\Delta_E) + (4\,X - 8\,\Delta_E\,(t_A + t_B))\,l}{12\,(t_A + t_B)}.$$

Furthermore, for different production costs, the differences between the ex-ante profits of firm A, $\underline{\pi}_A^{EA}(z_1) - \underline{\pi}_A^{EA}(z_2)$ is given by

$$\frac{(c_A^{z_2} - c_A^{z_1})\,(4\,(t_A + 2\,t_B)\,l + 3\,(2\,E(c_A) - c_A^{z_1} - c_A^{z_2}) - 4\,\Delta_E)}{12\,(t_A + t_B)}$$

and the differences between the ex-ante profits of firm B, $\underline{\pi}_B^{EA}(w_1) - \underline{\pi}_B^{EA}(w_2)$, is given by

$$\frac{(c_B^{w_2} - c_B^{w_1})\,(4\,(2\,t_A + t_B)\,l + 3\,(2\,E(c_B) - c_B^{w_1} - c_B^{w_2}) + 4\,\Delta_E)}{12\,(t_A + t_B)}$$

for all $z, z_1, z_2 \in I_A$ and $w, w_1, w_2 \in I_B$.

The difference between the ex-post and the ex-ante profit for a firm is the real deviation from the realized gain of the firm and the expected gain of the firm knowing its own production cost but being uncertain about the production cost of the other firm. It is the best measure of the risk involved for the firm given the uncertainty in the production costs of the other firm. The difference between the ex-post profit and the ex-ante profit for firm A is

$$\underline{\pi}_A^{EP}(z,w) - \underline{\pi}_A^{EA}(z) = \frac{\Delta_B}{12\,(t_A + t_B)}\;(2\,(t_A + 2\,t_B)\,l - 2\,\Delta_E - 3\,\Delta_A)\,.$$

The difference between the ex-post profit and the ex-ante profit for firm B is

$$\underline{\pi}_B^{EP}(z,w) - \underline{\pi}_B^{EA}(w) = \frac{\Delta_A}{12\,(t_A + t_B)}\;(2\,(2\,t_A + t_B)\,l + 2\,\Delta_E - 3\,\Delta_B)\,.$$

Definition 1.1.9. *The Hotelling model satisfies the* A-bounded uncertain costs $(A - BUC)$ *condition, if for all* $z \in I_A$

$$3\,\Delta_A + 2\,\Delta_E < 2\,(t_A + 2\,t_B)\,l.$$

The Hotelling model satisfies the B-bounded uncertain costs $(B - BUC)$ *condition, if for all* $w \in I_B$

$$3\,\Delta_B - 2\,\Delta_E < 2\,(2\,t_A + t_B)\,l.$$

The following corollary tells us that the sign of the risk of a firm has the opposite sign of the deviation of the competitor firm realized production cost from its average. Hence, under incomplete information the sign of the risk of a firm is not accessible to the firm. However, the probability of the sign of the risk of a firm to be positive or negative is accessible to the firm.

Corollary 1.1.3. *Under the A-bounded uncertain costs $(A - BUC)$ condition,*

$$\underline{\pi}_A^{EP}(z,w) < \underline{\pi}_A^{EA}(z) \quad \text{if and only if} \quad \Delta_B < 0. \tag{1.35}$$

Under the B-bounded uncertain costs $(B - BUC)$ condition,

$$\underline{\pi}_B^{EP}(z,w) < \underline{\pi}_B^{EA}(w) \quad \text{if and only if} \quad \Delta_A < 0. \tag{1.36}$$

The proof of the above corollary follows from a simple manipulation of the previous formulas for the ex-post and ex-ante profits.

The expected profit of the firm is the expected gain of the firm. We observe that the ex-ante and the ex-posts profits of both firms are strictly positive with respect to the local optimum price strategy. Hence, the expected profits of both firms are also strictly positive. Since the ex-ante profit $\underline{\pi}_A^{EA}(z)$ of firm A is equal to

$$\frac{9\,\Delta_A^2 - 12\,\Delta_A\,((t_A + 2\,t_B)\,l - \Delta_E) + 4\,((t_A + 2\,t_B)\,l - \Delta_E)^2}{36\,(t_A + t_B)},$$

from (1.13), we obtain that the expected profit of firm A is given by

$$E(\underline{\pi}_A^{EP}) = \frac{((t_A + 2\,t_B)\,l - \Delta_E)^2}{9\,(t_A + t_B)} + \frac{V_A}{4\,(t_A + t_B)}.$$

Similarly, the expected profit of firm B is given by

$$E(\underline{\pi}_B^{EP}) = \frac{((2\,t_A + t_B)\,l + \Delta_E)^2}{9\,(t_A + t_B)} + \frac{V_B}{4\,(t_A + t_B)}.$$

The difference between the ex-ante and the expected profit of a firm is the deviation from the expected realized gain of the firm given the realization of its own production cost and the expected gain in average for different realizations of its own production cost, but being in both cases uncertain about the production costs of the competitor firm. It is the best measure of the quality of its realized production cost in terms of the expected profit over its own production costs.

Corollary 1.1.4. *The difference between the ex-ante profit and the expected profit for firm A is*

$$E(\underline{\pi}_A^{EP}) - \underline{\pi}_A^{EA}(z) = \frac{\Delta_A\,(4\,(t_A + 2\,t_B)\,l - 3\,\Delta_A - 4\,\Delta_E) + 3\,V_A}{12\,(t_A + t_B)}. \qquad (1.37)$$

The difference between the ex-ante profit and the expected profit for firm B is

$$E(\underline{\pi}_B^{EP}) - \underline{\pi}_B^{EA}(w) = \frac{\Delta_B\,(4\,(2\,t_A + t_B)\,l - 3\,\Delta_B + 4\,\Delta_E) + 3\,V_B}{12\,(t_A + t_B)}. \qquad (1.38)$$

Proof. Let $X = (t_A + 2\,t_B)\,l - \Delta_E$. Hence,

$$\begin{aligned} E(\underline{\pi}_A^{EP}) - \underline{\pi}_A^{EA}(z) &= \frac{4\,X^2 - (2\,X - 3\,\Delta_A)^2 + 9\,V_A}{36\,(t_A + t_B)} \\ &= \frac{\Delta_A\,(4\,X - 3\,\Delta_A) + 3\,V_A}{12\,(t_A + t_B)} \end{aligned}$$

and so equality (1.37) holds. The proof of equality (1.38) follows similarly. □

1.1.6 Comparative consumer surplus and welfare analysis

The ex-post consumer surplus is the realized gain of the consumers community for given outcomes of the production costs of both firms. Under incomplete information, by equation (1.4), the ex-post consumer surplus is

$$\underline{CS}^{EP} = v_T\,l - \frac{5\,t_B + 4\,t_A}{6}\,l^2 - \frac{4\,E(c_B) + 2\,E(c_A) + 3\,\Delta_B}{6}\,l + K_1, \qquad (1.39)$$

where

$$K_1 = \frac{(2\,(t_A + 2\,t_B)\,l - 3\,\Delta_C + \Delta_E)^2}{72\,(t_A + t_B)}.$$

The expected value of the consumer surplus is the expected gain of the consumers community for all possible outcomes of the production costs of both firms. The expected value of the consumer surplus $E(\underline{CS}^{EP})$ is given by

$$\begin{aligned} E(\underline{CS}^{EP}) &= \int_{I_B}\int_{I_A} \underline{CS}^{EP} \mathrm{d}q_A(z)\,\mathrm{d}q_B(w) \\ &= v_T\,l - \frac{5\,t_B + 4\,t_A}{6}\,l^2 - \frac{l}{3}\,(2\,E(c_B) + E(c_A)) + U_1 \end{aligned}$$

where

$$U_1 = \frac{(2\,(t_A + 2\,t_B)\,l - 2\,\Delta_E)^2 + 9\,(V_A + V_B)}{72\,(t_A + t_B)}.$$

We note that, from equalities (1.13) and (1.16), the expected value of K_1 is

$$\begin{aligned} U_1 &= \frac{(2\,(t_A + 2\,t_B)\,l + \Delta_E)^2 - 6\,E(\Delta_C)\,(2\,(t_A + 2\,t_B)\,l + \Delta_E) + 9\,E(\Delta_C^2)}{72\,(t_A + t_B)} \\ &= \frac{(2(t_A + 2t_B)l + \Delta_E)^2 - 6\Delta_E(2(t_A + 2t_B)l + \Delta_E) + 9(V_A + V_B + \Delta_E^2)}{72(t_A + t_B)} \\ &= \frac{(2\,(t_A + 2\,t_B)\,l - 2\,\Delta_E)^2 + 9\,(V_A + V_B)}{72\,(t_A + t_B)}. \end{aligned}$$

The difference between the ex-post consumer surplus and the expected value of the consumer surplus measures the difference between the gain of the consumers for the realized outcomes of the production costs of both firms and the expected gain of the consumers for all possible outcomes of the production costs of both firms. Hence, it measures the risk taken by the consumers for different outcomes of the production costs of both firms.

Corollary 1.1.5. *The difference between the ex-post consumer surplus and* *the expected value of the consumer surplus,* $\underline{CS}^{EP} - E(\underline{CS}^{EP})$*, is*

$$-\frac{\Delta_B}{2}\, l + \frac{(t_A + 2\,t_B)\,(\Delta_B - \Delta_A)}{6\,(t_A + t_B)}\, l + \frac{(\Delta_E - 3\,\Delta_C)^2 - 4\,\Delta_E^2 - 9\,(V_A + V_B)}{72\,(t_A + t_B)}.$$

Proof. Let $X = 2\,(t_A + 2\,t_B)\,l$. Hence,

$$\begin{aligned}
&\underline{CS}^{EP} - E(\underline{CS}^{EP}) = \\
&= -\frac{\Delta_B}{2}\, l + \frac{(X - 3\,\Delta_C + \Delta_E)^2 - (X - 2\,\Delta_E)^2 - 9\,(V_A + V_B)}{72\,(t_A + t_B)} \\
&= -\frac{\Delta_B}{2}\, l + \frac{6\,X\,(\Delta_E - \Delta_C) + (\Delta_E - 3\,\Delta_C)^2 - 4\,\Delta_E^2 - 9\,(V_A + V_B)}{72\,(t_A + t_B)} \\
&= -\frac{\Delta_B}{2}\, l + \frac{X\,(\Delta_B - \Delta_A)}{12\,(t_A + t_B)} + \frac{(\Delta_E - 3\,\Delta_C)^2 - 4\,\Delta_E^2 - 9\,(V_A + V_B)}{72\,(t_A + t_B)}
\end{aligned}$$

□

The ex-post welfare is the realized gain of the state that includes the gains of the consumers community and the gains of the firms for a given outcomes of the production costs of both firms.

By equation (1.5), the ex-post welfare is

$$\underline{W}^{EP} = v_T\, l - \frac{(t_A + t_B)^2 + 5\,t_A\,t_B}{18\,(t_A + t_B)}\, l^2 - \frac{4\,E(c_B) + 2\,E(c_A) + 3\,\Delta_B}{6}\, l + K_2 + K_3, \tag{1.40}$$

where

$$K_2 = \frac{(4\,t_B - t_A)\,(\Delta_E - 3\,\Delta_C) - 3\,(\Delta_A\,(2\,t_A + t_B) + \Delta_B\,(t_A + 2\,t_B))}{18\,(t_A + t_B)}\, l$$

and

$$K_3 = \frac{3\,\Delta_C\,(9\,\Delta_C - 2\,\Delta_E) - \Delta_E^2}{72\,(t_A + t_B)}.$$

The expected value of the welfare is the expected gain of the state for all possible outcomes of the production costs of both firms. The expected value of the welfare $E(\underline{W}^{EP})$ is given by

$$E(\underline{W}^{EP}) = \int_{I_B}\int_{I_A} \underline{W}^{EP} \mathrm{d}q_A(z)\,\mathrm{d}q_B(w)$$
$$= v_T\,l - \frac{(t_A+t_B)^2 + 5\,t_A\,t_B}{18\,(t_A+t_B)}\,l^2 - \frac{2\,E(c_B)+E(c_A)}{3}\,l - \frac{\Delta_E\,(4\,t_B - t_A)}{9\,(t_A+t_B)}\,l + U_2, \tag{1.41}$$

where

$$U_2 = \frac{27\,(V_A+V_B) + 20\,\Delta_E^2}{72\,(t_A+t_B)}.$$

We note that, from equalities (1.13) and (1.16), the expected value of K_3 is

$$\begin{aligned} U_2 &= \frac{27\,E(\Delta_C^2) - 6\,E(\Delta_C)\,\Delta_E - \Delta_E^2}{72\,(t_A+t_B)} \\ &= \frac{27\,(V_A+V_B+\Delta_E^2) - 6\,\Delta_E^2 - \Delta_E^2}{72\,(t_A+t_B)} = \frac{27\,(V_A+V_B) + 20\,\Delta_E^2}{72\,(t_A+t_B)}. \end{aligned}$$

The difference between the ex-post welfare and the expected value of the welfare measures the difference in the gains of the state between the realized outcomes of the production costs of both firms and the expected gain of the state for all possible outcomes of the production costs of both firms. Hence, it measures the risk taken by the state for different outcomes of the production costs of both firms.

Corollary 1.1.6. *The difference between the ex-post welfare and the expected value of welfare, $\underline{W}^{EP} - E(\underline{W}^{EP})$, is*

$$-\frac{\Delta_A\,(t_A + 5\,t_B) + \Delta_B\,(5\,t_A + t_B)}{6\,(t_A+t_B)}\,l + \frac{9\,(\Delta_C^2 - V_A - V_B) - 2\,\Delta_C\,\Delta_E - 7\,\Delta_E^2}{24\,(t_A+t_B)}$$

Proof. From equalities (1.40) and (1.41) we obtain that

$$\underline{W}^{EP} - E(\underline{W}^{EP}) = -\frac{\Delta_B}{2}\,l + K_3 + K_4 + \frac{\Delta_E\,(4\,t_B - t_A)}{9\,(t_A+t_B)}\,l - U_2.$$

We note that

$$
\begin{aligned}
K_5 &= K_4 - U_2 = \frac{3\,\Delta_C\,(9\,\Delta_C - 2\,\Delta_E) - 21\,\Delta_E^2 - 27\,(V_A + V_B)}{72\,(t_A + t_B)} \\
&= \frac{9\,(\Delta_C^2 - V_A - V_B) - 2\,\Delta_C\,\Delta_E - 7\,\Delta_E^2}{24\,(t_A + t_B)}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\underline{W}^{EP} - E(\underline{W}^{EP}) = -\frac{\Delta_B}{2}\,l + \frac{2\,\Delta_E\,(4\,t_B - t_A)}{18\,(t_A + t_B)} + K_5 + \\
&+ \frac{(4\,t_B - t_A)\,(\Delta_E - 3\,\Delta_C) - 3\,(\Delta_A\,(2\,t_A + t_B) + \Delta_B\,(t_A + 2\,t_B))}{18\,(t_A + t_B)}\,l \\
&= -\frac{\Delta_B}{2}\,l + K_5 + \\
&+ \frac{(4\,t_B - t_A)\,(\Delta_E - \Delta_C) - 3\,(\Delta_A\,(2\,t_A + t_B) + \Delta_B\,(t_A + 2\,t_B))}{6\,(t_A + t_B)}\,l \\
&= -\frac{\Delta_B}{2}\,l + K_5 + \\
&+ \frac{(4\,t_B - t_A)\,(\Delta_B - \Delta_A) - 3\,(t_A\,(2\,\Delta_A + \Delta_B) + t_B\,(\Delta_A + 2\,\Delta_B))}{6\,(t_A + t_B)}\,l \\
&= -\frac{\Delta_B}{2}\,l + \frac{2\,\Delta_B\,(t_B - t_A) - \Delta_A\,(t_A + 5\,t_B)}{6\,(t_A + t_B)}\,l + K_5 \\
&= -\frac{\Delta_B\,(5\,t_A + t_B) + \Delta_A\,(t_A + 5\,t_B)}{6\,(t_A + t_B)}\,l + K_5.
\end{aligned}
$$

□

1.1.7 Complete versus Incomplete information

Let us consider the case where the production costs are revealed to both firms before they choose the prices. In this case, the competition between the firms is under complete information.

A *price strategy* (p_A^{CI}, p_B^{CI}) is given by a pair of functions $p_A^{CI} : I_A \times I_B \to \mathbb{R}_0^+$ and $p_B^{CI} : I_A \times I_B \to \mathbb{R}_0^+$ where $p_A^{CI}(z, w)$ denotes the price of firm A and

$p_B^{CI}(z,w)$ denotes the price of firm B when the type of firm A is $z \in I_A$ and the type of firm B is $w \in I_B$.

Under the BC condition, by equations (1.6) and (1.7), the Nash price strategy (p_A^{CI}, p_B^{CI}) is given by

$$\underline{p}_A^{CI}(z,w) = c_B + \frac{2}{3}(\Delta_C) + \frac{t_A + 2\,t_B}{3}\,l$$

and

$$\underline{p}_B^{CI}(z,w) = c_A - \frac{2}{3}(\Delta_C) + \frac{2\,t_A + t_B}{3}\,l.$$

By equation (1.8), the profit $\pi_A^{CI} : I_A \times I_B \to \mathbb{R}_0^+$ of firm A is given by

$$\underline{\pi}_A^{CI}(z,w) = \frac{((t_A + 2\,t_B)\,l - \Delta_C)^2}{9\,(t_A + t_B)}.$$

Similarly, by equation (1.9), the profit $\pi_B^{CI} : I_A \times I_B \to \mathbb{R}_0^+$ of firm B is given by

$$\underline{\pi}_B^{CI}(z,w) = \frac{((2\,t_A + t_B)\,l + \Delta_C)^2}{9\,(t_A + t_B)}.$$

Using equality (1.15), the expected profit $E_B(\underline{\pi}_A^{CI})$ for firm A is given by

$$E_B(\underline{\pi}_A^{CI}) = \frac{((t_A + 2\,t_B)\,l - \Delta_A - \Delta_E)^2 + V_B}{9\,(t_A + t_B)}$$

Similarly, using equality (1.14) the expected profit $E_A(\underline{\pi}_B^{CI})$ for firm B is given by

$$E_A(\underline{\pi}_B^{CI}) = \frac{((2\,t_A + t_B)\,l - \Delta_B + \Delta_E)^2 + V_A}{9\,(t_A + t_B)}$$

The expected profit $E(\underline{\pi}_A^{CI})$ for firm A is given by

$$E(\underline{\pi}_A^{CI}) = \frac{((t_A + 2\,t_B)\,l - \Delta_E)^2 + V_A + V_B}{9\,(t_A + t_B)}.$$

Similarly, the expected profit $E(\underline{\pi}_B^{CI})$ for firm B is given by

$$E(\underline{\pi}_B^{CI}) = \frac{((2\,t_A + t_B)\,l + \Delta_E)^2 + V_A + V_B}{9\,(t_A + t_B)}.$$

By equation (1.12), the consumer surplus is given by

$$\underline{CS}^{CI}(z,w) = v_T\,l - \frac{5\,t_B + 4\,t_A}{6}\,l^2 - \frac{\Delta_A + E(c_A) + 2\,\Delta_B + 2\,E(c_B)}{3}\,l + Z_1, \tag{1.42}$$

where

$$Z_1 = \frac{((t_A + 2\,t_B)\,l - \Delta_C)^2}{18\,(t_A + t_B)}.$$

The expected value of the consumer surplus $E(\underline{CS}^{CI})$ is

$$E(\underline{CS}^{CI}(z,w)) = v_T\,l - \frac{5\,t_B + 4\,t_A}{6}\,l^2 - \frac{E(c_A) + 2\,E(c_B)}{3}\,l + W_1$$

where

$$W_1 = \frac{((t_A + 2\,t_B)\,l - \Delta_E)^2 + V_A + V_B}{18\,(t_A + t_B)}.$$

We note that, from equalities (1.13) and (1.16), the expected value of Z_1 is

$$\begin{aligned} W_1 &= \frac{(t_A + 2\,t_B)^2\,l^2 - 2\,(t_A + 2\,t_B)\,l\,E(\Delta_C) + E(\Delta_C^2)}{18\,(t_A + t_B)} \\ &= \frac{(t_A + 2\,t_B)^2\,l^2 - 2\,\Delta_E\,(t_A + 2\,t_B)\,l + \Delta_E^2 + V_A + V_B}{18\,(t_A + t_B)} \\ &= \frac{((t_A + 2\,t_B)\,l - \Delta_E)^2 + V_A + V_B}{18\,(t_A + t_B)}. \end{aligned}$$

By equation (1.13), the welfare is given by

$$\underline{W}^{CI}(z,w) = v_T\,l - \frac{t_A + t_B}{18}\,l^2 - \frac{\Delta_A + E(c_A) + 2\,\Delta_B + 2\,E(c_B)}{3}\,l + Z_2,$$

where

$$Z_2 = \frac{-5\, t_A\, t_B\, l^2 + 2\,\Delta_C\, l\,(t_A - 4\, t_B) + 5\,\Delta_C^2}{18\,(t_A + t_B)}$$

The expected value of the welfare $E(\underline{W}^{CI})$ is given by

$$E(\underline{W}^{CI}(z,w)) = \upsilon_T\, l - \frac{t_A + t_B}{18}\, l^2 - \frac{E(c_A) + 2\,E(c_B)}{3}\, l + W_2$$

where

$$W_2 = \frac{-5\, t_A\, t_B\, l^2 + 2\,\Delta_E\, l\,(t_A - 4\, t_B) + 5\,(\Delta_E^2 + V_A + V_B)}{18\,(t_A + t_B)}.$$

We note that, from equalities (1.13) and (1.16), the expected value of Z_2 is

$$\begin{aligned} W_2 &= \frac{-5\, t_A\, t_B\, l^2 + 2\,E(\Delta_C)\, l\,(t_A - 4\, t_B) + 5\,E(\Delta_C^2)}{18\,(t_A + t_B)} \\ &= \frac{-5\, t_A\, t_B\, l^2 + 2\,\Delta_E\, l\,(t_A - 4\, t_B) + 5\,(\Delta_E^2 + V_A + V_B)}{18\,(t_A + t_B)}. \end{aligned}$$

Corollary 1.1.7. *The difference between the ex-post profit and the profit, under complete information, for firm A, $\underline{\pi}_A^{EP}(z,w) - \underline{\pi}_A^{CI}(z,w)$, is*

$$\frac{(\Delta_A - \Delta_B)(\Delta_A + 2\,\Delta_B) - (2\,(t_A + 2\,t_B)\,l - 2\,\Delta_C)\,(2\Delta_A + \Delta_B)}{36\,(t_A + t_B)}. \qquad (1.43)$$

The difference between the ex-post profit and the profit, under complete information, for firm B, $\underline{\pi}_B^{EP}(z,w) - \underline{\pi}_B^{CI}(z,w)$, is

$$\frac{(\Delta_B - \Delta_A)(\Delta_B + 2\,\Delta_A) - (2\,(2\,t_A + t_B)\,l + 2\,\Delta_C)\,(2\Delta_B + \Delta_A)}{36\,(t_A + t_B)}. \qquad (1.44)$$

Proof. Let $CI = (t_A + 2\,t_B)\,l - \Delta_C$. Hence,

$$\underline{\pi}_A^{EP}(z,w) - \underline{\pi}_A^{CI}(z,w) = \frac{(2\,CI + \Delta_B - \Delta_A)(2\,CI - \Delta_A - 2\,\Delta_B) - 4\,CI^2}{36\,(t_A + t_B)}$$
$$= \frac{(\Delta_B - \Delta_A)(-\Delta_A - 2\,\Delta_B) + 2\,CI(-2\Delta_A - \Delta_B)}{36\,(t_A + t_B)}$$

and so equality (1.43) holds. The proof of equality (1.44) follows similarly.

□

Corollary 1.1.8. *The difference between the ex-ante profit* $E_B(\underline{\pi}_A^{EP})$ *and* $E_B(\underline{\pi}_A^{CI})$ *for firm* A *is*

$$E_B(\underline{\pi}_A^{EP}) - E_B(\underline{\pi}_A^{CI}) = \frac{\Delta_A\;(5\,\Delta_A - 4\,((t_A + 2\,t_B)\,l - \Delta_E))}{36\,(t_A + t_B)} - \frac{V_B}{9\,(t_A + t_B)}.$$

The difference between the ex-ante profit $E_A(\underline{\pi}_B^{EP})$ *and* $E_A(\underline{\pi}_B^{CI})$ *for firm* B *is*

$$E_A(\underline{\pi}_B^{EP}) - E_A(\underline{\pi}_B^{CI}) = \frac{\Delta_B\;(5\,\Delta_B - 4\,((2\,t_A + t_B)\,l + \Delta_E))}{36\,(t_A + t_B)} - \frac{V_A}{9\,(t_A + t_B)}.$$

The proof of the above corollary follows from a simple manipulation of the previous formulas for the ex-post and ex-ante profits.

The difference between the expected profits of firm A with complete and incomplete information is given by

$$E(\underline{\pi}_A^{EP}) - E(\underline{\pi}_A^{CI}) = \frac{5\,V_A - 4\,V_B}{36\,(t_A + t_B)}. \tag{1.45}$$

The difference between the expected profits of firm B with complete and incomplete information is given by

$$E(\underline{\pi}_B^{EP}) - E(\underline{\pi}_B^{CI}) = \frac{5\,V_B - 4\,V_A}{36\,(t_A + t_B)}. \tag{1.46}$$

Corollary 1.1.9. *The difference between the ex-post consumer surplus and* *the consumer surplus, under complete information,* $\underline{CS}^{EP} - \underline{CS}^{CI}$, *is*

$$\frac{\Delta_A\,(5\,t_A + 4\,t_B) + \Delta_B\,(4\,t_A + 5\,t_B)}{18\,(t_A + t_B)}\,l + \frac{(\Delta_B - \Delta_A - 4\,\Delta_C)\,(\Delta_B - \Delta_A)}{72\,(t_A + t_B)}. \tag{1.47}$$

Therefore, equation (1.47) determines in which cases it is better to have uncertainty in the production costs instead of complete information in terms of consumer surplus $\underline{CS}^{EP} > \underline{CS}^{CI}$.

Proof. From equalities (1.39) and (1.42), we obtain that

$$\underline{CS}^{EP} - \underline{CS}^{CI} = \frac{2\,\Delta_A + \Delta_B}{6}\,l + K_1 - K_2,$$

where

$$K_1 = \frac{(2\,(t_A + 2\,t_B)\,l - 3\,\Delta_C + \Delta_E)^2}{72\,(t_A + t_B)}.$$

and

$$K_2 = \frac{((t_A + 2\,t_B)\,l - \Delta_C)^2}{18\,(t_A + t_B)}.$$

Let $X = (t_A + 2\,t_B)\,l$. We note that

$$\begin{aligned} K_1 - K_2 &= \frac{(2\,X - 3\,\Delta_C + \Delta_E)^2 - 4\,(X - \Delta_C)^2}{72\,(t_A + t_B)} \\ &= \frac{4\,X\,(\Delta_E - \Delta_C) + (\Delta_E - 3\,\Delta_C)^2 - 4\,\Delta_C^2}{72\,(t_A + t_B)} \\ &= \frac{4\,X\,(\Delta_B - \Delta_A) + (\Delta_B - \Delta_A - 2\,\Delta_C)^2 - 4\,\Delta_C^2}{72\,(t_A + t_B)} \\ &= \frac{(t_A + 2\,t_B)\,(\Delta_B - \Delta_A)}{18\,(t_A + t_B)}\,l + \frac{(\Delta_B - \Delta_A - 4\,\Delta_C)\,(\Delta_B - \Delta_A)}{72\,(t_A + t_B)}. \end{aligned}$$

Hence, $\underline{CS}^{EP} - \underline{CS}^{CI}$ is given by expression (1.47). □

The difference between expected value of the consumer surplus and the expected value of the consumer surplus under complete information, is

$$E(\underline{CS}^{EP}) - E(\underline{CS}^{CI}) = \frac{5\,(V_A + V_B)}{72\,(t_A + t_B)}. \tag{1.48}$$

Therefore, in expected value the consumer surplus is greater with incomplete information than with complete information.

The difference between the ex-post welfare and the welfare, under complete information, is given by

$$\underline{W}^{EP} - \underline{W}^{CI} = \frac{2\Delta_A + \Delta_B}{6}\,l - \frac{2\,\Delta_C\,l\,(t_A - 4\,t_B) + 5\,\Delta_C^2}{18\,(t_A + t_B)} + K_3 + K_4,$$

where

$$K_3 = \frac{(4\,t_B - t_A)\,(\Delta_E - 3\,\Delta_C) - 3\,(\Delta_A\,(2\,t_A + t_B) + \Delta_B\,(t_A + 2\,t_B))}{18\,(t_A + t_B)}\,l$$

and

$$K_4 = \frac{3\,\Delta_C\,(9\,\Delta_C - 2\,\Delta_E) - \Delta_E^2}{72\,(t_A + t_B)}.$$

Hence,

$$\underline{W}^{EP} - \underline{W}^{CI} = \frac{\Delta_A\,(t_A - t_B) + \Delta_B\,(t_B - t_A)}{18\,(t_A + t_B)}\,l + \frac{7\,\Delta_C^2 - 6\,\Delta_C\,\Delta_E - \Delta_E^2}{72\,(t_A + t_B)} \tag{1.49}$$

Therefore, equation (1.49) determines in which cases it is better to have uncertainty in the production costs instead of complete information in terms of welfare $\underline{W}^{EP} > \underline{W}^{CI}$.

The difference between expected value of the welfare and the expected value of the welfare under complete information, is

$$E(\underline{W}^{EP}) - E(\underline{W}^{CI}) = \frac{7\,(V_A + V_B)}{72\,(t_A + t_B)}. \tag{1.50}$$

Therefore, in expected value the welfare is greater with incomplete information than with complete information.

1.1.8 Example: Symmetric Hotelling

A Hotelling game is *symmetric*, if $(I_A, \Omega_A, q_A) = (I_B, \Omega_B, q_B)$ and $c = c_A = c_B$. Hence, we observe that all the formulas of this section hold with the following simplifications

$$\Delta_E = 0; \quad E(c) = E(c_A) = E(c_B) \text{ and } V = V_A = V_B.$$

The bounded uncertain costs in the symmetric case can be written in the following simple way.

Definition 1.1.10. *The symmetric Hotelling model satisfies the* bounded uncertain costs $(BUC1)$ *condition, if*

$$-2\,(2\,t_A + t_B)\,l < 3\,\Delta_C < 2\,(t_A + 2\,t_B)\,l$$

for all $z \in I_A$ and for all $w \in I_B$.

Definition 1.1.11. *The symmetric Hotelling model satisfies the* bounded uncertain costs $(BUC2)$ *condition, if*

$$(3\,c_M + 3\,E(c) - 6\,c_m + 2\,(t_B - t_A)\,l)\,l \leq \frac{(2\,(t_A + 2\,t_B)\,l + 3\,E(c) - 3\,c_M)^2}{6\,(t_A + t_B)}$$

and

$$(3\,c_M + 3\,E(c) - 6\,c_m + 2\,(t_A - t_B)\,l)\,l \leq \frac{(2\,(t_B + 2\,t_A)\,l + 3\,E(c) - 3\,c_M)^2}{6\,(t_A + t_B)}.$$

Under the $BUC1$ condition, the expected prices of the local optimum price strategy have the simple expression

$$E(\underline{p}_A) = \frac{t_A + 2\,t_B}{3}\,l + E(c) \quad \text{and} \quad E(\underline{p}_B) = \frac{2\,t_A + t_B}{3}\,l + E(c).$$

By Proposition 1.1.1, for the Hotelling game with incomplete symmetric information, the local optimum price strategy (p_A, p_B) has the form

$$\underline{p}_A^z = \frac{t_A + 2\,t_B}{3}\,l + E(c) + \frac{\Delta_A}{2}; \quad \underline{p}_B^w = \frac{2\,t_A + t_B}{3}\,l + E(c) + \frac{\Delta_B}{2}.$$

The ex-post profit of firm A and firm B are, respectively

$$\underline{\pi}_A^{EP}(z,w) = \frac{(2\,(t_A + 2\,t_B)\,l - 3\,\Delta_A)\,(2\,(t_A + 2\,t_B)\,l - 3\,\Delta_C)}{36\,(t_A + t_B)}$$

and

$$\underline{\pi}_B^{EP}(z,w) = \frac{(2\,(2\,t_A + t_B)\,l - 3\,\Delta_B)\,(2\,(2\,t_A + t_B)\,l + 3\,\Delta_C)}{36\,(t_A + t_B)}.$$

Let $X = \Delta_B\,(2\,t_A + t_B) - \Delta_A\,(t_A + 2\,t_B)$. The difference between the ex-post profits of both firms is given by

$$\underline{\pi}_A^{EP}(z,w) - \underline{\pi}_B^{EP}(z,w) = \frac{t_B - t_A}{3}\,l^2 + \frac{2\,X + 3\,\Delta_C\,(\Delta_A + \Delta_B - 4\,(t_A + t_B))}{12\,(t_A + t_B)}$$

Furthermore, for different production costs, the difference between the ex-post profit of firm A, $\underline{\pi}_A^{EP}(z_1,w) - \underline{\pi}_A^{EP}(z_2,w)$, is given by

$$\frac{(c_A^{z_2} - c_A^{z_1})\,(4\,(t_A + 2\,t_B)\,l + 3\,(c_B^w + E(c) - c_A^{z_1} - c_A^{z_2}))}{12\,(t_A + t_B)}$$

and the difference between the ex-post profit of firm B, $\underline{\pi}_B^{EP}(z,w_1) - \underline{\pi}_B^{EP}(z,w_2)$, is given by

$$\frac{(c_B^{w_2} - c_B^{w_1})\,(4\,(2\,t_A + t_B)\,l + 3\,(c_A^z + E(c) - c_B^{w_1} - c_B^{w_2}))}{12\,(t_A + t_B)}$$

for all $z, z_1, z_2 \in I_A$ and $w, w_1, w_2 \in I_B$.

The ex-ante profit profit of firm A and firm B are, respectively

$$\underline{\pi}_A^{EA}(z) = \frac{(2\,(t_A + 2\,t_B)\,l - 3\,\Delta_A)^2}{36\,(t_A + t_B)}$$

and

$$\underline{\pi}_B^{EA}(w) = \frac{(2\,(2\,t_A + t_B)\,l - 3\,\Delta_B)^2}{36\,(t_A + t_B)}.$$

The difference between the ex-ante profits of both firms is given by

$$\underline{\pi}_A^{EA}(z) - \underline{\pi}_B^{EA}(w) = \frac{t_B - t_A}{3}\,l^2 + \frac{3\,\Delta_C\,(\Delta_A + \Delta_B) + 4\,X\,l}{12\,(t_A + t_B)}$$

Furthermore, for different production costs, the difference between the ex-ante profits of firm A, $\underline{\pi}_A^{EA}(z_1) - \underline{\pi}_A^{EA}(z_2)$, is given by

$$\frac{(c_A^{z_2} - c_A^{z_1})\,(4\,(t_A + 2\,t_B)\,l + 3\,(2\,E(c) - c_A^{z_1} - c_A^{z_2}))}{12\,(t_A + t_B)}.$$

Similarly, $\underline{\pi}_B^{EA}(w_1) - \underline{\pi}_B^{EA}(w_2)$ is given by

$$\frac{(c_B^{w_2} - c_B^{w_1})\,(4\,(2\,t_A + t_B)\,l + 3\,(2\,E(c) - c_B^{w_1} - c_B^{w_2}))}{12\,(t_A + t_B)}$$

for all $z, z_1, z_2 \in I_A$ and $w, w_1, w_2 \in I_B$.

The difference between the ex-post profit and the ex-ante profit for firm A is

$$\underline{\pi}_A^{EP}(z,w) - \underline{\pi}_A^{EA}(z) = \frac{\Delta_B}{12\,(t_A + t_B)}\,(2\,(t_A + 2\,t_B)\,l - 3\,\Delta_A)\,.$$

The difference between the ex-post profit and the ex-ante profit for firm B is

$$\underline{\pi}_B^{EP}(z,w) - \underline{\pi}_B^{EA}(w) = \frac{\Delta_A}{12\,(t_A + t_B)}\,(2\,(2\,t_A + t_B)\,l - 3\,\Delta_B)\,.$$

We observe that that the $A - BUC$ and $B - BUC$ conditions are implied by the $BUC1$ condition. Hence, Corollary 1.1.3 can be rewritten without any restriction, i.e.

$$\underline{\pi}_A^{EP}(z,w) < \underline{\pi}_A^{EA}(z) \quad \text{if and only if} \quad \Delta_B < 0;$$

and

$$\underline{\pi}_B^{EP}(z,w) < \underline{\pi}_B^{EA}(w) \quad \text{if and only if} \quad \Delta_A < 0.$$

The expected profit of firm A and firm B are, respectively,

$$E(\underline{\pi}_A) = \frac{((t_A + 2\,t_B)\,l)^2}{9\,(t_A + t_B)} + \frac{V}{4\,(t_A + t_B)}.$$

and

$$E(\underline{\pi}_B) = \frac{((2\,t_A + t_B)\,l)^2}{9\,(t_A + t_B)} + \frac{V}{4\,(t_A + t_B)}.$$

The difference between the ex-ante profit and the expected profit for firm A is

$$E(\underline{\pi}_A^{EP}) - \underline{\pi}_A^{EA}(z) = \frac{\Delta_A\,(4\,(t_A + 2\,t_B)\,l - 3\,\Delta_A) + 3\,V}{12\,(t_A + t_B)}.$$

The difference between the ex-ante profit and the expected profit for firm B is

$$E(\underline{\pi}_B^{EP}) - \underline{\pi}_B^{EA}(w) = \frac{\Delta_B\,(4\,(2\,t_A + t_B)\,l - 3\,\Delta_B) + 3\,V}{12\,(t_A + t_B)}.$$

The ex-post consumer surplus is

$$\underline{CS}^{EP} = v\,l - \frac{5\,t_B + 4\,t_A}{6}\,l^2 - \frac{2\,E(c) + \Delta_B}{2}\,l + \frac{(2\,(t_A + 2\,t_B)\,l - 3\,\Delta_C)^2}{72\,(t_A + t_B)}.$$

The expected value of the consumer surplus is

$$E(\underline{CS}^{EP}) = v_T\,l - \frac{5\,t_B + 4\,t_A}{6}\,l^2 - E(c)\,l + \frac{4\,(t_A + 2\,t_B)^2\,l^2 + 18\,V}{72\,(t_A + t_B)}.$$

The difference between the ex-post consumer surplus and the expected value of the consumer surplus is

$$\underline{CS}^{EP} - E(\underline{CS}^{EP}) = -\frac{\Delta_A\,(t_A + 2\,t_B) + \Delta_B\,(2\,t_A + t_B)}{6\,(t_A + t_B)}\,l + \frac{\Delta_C^2 - 2\,V}{8\,(t_A + t_B)}.$$

The ex-post welfare is

$$\underline{W}^{EP} = \upsilon_T\,l - \frac{(t_A + t_B)^2 + 5\,t_A\,t_B}{18\,(t_A + t_B)}\,l^2 - E(c) + \frac{3\,\Delta_C^2}{8\,(t_A + t_B)} - W_1,$$

where

$$W_1 = \frac{\Delta_A\,(t_A + 5\,t_B) + \Delta_B\,(t_B + 5\,t_A)}{6\,(t_A + t_B)}\,l.$$

The expected value of the welfare $E(\underline{W}^{EP})$ is given by

$$E(\underline{W}^{EP}) = \upsilon_T\,l - \frac{(t_A + t_B)^2 + 5\,t_A\,t_B}{18\,(t_A + t_B)}\,l^2 - E(c)\,l - \frac{3\,V}{4\,(t_A + t_B)}.$$

The difference between the ex-post welfare and the expected value of welfare is

$$\underline{W}^{EP} - E(\underline{W}^{EP}) = -\frac{\Delta_A\,(t_A + 5\,t_B) + \Delta_B\,(5\,t_A + t_B)}{6\,(t_A + t_B)}\,l + \frac{3\,(\Delta_C^2 - 2\,V)}{8\,(t_A + t_B)}$$

The expected profits $E_B(\underline{\pi}_A^{CI})$ for firm A and $E_A(\underline{\pi}_B^{CI})$ for firm B are given by

$$E_B(\underline{\pi}_A^{CI}) = \frac{((t_A + 2\,t_B)\,l - \Delta_A)^2 + V}{9\,(t_A + t_B)}$$

and

$$E_A(\underline{\pi}_B^{CI}) = \frac{((2\,t_A + t_B)\,l - \Delta_B)^2 + V}{9\,(t_A + t_B)}$$

The expected profits for firm A and B are given, respectively by

$$E(\underline{\pi}_A^{CI}) = \frac{(t_A + 2\,t_B)^2\,l^2 + 2\,V}{9\,(t_A + t_B)}.$$

and

$$E(\underline{\pi}_B^{CI}) = \frac{(2\,t_A + t_B)^2\,l^2 + 2\,V}{9\,(t_A + t_B)}.$$

The expected value of the consumer surplus $E(\underline{CS}^{CI})$ is

$$E(\underline{CS}^{CI}(z,w)) = v_T\,l - \frac{5\,t_B + 4\,t_A}{6}\,l^2 - E(c)\,l + \frac{(t_A + 2\,t_B)^2\,l^2 + 2\,V}{18\,(t_A + t_B)}.$$

The expected value of the welfare $E(\underline{W}^{CI})$ is given by

$$E(\underline{W}^{CI}(z,w)) = v_T\,l - \frac{t_A + t_B}{18}\,l^2 - E(c)\,l + \frac{10\,V - 5\,t_A\,t_B\,l^2}{18\,(t_A + t_B)}.$$

The difference between the ex-post profit and the profit, under complete information, for firm A, is

$$\underline{\pi}_A^{EP}(z,w) - \underline{\pi}_A^{CI}(z,w) = \frac{\Delta_C\,(5\,\Delta_A + 4\,\Delta_B) - 2\,(t_A + 2\,t_B)\,l\,(2\Delta_A + \Delta_B)}{36\,(t_A + t_B)}.$$

The difference between the ex-post profit and the profit, under complete information, for firm B, is

$$\underline{\pi}_B^{EP}(z,w) - \underline{\pi}_B^{CI}(z,w) = \frac{-\Delta_C(5\,\Delta_B + 4\,\Delta_A) - 2\,(2\,t_A + t_B)\,l\,(2\Delta_B + \Delta_A)}{36\,(t_A + t_B)}.$$

The difference between the ex-ante profit and the expected profit, under complete information, for firm A is

$$E_B(\underline{\pi}_A^{EP}) - E_B(\underline{\pi}_A^{CI}) = \frac{\Delta_A\,\left(5\,\Delta_A - 4\,(t_A + 2\,t_B)\,l\right)}{36\,(t_A + t_B)} - \frac{V}{9\,(t_A + t_B)}.$$

The difference between the ex-ante profit and the expected profit, under complete information, for firm B is

$$E_A(\underline{\pi}_B^{EP}) - E_A(\underline{\pi}_B^{CI}) = \frac{\Delta_B\,\left(5\,\Delta_B - 4\,(2\,t_A + t_B)\,l\right)}{36\,(t_A + t_B)} - \frac{V}{9\,(t_A + t_B)}.$$

The differences between the expected profits with complete and incomplete information for firm A and firm B are given by

$$E(\underline{\pi}_A^{EP}) - E(\underline{\pi}_A^{CI}) = E(\underline{\pi}_B^{EP}) - E(\underline{\pi}_B^{CI}) = \frac{V}{36\,(t_A + t_B)}.$$

The difference between the ex-post consumer surplus and the consumer surplus, under complete information, is

$$\underline{CS}^{EP} - \underline{CS}^{CI} = \frac{\Delta_A\,(5\,t_A + 4\,t_B) + \Delta_B\,(4\,t_A + 5\,t_B)}{18\,(t_A + t_B)}\,l + \frac{5\,\Delta_C^2}{72\,(t_A + t_B)}.$$

The difference between expected value of the consumer surplus and the expected value of the consumer surplus under complete information, is

$$E(\underline{CS}^{EP}) - E(\underline{CS}^{CI}) = \frac{10\,V}{72\,(t_A + t_B)}.$$

The difference between the ex-post welfare and the welfare, under complete information, is

$$\underline{W}^{EP} - \underline{W}^{CI} = \frac{\Delta_A\,(t_A - t_B) + \Delta_B\,(t_B - t_A)}{18\,(t_A + t_B)}\,l + \frac{7\,\Delta_C^2}{72\,(t_A + t_B)}.$$

The difference between expected value of the welfare and the expected value of the welfare under complete information, is

$$E(\underline{W}^{EP}) - E(\underline{W}^{CI}) = \frac{7\,V}{36\,(t_A + t_B)}.$$

1.1.9 Firms with the same transportation cost

In this subsection we present the results of the section where the linear transportation costs are equal to both firms, $t_A = t_B = t$, as originally presented by Hotelling.

The point x is the location of the *indifferent consumer* to buy from firm

A or firm B, and it is given by

$$x = \frac{p_B - p_A + t\,l}{2\,t}.$$

Definition 1.1.12. *The Hotelling model satisfies the* bounded costs (BC) *condition, if*

$$|c_A - c_B| < 3\,t\,l.$$

Under the BC condition, the local optimum price strategy $(\underline{p}_A, \underline{p}_B)$ is given by

$$\underline{p}_A = t\,l + \frac{1}{3}\,(2\,c_A + c_B) \quad \text{and} \quad \underline{p}_B = t\,l + \frac{1}{3}\,(c_A + 2\,c_B).$$

and the corresponding equilibrium profits are given by

$$\underline{\pi}_A = \frac{(3\,t\,l + c_B - c_A)^2}{18\,t} \quad \text{and} \quad \underline{\pi}_B = \frac{(3\,t\,l + c_A - c_B)^2}{18\,t}.$$

We note that if $(\underline{p}_A, \underline{p}_B)$ is a local optimum price strategy then $(\underline{p}_A, \underline{p}_B)$ is a Nash price equilibrium.

The consumer surplus $\underline{CS}$ with respect to the local optimum price strategy $(\underline{p}_A, \underline{p}_B)$ is given by

$$\underline{CS} = v_T\,l - \frac{3}{2}\,t\,l^2 - \frac{c_A + 2\,c_B}{3}\,l + \frac{(c_B - c_A + 3\,t\,l)^2}{36\,t}$$

and the welfare $\underline{W}$ is given by

$$\underline{W} = v_T\,l - \frac{1}{4}\,t\,l^2 - \frac{c_A + c_B}{2}\,l + \frac{5\,(c_A - c_B)^2}{36t}.$$

Definition 1.1.13. *The Hotelling model satisfies the* bounded uncertain costs ($BUC1$) *condition, if*

$$|3\,\Delta_C - \Delta_E| < 6\,t\,l,$$

for all $z \in I_A$ *and for all* $w \in I_B$.

Definition 1.1.14. *The Hotelling model satisfies the* strong bounded uncertain costs ($SBUC1$) *condition, if*

$$\overline{\Delta} < 3\,t\,l.$$

Corollary 1.1.10. *There is a local optimum price strategy* $(\underline{p}_A, \underline{p}_B)$ *if and only if the* $BUC1$ *condition holds. Under the* $BUC1$ *condition, the expected prices of the local optimum price strategy are given by*

$$E(\underline{p}_A) = t\,l + E(c_A) - \frac{\Delta_E}{3};$$
$$E(\underline{p}_B) = t\,l + E(c_B) + \frac{\Delta_E}{3}.$$

Furthermore, the local optimum price strategy $(\underline{p}_A, \underline{p}_B)$ *is unique and it is given by*

$$\underline{p}_A^z = E(\underline{p}_A) + \frac{\Delta_A}{2}; \quad \underline{p}_B^w = E(\underline{p}_B) + \frac{\Delta_B}{2}.$$

The ex-post profit of firm A is

$$\underline{\pi}_A^{EP}(z, w) = \frac{(6\,t\,l + \Delta_E - 3\,\Delta_C)(6\,t\,l + \Delta_E - 3\,\Delta_C - 3\,\Delta_B)}{72\,t}$$

and the ex-post profit of firm B is

$$\underline{\pi}_B^{EP}(z, w) = \frac{(6\,t\,l - \Delta_E + 3\,\Delta_C)(6\,t\,l - \Delta_E + 3\,\Delta_C - 3\,\Delta_A)}{72\,t}.$$

The ex-ante profit of firm A is

$$\underline{\pi}_A^{EA}(z) = \frac{(6\,t\,l - 3\,\Delta_A - 2\,\Delta_E)^2}{72\,t}.$$

and the ex-ante profit of firm B is

$$\underline{\pi}_B^{EA}(w) = \frac{(6\,t\,l - 3\,\Delta_B + 2\,\Delta_E)^2}{72\,t}.$$

Definition 1.1.15. *The Hotelling model satisfies the* bounded uncertain costs $(BUC2)$ *condition, if*

$$3\,\left(c_A^M + c_B^M - 2\,c_A^m\right) + E(c_A) - E(c_B) \le 3\,t\,l + \frac{\left(E(c_A) + 2\,E(c_B) - 3\,c_A^M\right)^2}{12\,t\,l}$$

and

$$3\,\left(c_A^M + c_B^M - 2\,c_B^m\right) + E(c_B) - E(c_A) \le 3\,t\,l + \frac{\left(2\,E(c_A) + E(c_B) - 3\,c_B^M\right)^2}{12\,t\,l}.$$

Definition 1.1.16. *The Hotelling model satisfies the* strong bounded uncertain costs $(SBUC2)$ *condition, if*

$$7\,\overline{\Delta} < 3\,t\,l$$

Theorem 1.1.3. *If the Hotelling model satisfies the $BUC1$ and $BUC2$ conditions the local optimum price strategy $(\underline{p}_A, \underline{p}_B)$ is a Bayesian-Nash equilibrium.*

Corollary 1.1.11. *If the Hotelling model satisfies $SBUC2$ condition the local optimum price strategy $(\underline{p}_A, \underline{p}_B)$ is a Bayesian-Nash equilibrium.*

Now, we present some results of comparative analysis of profits, consumer surplus and welfare.

The difference between the ex-post profits of both firms is given by

$$\underline{\pi}_A^{EP}(z, w) - \underline{\pi}_B^{EP}(z, w) = \frac{6\,t\,l\,(\Delta_A - \Delta_B) + (\Delta_E - 3\,\Delta_C)\,(8\,t\,l - \Delta_A - \Delta_B)}{24\,t}.$$

Furthermore, for different production costs, the difference between the ex-post profit of firm A, $\underline{\pi}_A^{EP}(z_1, w) - \underline{\pi}_A^{EP}(z_2, w)$ is given by

$$\frac{(c_A^{z_2} - c_A^{z_1})\,(12\,t\,l - \Delta_E + 3\,(c_B^{w} + E(c_A) - c_A^{z_1} - c_A^{z_2}))}{24\,t}.$$

The difference between the ex-post profit of firm B, $\underline{\pi}_B^{EP}(z, w_1) - \underline{\pi}_B^{EP}(z, w_2)$, is given by

$$\frac{(c_B^{w_2} - c_B^{w_1})\,(12\,t\,l + \Delta_E + 3\,(c_A^{z} + E(c_B) - c_B^{w_1} - c_B^{w_2}))}{24\,t}$$

for all $z, z_1, z_2 \in I_A$ and $w, w_1, w_2 \in I_B$. The difference between the ex-ante profits of both firms is given by

$$\underline{\pi}_A^{EA}(z) - \underline{\pi}_B^{EA}(w) = \frac{(4\,t\,l - \Delta_A - \Delta_B)\,(3\,(\Delta_B - \Delta_A) - 4\,\Delta_E)}{24\,t}.$$

Furthermore, for different production costs, the differences between the ex-ante profits of a firm are given by

$$\underline{\pi}_A^{EA}(z_1) - \underline{\pi}_A^{EA}(z_2) = \frac{(c_A^{z_2} - c_A^{z_1})\,(3\,(4\,t\,l + 2\,E(c_A) - c_A^{z_1} - c_A^{z_2}) - 4\,\Delta_E)}{24\,t}$$

and

$$\underline{\pi}_B^{EA}(w_1) - \underline{\pi}_B^{EA}(w_2) = \frac{(c_B^{w_2} - c_B^{w_1})\,(3\,(4\,t\,l + 2\,E(c_B) - c_B^{w_1} - c_B^{w_2}) + 4\,\Delta_E)}{24\,t}$$

for all $z, z_1, z_2 \in I_A$ and $w, w_1, w_2 \in I_B$. The difference between the ex-post profit and the ex-ante profit for firm A is

$$\underline{\pi}_A^{EP}(z, w) - \underline{\pi}_A^{EA}(z) = \frac{\Delta_B}{24\,t}\,(6\,t\,l - 2\,\Delta_E - 3\,\Delta_A)$$

and the difference between the ex-post profit and the ex-ante profit for firm B is

$$\underline{\pi}_B^{EP}(z, w) - \underline{\pi}_B^{EA}(w) = \frac{\Delta_A}{24\,t}\,(6\,t\,l + 2\,\Delta_E - 3\,\Delta_B)\,.$$

Definition 1.1.17. *The Hotelling model satisfies the* A-bounded uncertain costs $(A - BUC)$ *condition, if for all* $z \in I_A$

$$3\,\Delta_A + 2\,\Delta_E < 6\,t\,l.$$

The Hotelling model satisfies the B-bounded uncertain costs $(B - BUC)$ *condition, if for all* $w \in I_B$

$$3\,\Delta_B - 2\,\Delta_E < 6\,t\,l.$$

Under the A-bounded uncertain costs $(A - BUC)$ condition,

$$\underline{\pi}_A^{EP}(z,w) < \underline{\pi}_A^{EA}(z) \quad \text{if and only if} \quad \Delta_B < 0.$$

Under the B-bounded uncertain costs $(B - BUC)$ condition,

$$\underline{\pi}_B^{EP}(z,w) < \underline{\pi}_B^{EA}(w) \quad \text{if and only if} \quad \Delta_A < 0.$$

The expected profit of firm A is given by

$$E(\underline{\pi}_A^{EP}) = \frac{(3\,t\,l - \Delta_E)^2}{18\,t} + \frac{V_A}{8\,t}$$

and the expected profit of firm B is given by

$$E(\underline{\pi}_B^{EP}) = \frac{(3\,t\,l + \Delta_E)^2}{18\,t} + \frac{V_B}{8\,t}.$$

The difference between the ex-ante profit and the expected profit for firm A is

$$E(\underline{\pi}_A^{EP}) - \underline{\pi}_A^{EA}(z) = \frac{\Delta_A\,(12\,t\,l - 3\,\Delta_A - 4\,\Delta_E) + 3\,V_A}{24\,t}$$

and the difference between the ex-ante profit and the expected profit for firm

B is

$$E(\underline{\pi}_B^{EP}) - \underline{\pi}_B^{EA}(w) = \frac{\Delta_B\,(12\,t\,l - 3\,\Delta_B + 4\,\Delta_E) + 3\,V_B}{24\,t}.$$

Under incomplete information, the ex-post consumer surplus is

$$\underline{CS}^{EP} = v_T\,l - \frac{3}{2}\,t\,l^2 - \frac{l}{3}\,(2\,E(c_B) + E(c_A)) - \frac{\Delta_B\,l}{2} + \frac{(6\,t\,l - 3\,\Delta_C + \Delta_E)^2}{144\,t},$$

and the expected value of the consumer surplus is given by

$$E(\underline{CS}^{EP}) = v_T\,l - \frac{3}{2}\,t\,l^2 - \frac{l}{3}\,(2\,E(c_B) + E(c_A)) + \frac{(6\,t\,l - 2\,\Delta_E)^2 + 9\,(V_A + V_B)}{144\,t}$$

and the difference between the ex-post consumer surplus and the expected value of the consumer surplus is

$$\underline{CS}^{EP} - E(\underline{CS}^{EP}) = -\frac{\Delta_A + \Delta_B}{4}\,l + \frac{(\Delta_E - 3\,\Delta_C)^2 - 4\,\Delta_E^2 - 9\,(V_A + V_B)}{144\,t}.$$

The ex-post welfare, $\underline{W}^{EP}$, is

$$v_T\,l - \frac{1}{4}\,t\,l^2 - \frac{E(c_A) + E(c_B) + \Delta_A + \Delta_B}{2}\,l - \frac{3\,\Delta_C(2\,\Delta_E - 9\,\Delta_C) + (\Delta_E)^2}{144\,t},$$

the expected value of the welfare is given by

$$E(\underline{W}^{EP}) = v_T\,l - \frac{1}{4}\,t\,l^2 - \frac{E(c_A) + E(c_B)}{2}\,l + \frac{27\,(V_A + V_B) + 20\,\Delta_E^2}{144\,t}$$

and the difference between the ex-post welfare and the expected value of welfare is

$$\underline{W}^{EP} - E(\underline{W}^{EP}) = -\frac{\Delta_A + \Delta_B}{2}\,l + \frac{9\,(\Delta_C^2 - V_A - V_B) - 2\,\Delta_C\,\Delta_E - 7\,\Delta_E^2}{48\,t}.$$

Under complete information, the expected profit, $E_B(\underline{\pi}_A^{CI})$, for firm A is given by

$$E_B(\underline{\pi}_A^{CI}) = \frac{(3\,t\,l - \Delta_A - \Delta_E)^2 + V_B}{18\,t}$$

and the expected profit, $E_A(\underline{\pi}_B^{CI})$, for firm B is given by

$$E_A(\underline{\pi}_B^{CI}) = \frac{(3\,t\,l - \Delta_B + \Delta_E)^2 + V_A}{18\,t}.$$

The expected profit $E(\underline{\pi}_A^{CI})$ for firm A is given by

$$E(\underline{\pi}_A^{CI}) = \frac{(3\,t\,l - \Delta_E)^2 + V_A + V_B}{18\,t}$$

and the expected profit $E(\underline{\pi}_B^{CI})$ for firm B is given by

$$E(\underline{\pi}_B^{CI}) = \frac{(3\,t\,l + \Delta_E)^2 + V_A + V_B}{18\,t}.$$

Under complete information, the consumer surplus is given by

$$\underline{CS}^{CI}(z,w) = v_T\,l - \frac{3}{2}\,t\,l^2 - \frac{E(c_A) + 2E(c_B) + \Delta_A + 2\Delta_B}{3}\,l + \frac{(3\,t\,l - \Delta_C)^2}{36\,t}$$

and expected value of the consumer surplus $E(\underline{CS}^{CI})$ is

$$E(\underline{CS}^{CI}(z,w)) = v_T\,l - \frac{3}{2}\,t\,l^2 - \frac{E(c_A) + 2E(c_B)}{3}\,l + \frac{(3\,t\,l - \Delta_E)^2 + V_A + V_B}{36\,t}.$$

The welfare is given by

$$\underline{W}^{CI}(z,w) = v_T\,l - \frac{1}{4}\,t\,l^2 - \frac{E(c_A) + E(c_B) + \Delta_A + \Delta_B}{2}\,l + \frac{5\,\Delta_C^2}{36t}$$

and the expected value of the welfare $E(\underline{W}^{CI})$ is

$$E(\underline{W}^{CI}(z,w)) = v_T\,l - \frac{1}{4}\,t\,l^2 - \frac{E(c_A) + E(c_B)}{2}\,l + \frac{5\,(V_A + V_B + \Delta_E^2)}{36t}.$$

The difference between the ex-post profit and the profit, under complete information, for firm A, $\underline{\pi}_A^{EP}(z,w) - \underline{\pi}_A^{CI}(z,w)$, is

$$\frac{(\Delta_A - \Delta_B)(\Delta_A + 2\,\Delta_B) - 2\,(3\,t\,l - \Delta_C)\,(2\Delta_A + \Delta_B)}{72\,t}$$

and the difference between the ex-post profit and the profit, under complete information, for firm B, $\underline{\pi}_B^{EP}(z,w) - \underline{\pi}_B^{CI}(z,w)$, is

$$\frac{(\Delta_B - \Delta_A)(\Delta_B + 2\,\Delta_A) - 2\,(3\,t\,l + \Delta_C)\,(2\Delta_B + \Delta_A)}{72\,t}.$$

The difference between the ex-ante profit $E_B(\underline{\pi}_A^{EP})$ and $E_B(\underline{\pi}_A^{CI})$ for firm A is

$$E_B(\underline{\pi}_A^{EP}) - E_B(\underline{\pi}_A^{CI}) = \frac{\Delta_A\;(5\,\Delta_A - 4\,(3\,t\,l - \Delta_E))}{72\,t} - \frac{V_B}{18\,t}$$

and the difference between the ex-ante profit $E_A(\underline{\pi}_B^{EP})$ and $E_A(\underline{\pi}_B^{CI})$ for firm B is

$$E_A(\underline{\pi}_B^{EP}) - E_A(\underline{\pi}_B^{CI}) = \frac{\Delta_B\;(5\,\Delta_B - 4\,(3\,t\,l + \Delta_E))}{72\,t} - \frac{V_A}{18\,t}.$$

The differences between the expected profits of the firms with complete and incomplete information are given by

$$E(\underline{\pi}_A^{EP}) - E(\underline{\pi}_A^{CI}) = \frac{5\,V_A - 4\,V_B}{72\,t}; \quad \text{and} \quad E(\underline{\pi}_B^{EP}) - E(\underline{\pi}_B^{CI}) = \frac{5\,V_B - 4\,V_A}{72\,t}.$$

The difference between the ex-post consumer surplus and the consumer surplus, under complete information, is

$$\underline{CS}^{EP} - \underline{CS}^{CI} = \frac{(\Delta_A + \Delta_B)\,l}{4} + \frac{(\Delta_B - \Delta_A)(\Delta_B - \Delta_A - 4\,\Delta_C)}{144\,t}$$

and the difference between expected value of the consumer surplus and the expected value of the consumer surplus under complete information, is

$$E(\underline{CS}^{EP}) - E(\underline{CS}^{CI}) = \frac{5\,(V_A + V_B)}{144\,t}.$$

The difference between the ex-post welfare and the welfare, under complete information, is

$$\underline{W}^{EP} - \underline{W}^{CI} = \frac{7\,(\Delta_C)^2 - 6\,\Delta_C \Delta_E - (\Delta_E)^2}{144\,t}$$

and the difference between expected value of the welfare and the expected value of the welfare under complete information, is

$$E(\underline{W}^{EP}) - E(\underline{W}^{CI}) = \frac{7\,(V_A + V_B)}{144\,t}.$$

1.2 Quadratic transportation costs

In this section, we study the Hotelling model [25] with uncertainty in the production costs of both firms with quadratic transportation costs as presented by d'Aspremont el at. [2].

We introduce the bounded uncertain costs and location $BUCL1$ condition that defines a bound for the costs in terms of the transportation cost, the road length of the line and the location of the firms. Under the bounded costs $BUCL1$ condition we compute the unique local optimum price strategy for the firms with the property that the market shares of both firms are not empty for any outcome of production costs. We introduce a mild additional bounded uncertain costs $BUCL2$ and, under the $BUCL1$ and $BUCL2$ conditions, we prove that the local optimum price strategy is a Bayesian-Nash price strategy.

We introduce the $BUCL3$ condition and we study the optimal localization and price strategies under incomplete information on the production costs of the firms and. Under the $BUCL3$, and assuming that the firms choose the Bayesian-Nash price strategy, we show that the maximal differentiation is a local optimum for the localization strategy of both firms. Finally, we do a complete analysis of profits, consumer surplus and welfare under complete

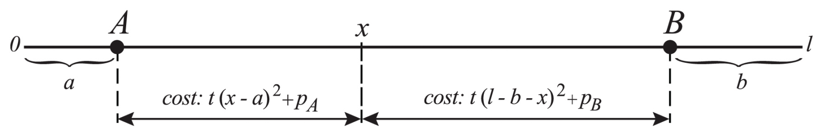


Figure 1.2: Hotelling's linear city with quadratic transportation costs

and incomplete information.

1.2.1 Hotelling model under complete information

The buyers of a commodity will be supposed uniformly distributed along a line with length l, where two firms A and B located at respective distances a and b from the endpoints of the line sell the same commodity with unitary *production costs* c_A and c_B. We assume without loss of generality that $a \geq 0$, $b \geq 0$ and $l - a - b \geq 0$. No customer has any preference for either seller except on the ground of price plus *transportation cost* t.

Denote A's *price* by p_A and B's *price* by p_B. The point of division $x = x(p_A, p_B) \in]0, l[$ between the regions served by the two entrepreneurs is determined by the condition that at this place it is a matter of indifference whether one buys from A or from B (see Figure 1.2). The point x is the location of the *indifferent consumer* to buy from firm A or firm B, if

$$p_A + t\,(x - a)^2 = p_B + t\,(l - b - x)^2$$

Let

$$m = l - a - b \qquad \text{and} \qquad \Delta_l = a - b.$$

Solving for x, we obtain

$$x = \frac{p_B - p_A}{2\,t\,m} + \frac{l + \Delta_l}{2}.$$

Both firms have a non-empty market share if, and only if, $x \in]0, l[$. Hence, the prices will have to satisfy

$$|p_A - p_B - t\,m\,\Delta_l| < t\,m\,l \tag{1.51}$$

Assuming inequality (1.51), both firms A and B have a non-empty demand (x and $l - x$) and the *profits* of the two firms are defined respectively by

$$\pi_A = (p_A - c_A)\,x = (p_A - c_A)\left(\frac{p_B - p_A}{2\,t\,m} + \frac{l + \Delta_l}{2}\right) \tag{1.52}$$

and

$$\pi_B = (p_B - c_B)\,(l - x) = (p_B - c_B)\left(\frac{p_A - p_B}{2\,t\,m} + \frac{l - \Delta_l}{2}\right). \tag{1.53}$$

Two of the fundamental economic quantities in oligopoly theory are the consumer surplus CS and the welfare W. The consumer surplus is the gain of the consumers community for given price strategies of both firms. The welfare is the gain of the state that includes the gains of the consumers community and the gains of the firms for given price strategies of both firms.

Let us denote by v_T the total amount that consumers are willing to pay for the commodity. The total amount $v(y)$ that a consumer located at y pays for the commodity is given by

$$v(y) = \begin{cases} p_A + t\,(y - a)^2 & \text{if } 0 < y < x; \\ p_B + t\,(l - b - y)^2 & \text{if } x < y < l. \end{cases}$$

The *consumer surplus* CS is the difference between the total amount that a consumer is willing to pay v_T and the total amount that the consumer pays $v(y)$

$$CS = \int_0^l v_T - v(y)\mathrm{d}y. \tag{1.54}$$

The *welfare* W is given by adding the profits of firms A and B with the consumer surplus

$$W = CS + \pi_A + \pi_B. \tag{1.55}$$

Definition 1.2.1. *A price strategy* $(\underline{p}_A, \underline{p}_B)$ *for both firms is a* local optimum price strategy *if (i) for every small deviation of the price* $\underline{p}_A$ *the profit* π_A *of firm A decreases, and for every small deviation of the price* p_B *the profit* π_B *of firm B decreases (local optimum property); and (ii) the indifferent consumer exists, i.e.* $0 < \underline{x} < l$ *(duopoly property).*

Let us compute the local optimum price strategy $(\underline{p}_A, \underline{p}_B)$. Differentiating π_A with respect to p_A and π_B with respect to p_B and equalizing to zero, we obtain the first order conditions (FOC). The FOC imply that

$$\underline{p}_A = t\,m\left(l + \frac{\Delta_l}{3}\right) + \frac{1}{3}\,(2\,c_A + c_B) \tag{1.56}$$

and

$$\underline{p}_B = t\,m\left(l - \frac{\Delta_l}{3}\right) + \frac{1}{3}\,(c_A + 2\,c_B). \tag{1.57}$$

We note that the first order conditions refer to jointly optimizing the profit function (1.52) with respect to the price p_A and the profit function (1.53) with respect to the price p_B.

Since the profit functions (1.52) and (1.53) are concave, the second-order conditions for this maximization problem are satisfied and so the prices (1.56) and (1.57) are indeed maxima for the functions (1.52) and (1.53), respectively. The corresponding equilibrium profits are given by

$$\underline{\pi}_A = \frac{(m\,(3\,l + \Delta_l)\,t + c_B - c_A)^2}{18\,t\,m} \tag{1.58}$$

and

$$\underline{\pi}_B = \frac{(m\,(3\,l - \Delta_l)\,t + c_A - c_B)^2}{18\,t\,m}. \tag{1.59}$$

Furthermore, the indifferent consumer location corresponding to the maximizers $\underline{p}_A$ and $\underline{p}_B$ of the profit functions π_A and π_B is

$$\underline{x} = \frac{l}{2} + \frac{\Delta_l}{6} + \frac{c_B - c_A}{6\,t\,m}.$$

Finally, for the pair of prices $(\underline{p}_A, \underline{p}_B)$ to be a local optimum price strategy, we need assumption (1.51) to be satisfied with respect to these pair of prices. We observe that assumption (1.51) is satisfied with respect to the pair of prices $(\underline{p}_A, \underline{p}_B)$ if and only if the following condition with respect to the production costs is satisfied.

Definition 1.2.2. *The Hotelling model satisfies the* bounded costs and location (BCL) *condition, if*

$$|c_A - c_B - t\,m\,\Delta_l| < 3\,t\,m\,l.$$

We note that under the BCL condition the prices are higher than the production costs $\underline{p}_A > c_A$ and $\underline{p}_B > c_B$. Hence, there is a local optimum price strategy if and only if the BCL condition holds. Furthermore, under the BCL condition, the pair of prices $(\underline{p}_A, \underline{p}_B)$ is the local optimum price strategy.

A strong restriction that the BCL condition imposes is that Δ_C converges to 0 when m tends to 0, i.e. when the differentiation in the localization tends to vanish.

We note that, if a Nash price equilibrium satisfies the duopoly property then it is a local optimum price strategy. However, a local optimum price strategy is only a local strategic maximum. Hence, the local optimum price strategy to be a Nash equilibrium must also be global strategic maximum. In this section, we are going to show that this is the case.

Following D'Aspremont et al. [2], we note that the profits of the two firms, valued at local optimum price strategy are globally optimal if they are at least as great as the payoffs that firms would earn by undercutting the rivals' price and supplying the whole market.

Let (p_A, p_B) be the local optimum price strategy. Firm A may gain the whole market, undercutting its rival by setting

$$p_A^M = \underline{p}_B - t\,m\,(l - \Delta_l).$$

In this case the profit amounts to

$$\pi_A^M = \frac{2}{3}\,(c_B - c_A + t\,m\,\Delta_l)\;l.$$

A similar argument is valid for store B. Undercutting this rival, setting

$$p_B^M = \underline{p}_A - t\,m\,(l + \Delta_l),$$

it would earn

$$\pi_B^M = \frac{2}{3}\,(c_A - c_B - t\,m\,\Delta_l)\;l.$$

The conditions for such undercutting not to be profitable are $\underline{\pi}_A \geq \pi_A^M$ and $\underline{\pi}_B \geq \pi_B^M$. Hence, proving that

$$\frac{(m\,(3\,l + \Delta_l)\,t + c_B - c_A)^2}{18\,t\,m} \geq \frac{2}{3}\,(t\,m\,\Delta_l - \Delta_C)\;l \tag{1.60}$$

is sufficient to prove that $\underline{\pi}_A \geq \pi_A^M$. Similarly, proving that

$$\frac{(m\,(3\,l - \Delta_l)\,t + c_A - c_B)^2}{18\,t\,m} \geq \frac{2}{3}\,(\Delta_C - t\,m\,\Delta_l)\;l \tag{1.61}$$

is sufficient to prove that $\underline{\pi}_B \geq \pi_B^M$.

However, conditions (1.60) and (1.61) are satisfied because they are equivalent to

$$(m\,(3\,l - \Delta_l)\,t + c_A - c_B)^2 \geq 0$$

and

$$(m\,(3\,l + \Delta_l)\,t + c_B - c_A)^2 \geq 0.$$

Therefore, if $(\underline{p}_A, \underline{p}_B)$ is a local optimum price strategy then $(\underline{p}_A, \underline{p}_B)$ is a Nash price equilibrium.

We are going to find when the maximal differentiation is a local optimum strategy assuming that the firms in second subgame choose the Nash price equilibrium strategy. For a complete discussion see Ziss [41].

We note that from (1.56) and (1.58), we can write the profit of firm A as

$$\underline{\pi}_A = \frac{(\underline{p}_A - c_A)^2}{2\,t\,(l-a-b)}.$$

Since

$$\frac{\partial \underline{p}_A}{\partial a} = -\frac{2}{3}\,t\,(l+a),$$

we obtain that

$$\frac{\partial \underline{\pi}_A}{\partial a} = -\frac{\underline{p}_A - c_A}{6\,t\,(l-a-b)^2}\,\left(c_A - c_B + t\,(l-a-b)\,(l+3\,a+b)\right).$$

Similarly, we obtain that

$$\frac{\partial \underline{\pi}_B}{\partial b} = \frac{\underline{p}_B - c_B}{6\,t\,(l-a-b)^2}\,\left(c_A - c_B - t\,(l-a-b)\,(l+a+3\,b)\right).$$

Therefore, the maximal differentiation $(a,b) = (0,0)$ is a local optimum strategy if and only if

$$\frac{\partial \underline{\pi}_A}{\partial a}(0,0) = -\frac{\underline{p}_A - c_A}{6\,t\,l^2}\,\left(c_A - c_B + t\,l^2\right) < 0$$

and

$$\frac{\partial \underline{\pi}_B}{\partial b}(0,0) = \frac{\underline{p}_B - c_B}{6\,t\,l^2}\,\left(c_A - c_B - t\,l^2\right) < 0$$

Since

$$\frac{\underline{p}_A - c_A}{6\,t\,l^2} > 0 \quad \text{and} \quad \frac{\underline{p}_B - c_B}{6\,t\,l^2} > 0$$

the maximal differentiation $(a,b) = (0,0)$ is a local optimum strategy if and only if

$$|c_A - c_B| < t\,l^2.$$

Throughout this section, consider

$$X_1 = v_T\,l - \frac{t}{3}\,l^3 + t\,l\,b\,(l-b) - t\,m\,l\left(l - \frac{\Delta_l}{3}\right)$$

and

$$X_2 = \frac{m\,t}{36}\,(45\,l^2 + 6\,l\,\Delta_l + 5\,\Delta_l^2).$$

By equation (1.54), the consumer surplus $\underline{CS}$ with respect to the local optimum price strategy $(\underline{p}_A, \underline{p}_B)$ is given by

$$\begin{aligned}\underline{CS} &= \int_0^x v_T - \underline{p}_A - t\,(y-a)^2\,\mathrm{d}y + \int_x^l v - \underline{p}_B - t\,(l-b-y)^2\,\mathrm{d}y \\ &= v_T\,l + \underline{x}^2\,(l-a-b)\,t + (b\,(l-b)\,t - p_B)\,l - \frac{t}{3}\,l^3\end{aligned}$$

Hence,

$$\underline{CS} = X_1 - \frac{c_A + 2\,c_B}{3}\,l + \frac{(t\,m\,(3\,l+\Delta_l) + c_B - c_A)^2}{36\,t\,m}. \qquad (1.62)$$

Adding (1.58), (1.59) and (1.62), we obtain the welfare

$$\underline{W} = X_1 - \frac{c_A + c_B}{2}\,l - \frac{5\,(c_A - c_B)}{18}\,\Delta_l + \frac{5\,(c_A - c_B)^2}{36\,t\,m} + X_2. \qquad (1.63)$$

1.2.2 Incomplete information on the production costs

The incomplete information consists in each firm to know its production cost but to be uncertain about the competitor's cost. In this subsection, we introduce a simple notation that is fundamental for the elegance and understanding of the results presented in this section. This notation has already been introduced in subsection 1.1.2. However, we duplicate the information in order to guarantee the independence of the sections.

Let the triples (I_A, Ω_A, q_A) and (I_B, Ω_B, q_B) represent (finite, countable or uncountable) sets of types I_A and I_B with σ-algebras Ω_A and Ω_B and probability measures q_A and q_B, over I_A and I_B, respectively.

We define the expected values $E_A(f)$, $E_B(f)$ and $E(f)$ with respect to the probability measures q_A and q_B as follows:

$$E_A(f) = \int_{I_A} f(z,w)\, \mathrm{d}q_A(z); \quad E_B(f) = \int_{I_B} f(z,w)\, \mathrm{d}q_B(w)$$

and

$$E(f) = \int_{I_A} \int_{I_B} f(z,w)\, \mathrm{d}q_B(w) \mathrm{d}q_A(z).$$

Let $c_A : I_A \to \mathbb{R}_0^+$ and $c_B : I_B \to \mathbb{R}_0^+$ be measurable functions where $c_A^z = c_A(z)$ denotes the production cost of firm A when the type of firm A is $z \in I_A$ and $c_B^w = c_B(w)$ denotes the production cost of firm B when the type of firm B is $w \in I_B$. Furthermore, we assume that the expected values of c_A and c_B are finite

$$E(c_A) = E_A(c_A) = \int_{I_A} c_A^z\, \mathrm{d}q_A(z) < \infty;$$
$$E(c_B) = E_B(c_B) = \int_{I_B} c_B^w\, \mathrm{d}q_B(w) < \infty.$$

We assume that $\mathrm{d}q_A(z)$ denotes the probability of the *belief* of the firm B on the production costs of the firm A to be c_A^z. Similarly, we assume that $\mathrm{d}q_B(w)$ denotes the probability of the belief of the firm A on the production costs of the firm B to be c_B^w.

The simplicity of the following cost deviation formulas is crucial to express the main results of this section in a clear and understandable way. The *cost deviations* of firm A and firm B

$$\Delta_A : I_A \to \mathbb{R}_0^+ \quad \text{and} \quad \Delta_B : I_B \to \mathbb{R}_0^+$$

are given respectively by $\Delta_A(z) = c_A^z - E(c_A)$ and $\Delta_B(w) = c_B^w - E(c_B)$. The *cost deviation* between the firms

$$\Delta_C : I_A \times I_B \to \mathbb{R}_0^+$$

is given by $\Delta_C(z, w) = c_A^z - c_B^w$. Since the meaning is clear, we will use through the section the following simplified notation:

$$\Delta_A = \Delta_A(z); \quad \Delta_B = \Delta_B(w) \text{ and } \Delta_C = \Delta_C(z, w).$$

The *expected cost deviation* Δ_E between the firms is given by $\Delta_E = E(c_A) - E(c_B)$. Hence,

$$\Delta_C - \Delta_E = \Delta_A - \Delta_B.$$

Let V_A and V_B be the variances of the production costs c_A and c_B, respectively. We observe that

$$E(\Delta_C) = \Delta_E; \quad E(\Delta_A^2) = E_A(\Delta_A^2) = V_A; \quad E(\Delta_B^2) = E_B(\Delta_B^2) = V_B. \quad (1.64)$$

Furthermore,

$$E_A(\Delta_C^2) = \Delta_B^2 + V_A + \Delta_E (\Delta_E - 2\,\Delta_B); \quad (1.65)$$

$$E_B(\Delta_C^2) = \Delta_A^2 + V_B + \Delta_E (\Delta_E + 2\,\Delta_A); \quad (1.66)$$

$$E(\Delta_C^2) = \Delta_E^2 + V_A + V_B. \quad (1.67)$$

1.2.3 Local optimal price strategy under incomplete information

In this section, we introduce incomplete information in the classical Hotelling game and we find the local optimal price strategy. We introduce the bounded uncertain costs condition that allows us to find the local optimum price strategy.

A *price strategy* (p_A, p_B) is given by a pair of functions $p_A : I_A \to \mathbb{R}_0^+$ and $p_B : I_B \to \mathbb{R}_0^+$ where $p_A^z = p_A(z)$ denotes the price of firm A when the type of firm A is $z \in I_A$ and $p_B^w = p_B(w)$ denotes the price of firm B when the type of firm B is $w \in I_B$. We note that $E(p_A) = E_A(p_A)$ and $E(p_B) = E_B(p_B)$. The *indifferent consumer* $x : I_A \times I_B \to (0, l)$ is given by

$$x^{z,w} = \frac{p_B^w - p_A^z + t\,m\,(l + \Delta_l)}{2\,t\,m}. \tag{1.68}$$

The ex-post profit of the firms is the effective profit of the firms given a realization of the production costs for both firm. Hence, it is the main economic information for both firms. However, the incomplete information prevents the firms to have access to their ex-post profits except after the firms have already decided their price strategies. The *ex-post profits* $\pi_A^{EP} : I_A \times I_B \to \mathbb{R}_0^+$ and $\pi_B^{EP} : I_A \times I_B \to \mathbb{R}_0^+$ are given by

$$\pi_A^{EP}(z, w) = \pi_A(z, w) = (p_A^z - c_A^z)\, x^{z,w}$$

and

$$\pi_B^{EP}(z, w) = \pi_B(z, w) = (p_B^w - c_B^w)\,(l - x^{z,w}).$$

The ex-ante profit of the firms is the expected profit of the firm that knows its production cost but are uncertain about the production cost of the competitor firm. The *ex-ante profits* $\pi_A^{EA} : I_A \to \mathbb{R}_0^+$ and $\pi_B^{EA} : I_B \to \mathbb{R}_0^+$ are given by

$$\pi_A^{EA}(z) = E_B(\pi_A^{EP}) \quad \text{and} \quad \pi_B^{EA}(w) = E_A(\pi_B^{EP}). \tag{1.69}$$

We note that, the *expected profit* $E(\pi_A^{EP})$ of firm A is equal to $E_A(\pi_A^{EA})$ and the *expected profit* $E(\pi_B^{EP})$ of firm B is equal to $E_B(\pi_B^{EA})$.

The incomplete information forces the firms to have to choose their price strategies using their knowledge of their ex-ante profits, to which they have access, instead of the ex-post profits, to which they do not have access except after the price strategies are decided.

Definition 1.2.3. *A price strategy* $(\underline{p}_A, \underline{p}_B)$ *for both firms is a* local optimum price strategy *if (i) for every* $z \in I_A$ *and for every small deviation of the price* $\underline{p}_A^z$ *the ex-ante profit* $\pi_A^{EA}(z)$ *of firm A decreases, and for every* $w \in I_B$ *and for every small deviation of the price* $\underline{p}_B^w$ *the ex-ante profit* $\pi_B^{EA}(w)$ *of firm B decreases (local optimum property); and (ii) for every* $z \in I_A$ *and* $w \in I_B$ *the indifferent consumer exists, i.e.* $0 < \underline{x}^{z,w} < l$ *(duopoly property).*

We introduce the $BUCL1$ condition that has the crucial economical information that can be extracted from the exogenous variables. The $BUCL1$ condition allow us to know if there is, or not, a local optimum price strategy in the presence of uncertainty for the production costs of both firms.

Definition 1.2.4. *The Hotelling model satisfies the* bounded uncertain costs and location ($BUCL1$) *condition, if*

$$|\Delta_E - 3\,\Delta_C + 2\,\Delta_l\, t\, m| < 6\, t\, m\, l.$$

for all $z \in I_A$ *and for all* $w \in I_B$.

A strong restriction that the $BUCL1$ condition imposes is that Δ_C converges to 0 when m tends to 0, i.e. when the differentiation in the localization tends to vanish.

For $i \in \{A, B\}$, we define

$$c_i^m = \min_{z \in I_i}\{c_i^z\} \quad \text{and} \quad c_i^M = \max_{z \in I_i}\{c_i^z\}.$$

Let

$$\overline{\Delta} = \max_{i,j \in \{A,B\}}\{c_i^M - c_j^m\}$$

Thus, the bounded uncertain costs and location $BUCL1$ is implied by the following stronger $SBUCL1$ condition.

Definition 1.2.5. *The Hotelling model satisfies the* bounded uncertain costs

and location $(SBUCL1)$ *condition, if*

$$\overline{\Delta} < t\, l\, m.$$

The following theorem is a key economical result in oligopoly theory. First, it tells us about the existence, or not, of a local optimum price strategy only by accessing a simple inequality in the exogenous variables and so available to both firms. Secondly, it gives us explicit and simple formulas that allow the firms to know the relevance of the exogenous variables in their price strategies and corresponding profits.

Theorem 1.2.1. *There is a local optimum price strategy* $(\underline{p}_A, \underline{p}_B)$ *if and only if the* $BUCL1$ *condition holds. Under the* $BUCL1$ *condition, the expected prices of the local optimum price strategy are given by*

$$E(\underline{p}_A) = t\, m \left(l + \frac{\Delta_l}{3} \right) + E(c_A) - \frac{\Delta_E}{3}; \tag{1.70}$$

$$E(\underline{p}_B) = t\, m \left(l - \frac{\Delta_l}{3} \right) + E(c_B) + \frac{\Delta_E}{3}. \tag{1.71}$$

Furthermore, the local optimum price strategy $(\underline{p}_A, \underline{p}_B)$ *is unique and it is given by*

$$\underline{p}_A^z = E(\underline{p}_A) + \frac{\Delta_A}{2}; \quad \underline{p}_B^w = E(\underline{p}_B) + \frac{\Delta_B}{2}. \tag{1.72}$$

We observe that the difference between the expected prices of both firms has a very useful and clear economical interpretation in terms of the localization and expected cost deviations.

$$E(\underline{p}_A) - E(\underline{p}_B) = \frac{2\, t\, m\, \Delta_l + \Delta_E}{3}.$$

Furthermore, for different production costs, the differences between the optimal prices of a firm are proportional to the differences of the production costs

$$\underline{p}_A^{z_1} - \underline{p}_A^{z_2} = \frac{c_A^{z_1} - c_A^{z_2}}{2}.$$

and

$$\underline{p}_B^{w_1} - \underline{p}_B^{w_2} = \frac{c_B^{w_1} - c_B^{w_2}}{2}.$$

for all $z_1, z_2 \in I_A$ and $w_1, w_2 \in I_B$. Hence, half of the production costs value is incorporated in the price.

The ex-post profit of the firms is the effective profit of the firms given a realization of the production costs for both firms. Hence it is the main economic information for both firms. By equation (1.72), the ex-post profit of firm A is

$$\underline{\pi}_A^{EP}(z,w) = \frac{(2\,t\,m\,(3\,l + \Delta_l) - 3\,\Delta_A - 2\,\Delta_E)\,(2\,t\,m\,(3\,l + \Delta_l) + \Delta_E - 3\,\Delta_C)}{72\,t\,m}$$

and the ex-post profit of firm B is

$$\underline{\pi}_B^{EP}(z,w) = \frac{(2\,t\,m\,(3\,l - \Delta_l) - 3\,\Delta_B + 2\,\Delta_E)\,(2\,t\,m\,(3\,l - \Delta_l) - \Delta_E + 3\,\Delta_C)}{72\,t\,m}.$$

The ex-ante profit of a firm is the expected profit of the firm that knows its production cost but is uncertain about the production costs of the competitor firm. Since $\underline{\pi}_A^{EP}(z,w)$ is given by

$$\frac{(2\,t\,m\,(3\,l + \Delta_l) - 3\,\Delta_A - 2\,\Delta_E)\,(2\,t\,m\,(3\,l + \Delta_l) + \Delta_E + 3\,(c_B^w - c_A^z))}{72\,t\,m},$$

the ex-ante profit of firm A, $\underline{\pi}_A^{EA}(z)$, is

$$\frac{(2\,t\,m\,(3\,l + \Delta_l) - 3\,\Delta_A - 2\,\Delta_E)\,(2\,t\,m\,(3\,l + \Delta_l) + \Delta_E + 3\,(E(c_B) - c_A^z))}{72\,t\,m}$$

Hence,

$$\underline{\pi}_A^{EA}(z) = \frac{(2\,t\,m\,(3\,l + \Delta_l) - 3\,\Delta_A - 2\,\Delta_E)^2}{72\,t\,m}. \tag{1.73}$$

Similarly, the ex-ante profit of firm B is

$$\underline{\pi}_B^{EA}(w) = \frac{(2\,t\,m\,(3\,l - \Delta_l) - 3\,\Delta_B + 2\,\Delta_E)^2}{72\,t\,m}. \tag{1.74}$$

Let α_A and α_B be given by

$$\alpha_A = \max\{E(c_B) - c_B^w : w \in I_B\} \ \text{ and } \ \alpha_B = \max\{E(c_A) - c_A^z : z \in I_A\}.$$

The following corollary gives us the information of the market size of both firms by giving the explicit localization of the indifferent consumer with respect to the local optimum price strategy.

Corollary 1.2.1. *Under the $BUCL1$ condition, the indifferent consumer $x^{z,w}$ is given by*

$$\underline{x}^{z,w} = \frac{1}{2}\left(l + \frac{\Delta_l}{3}\right) + \frac{\Delta_E - 3\,\Delta_C}{12\,t\,m}. \tag{1.75}$$

The pair of prices $(\underline{p}_A, \underline{p}_B)$ satisfies

$$\underline{p}_A^z - c_A^z \geq \alpha_A/2; \qquad \underline{p}_B^w - c_B^w \geq \alpha_B/2. \tag{1.76}$$

Proof of Theorem 1.2.1 and Corollary 1.2.1.
Under incomplete information, each firm seeks to maximize its ex-ante profit. From (1.69), the ex-ante profit for firm A is given by

$$\begin{aligned} \pi_A^{EA}(z) &= \int_{I_B} (p_A^z - c_A^z)\left(\frac{p_B^w - p_A^z}{2\,t\,m} + \frac{l + \Delta_l}{2}\right) \mathrm{d}q_B(w) \\ &= (p_A^z - c_A^z)\left(\frac{E(p_B) - p_A^z}{2\,t\,m} + \frac{l + \Delta_l}{2}\right). \end{aligned} \tag{1.77}$$

From the first order condition FOC applied to the ex-ante profit of firm A we obtain

$$p_A^z = \frac{c_A^z + E(p_B) + t\,m\,(l + \Delta_l)}{2}. \tag{1.78}$$

Similarly,

$$\pi_B^{EA}(w) = (p_B^w - c_B^w)\left(\frac{E(p_A) - p_B^w}{2\,t\,m} + \frac{l - \Delta_l}{2}\right),$$

and, by the FOC, we obtain

$$p_B^w = \frac{c_B^w + E(p_A) + t\,m\,(l - \Delta_l)}{2}. \tag{1.79}$$

Then, from (1.78) and (1.79),

$$E(p_A) = \frac{E(c_A) + E(p_B) + t\,m\,(l + \Delta_l)}{2};$$
$$E(p_B) = \frac{E(c_B) + E(p_A) + t\,m\,(l - \Delta_l)}{2}.$$

Solving the system of two equations, we obtain that

$$E(p_A) = t\,m\left(l + \frac{\Delta_l}{3}\right) + \frac{E(c_B) + 2\,E(c_A)}{3};$$
$$E(p_B) = t\,m\left(l - \frac{\Delta_l}{3}\right) + \frac{E(c_A) + 2\,E(c_B)}{3}.$$

Hence, equalities (1.70) and (1.71) are satisfied. Replacing (1.71) in (1.78) and replacing (1.70) in (1.79) we obtain that

$$p_A^z = t\,m\left(l + \frac{\Delta_l}{3}\right) + \frac{c_A^z}{2} + \frac{E(c_A) + 2\,E(c_B)}{6};$$
$$p_B^w = t\,m\left(l - \frac{\Delta_l}{3}\right) + \frac{c_B^w}{2} + \frac{2\,E(c_A) + E(c_B)}{6}.$$

Hence, equation (1.72) is satisfied.

Replacing in equation (1.68) the values of $\underline{p}_A$ and $\underline{p}_B$ given by the equation (1.72) we obtain that the indifferent consumer $x^{z,w}$ is given by

$$x^{z,w} = \frac{1}{2}\left(l + \frac{\Delta_l}{3}\right) + \frac{3(c_B^w - c_A^z) + E(c_A) - E(c_B)}{12\,t\,m}.$$

Hence, equation (1.75) is satisfied. Therefore, $(\underline{p}_A, \underline{p}_B)$ satisfies property (ii) if and only if the $BUCL1$ condition holds.

Since the ex-ante profit functions (1.77) and (1.2.3) are concave, the second-order conditions for this maximization problem are satisfied and so the prices $\underline{p}_A^z$ and $\underline{p}_B^w$ are indeed maxima for the functions (1.77) and (1.2.3), respectively. Therefore, the pair $(\underline{p}_A^z, \underline{p}_B^w)$ satisfies property (i) and so $(\underline{p}_A^z, \underline{p}_B^w)$ is a local optimum price strategy.

Let us prove that $\underline{p}_A^z$ and $\underline{p}_B^w$ satisfy inequalities (1.76). By equation (1.72),

$$\underline{p}_A^z - c_A^z = t\,m\left(l + \frac{\Delta_l}{3}\right) - \frac{c_A^z}{2} + \frac{E(c_A) + 2\,E(c_B)}{6};$$
$$\underline{p}_B^w - c_B^w = t\,m\left(l - \frac{\Delta_l}{3}\right) - \frac{c_B^w}{2} + \frac{2\,E(c_A) + E(c_B)}{6}.$$

By the $BUCL1$ condition, for every $w \in I_B$, we obtain

$$\begin{aligned}&6\left(\underline{p}_A^z - c_A^z - t\,m\left(l + \frac{\Delta_l}{3}\right)\right) = -3\,c_A^z + E(c_A) + 2\,E(c_B)\\ &= 3\,(E(c_B) - c_B^w) - 3\,(c_A^z - c_B^w) + E(c_A) - E(c_B)\\ &> 3\,(E(c_B) - c_B^w) - 6\,t\,l - 2\,\Delta_l\,t\,m.\end{aligned}$$

Similarly, by the $BUCL1$ condition, for every $z \in I_A$, we obtain

$$\begin{aligned}&6\left(\underline{p}_B^w - c_B^w - t\,m\left(l - \frac{\Delta_l}{3}\right)\right) = -3\,c_B^w + 2\,E(c_A) + E(c_B)\\ &= 3\,(E(c_A) - c_A^z) - 3\,(c_B^w - c_A^z) - E(c_A) + E(c_B)\\ &> 3\,(E(c_A) - c_A^z) - 6\,t\,l + 2\,\Delta_l\,t\,m.\end{aligned}$$

Hence, inequalities (1.76) are satisfied. □

1.2.4 Bayesian Nash equilibrium

We note that, if a Bayesian-Nash price equilibrium satisfies the duopoly property then it is a local optimum price strategy. However, a local optimum price strategy is only a local strategic maximum. Hence, the local optimum price strategy to be a Bayesian-Nash equilibrium must also be global strategic maximum. In this subsection, we are going to show that this is the case.

Following D'Aspremont et al. [2], we note that the profits of the two firms, valued at local optimum price strategy are globally optimal if they are at least as great as the payoffs that firms would earn by undercutting the rivals's price and supplying the whole market for all admissible subsets of types I_A and I_B.

Definition 1.2.6. *A price strategy* $(\underline{p}_A, \underline{p}_B)$ *for both firms is a* Bayesian-Nash*, if for every* $z \in I_A$ *and for every deviation of the price* $\underline{p}_A^z$ *the ex-ante profit* $\pi_A^{EA}(z)$ *of firm* A *decreases, and for every* $w \in I_B$ *and for every deviation of the price* $\underline{p}_B^w$ *the ex-ante profit* $\pi_B^{EA}(w)$ *of firm* B *decreases.*

Let $(\underline{p}_A, \underline{p}_B)$ be the local optimum price strategy. Given the type w_0 of firm B, firm A may gain the whole market, undercutting its rival by setting

$$p_A^M(w_0) = \underline{p}_B^{w_0} - t\,m\,(l - \Delta_l) - \epsilon, \text{ with } \epsilon > 0.$$

Hence, by $BUCL1$ condition $p_A^M(w_0) \leq p_A^z$ for all $z \in I_A$. We observe that if firm A chooses the price $p_A^M(w_0)$ then by equalities (1.68) and (1.72) the whole market belongs to Firm A for all types w of firm B with $c^w \geq c^{w_0}$. Let

$$x(w; w_0) = \min\left\{l, \frac{p_B^w - p_A^M(w_0)}{2\,t\,m} + \frac{l + \Delta_l}{2}\right\}.$$

Thus, the *expected profit* with respect to the price $p_A^M(w_0)$ for firm A is

$$\pi_A^{EA,M}(w_0) = \int_{I_B} \left(p_A^M(w_0) - c_A^z\right) x(w; w_0)\, \mathrm{d}q_B(w).$$

Let $w_M \in I_B$ such that $c^{w_M} = c_B^M$. Since $c^{w_M} \geq c_B^{w_0}$ for every $w_0 \in I_B$, we obtain

$$\pi_A^{EA,M}(w_0) \leq \left(p_A^M(w_0) - c_A^z\right)\, l \leq (p_A^M(w_M) - c_A^z)\, l \qquad (1.80)$$

Given the type z_0 of firm A, firm B may gain the whole market, undercutting its rival by setting

$$p_B^M(z_0) = \underline{p}_A^{z_0} - t\,m\,(l + \Delta_l) - \epsilon, \text{ with } \epsilon > 0.$$

Hence, by $BUCL1$ condition $p_B^M(z_0) \leq p_B^w$ for all $w \in I_B$. We observe that if firm B chooses the price $p_B^M(z_0)$ then by equalities (1.68) and (1.72) the whole market belongs to Firm B for all types z of firm A with $c^z \geq c^{z_0}$. Let

$$x(z; z_0) = \max\left\{0, \frac{p_B^M(z_0) - p_A^z}{2\,t\,m} + \frac{l + \Delta_l}{2}\right\}.$$

Thus, the *expected profit* with respect to the price $p_B^M(z_0)$ of firm B is

$$\pi_B^{EA,M}(z_0) = \int_{I_A} \left(p_B^M(z_0) - c_B^w\right)\, (l - x(z; z_0))\, \mathrm{d}q_A(z).$$

Let $z_M \in I_A$ such that $c_A^{z_M} = c_A^M$. Since $c^{z_M} \geq c^{z_0}$ for every $z_0 \in I_A$, we obtain

$$\pi_B^{EA,M}(z_0) \leq \left(p_B^M(z_0) - c_B^w\right)\, l \leq (p_B^M(z_M) - c_B^w)\, l. \qquad (1.81)$$

Remark 1.2.1. *Under the $BUCL1$ condition, the strategic equilibrium $(\underline{p}_A, \underline{p}_B)$ is the unique pure Bayesian Nash equilibrium with the duopoly property if for every $z \in I_A$ and every $w \in I_B$,*

$$\pi_A^{EA,M}(w) \leq \underline{\pi}_A^{EA}(z) \quad \textit{and} \quad \pi_B^{EA,M}(z) \leq \underline{\pi}_B^{EA}(w). \qquad (1.82)$$

Definition 1.2.7. *The Hotelling model satisfies the* bounded uncertain costs

and location ($BUCL2$) *condition, if*

$$\Delta_E + 3\left(c_A^M + c_B^M - 2\,c_A^m\right) + \frac{\Delta_l\,(3\,c_A^M - E(c_A) - 2\,E(c_B))}{3\,l} \leq$$
$$\leq \frac{t\,m\,(3\,l - \Delta_l)^2}{3\,l} + \frac{\left(3\,c_A^M - E(c_A) - 2\,E(c_B)\right)^2}{12\,t\,m\,l} \tag{1.83}$$

and

$$-\Delta_E + 3\left(c_A^M + c_B^M - 2\,c_B^m\right) - \frac{\Delta_l\,(3\,c_B^M - E(c_B) - 2\,E(c_A))}{3\,l} \leq$$
$$\leq \frac{t\,m\,(3\,l + \Delta_l)^2}{3\,l} + \frac{\left(3\,c_B^M - E(c_B) - 2\,E(c_A)\right)^2}{12\,t\,m\,l}. \tag{1.84}$$

Thus, the bounded uncertain costs condition $BUCL2$ is implied by the following stronger $SBUCL2$ condition.

Definition 1.2.8. *The Hotelling model satisfies the* strong bounded uncertain costs and location ($SBUCL2$) *condition, if*

$$6\,\overline{\Delta} < l\,t\,m$$

We observe that the $SBUCL2$ condition implies $SBUCL1$ condition and so implies the $BUCL1$ condition.

Theorem 1.2.2. *If the Hotelling model satisfies the $BUCL1$ and $BUCL2$ conditions the local optimum price strategy $(\underline{p}_A, \underline{p}_B)$ is a Bayesian Nash equilibrium.*

Corollary 1.2.2. *If the Hotelling model satisfies $SBUCL2$ condition the local optimum price strategy $(\underline{p}_A, \underline{p}_B)$ is a Bayesian Nash equilibrium.*

Proof. By equalities (1.73) and (1.74), we obtain that $\underline{\pi}_A^{EA}(z_M) \leq \underline{\pi}_A^{EA}(z)$ and $\underline{\pi}_B^{EA}(w_M) \leq \underline{\pi}_B^{EA}(w)$ for all $z \in I_A$ and for all $w \in I_B$. Hence, putting conditions (1.80), (1.81) and (1.82) together, we obtain the following sufficient condition for the local optimal strategic prices $(\underline{p}_A, \underline{p}_B)$ to be a Bayesian

Nash equilibrium:

$$(p_A^M(w_M) - c_A^m)\, l \leq \underline{\pi}_A^{EA}(z_M) \quad \text{and} \quad (p_B^M(z_M) - c_B^m)\, l \leq \underline{\pi}_B^{EA}(w_M). \quad (1.85)$$

By equalities (1.73) and (1.74) we obtain that

$$\underline{\pi}_A^{EA}(z_M) = \frac{(2\,t\,m\,(3\,l + \Delta_l) + E(c_A) + 2\,E(c_B) - 3\,c_A^M)^2}{72\,t\,m}$$

and

$$\underline{\pi}_B^{EA}(w_M) = \frac{(2\,t\,m\,(3\,l - \Delta_l) + 2\,E(c_A) + E(c_B) - 3\,c_B^M)^2}{72\,t\,m}.$$

Also, from (1.72), we know that

$$\begin{aligned} p_A^M(w_M) - c_A^m &= \underline{p}_B^{w_M} - t\,m\,(l - \Delta_l) - \epsilon - c_A^m \\ &= \frac{1}{6}\,(4\,t\,m\,\Delta_l + 3\,c_B^M + 2\,E(c_A) + E(c_B) - 6\,c_A^m) - \epsilon. \end{aligned}$$

and

$$\begin{aligned} p_B^M(z_M) - c_B^m &= \underline{p}_A^{z_M} - t\,m\,(l + \Delta_l) - \epsilon - c_B^m \\ &= \frac{1}{6}\,(-4\,t\,m\,\Delta_l + 3\,c_A^M + E(c_A) + 2\,E(c_B) - 6\,c_B^m) - \epsilon. \end{aligned}$$

Hence, condition (1.85) holds if

$$\begin{aligned} 12\,t\,m\,l\,(4\,t\,m\,\Delta_l + 3\,c_B^M + 2\,E(c_A) + E(c_B) - 6\,c_A^m) \leq \\ \leq (2\,t\,m\,(3\,l + \Delta_l) + E(c_A) + 2\,E(c_B) - 3\,c_A^M)^2 \end{aligned} \quad (1.86)$$

and

$$\begin{aligned} 12\,t\,m\,l\,(-4\,t\,m\,\Delta_l + 3\,c_A^M + E(c_A) + 2\,E(c_B) - 6\,c_B^m) \leq \\ (2\,t\,m\,(3\,l - \Delta_l) + 2\,E(c_A) + E(c_B) - 3\,c_B^M)^2. \end{aligned} \quad (1.87)$$

Finally, we note that inequality (1.86) is equivalent to inequality (1.83) and that inequality (1.87) is equivalent to inequality (1.84). □

1.2.5 Optimum localization equilibrium under incomplete information

We note that from (1.72) and (1.73), we can write the profit of firm A as

$$\underline{\pi}_A^{EA}(z) = \frac{(\underline{p}_A^z - c_A)^2}{2\,t\,(l-a-b)}.$$

Since

$$\frac{\partial \underline{p}_A^z}{\partial a} = -\frac{2}{3}\,t\,(l+a)$$

we have

$$\frac{\partial \underline{\pi}_A^{EA}}{\partial a} = \frac{\underline{p}_A - c_A}{12\,t\,(l-a-b)^2}\,\left(-2\,t\,(l-a-b)\,(l+3\,a+b) - 3\,\Delta_A - 2\,\Delta_E\right).$$

Similarly, we obtain that

$$\frac{\partial \underline{\pi}_B^{EA}}{\partial b} = \frac{\underline{p}_B - c_B}{12\,t\,(l-a-b)^2}\,\left(-2\,t\,(l-a-b)\,(l+3\,b+a) - 3\,\Delta_B + 2\,\Delta_E\right).$$

Therefore, the maximal differentiation $(a, b) = (0, 0)$ is a local optimum strategy if and only if

$$\frac{\partial \underline{\pi}_A^{EA}}{\partial a}(0,0) = -\frac{\underline{p}_A - c_A}{12\,t\,l^2}\,\left(2\,t\,l^2 + 3\,\Delta_A + 2\,\Delta_E\right) < 0$$

and

$$\frac{\partial \underline{\pi}_B^{EA}}{\partial b}(0,0) = -\frac{\underline{p}_B - c_B}{12\,t\,l^2}\,\left(2\,t\,l^2 + 3\,\Delta_B - 2\,\Delta_E\right) < 0$$

Since

$$\frac{\underline{p}_A - c_A}{6\,t\,l^2} > 0 \quad \text{and} \quad \frac{\underline{p}_B - c_B}{6\,t\,l^2} > 0$$

the maximal differentiation $(a, b) = (0, 0)$ is a local optimum strategy if and only if the following condition holds.

Definition 1.2.9. *The Hotelling model satisfies the* bounded uncertain costs and location ($BUCL3$) *condition, if*

$$2\,t\,l^2 + 3\,\Delta_A + 2\,\Delta_E > 0$$

for all $z \in I_A$ *and*

$$2\,t\,l^2 + 3\,\Delta_B - 2\,\Delta_E > 0$$

for all $w \in I_B$ *.*

1.2.6 Comparative profit analysis

From now on, we assume that the $BUCL1$ condition holds and that the price strategy $(\underline{p}_A, \underline{p}_B)$ is the local optimum price strategy determined in Theorem 1.2.1.

Let $\Delta_1 = \Delta_A + \Delta_B$ and $\Delta_2 = \Delta_A - \Delta_B$. We observe that the difference between the ex-post profits of both firms, $\underline{\pi}_A^{EP}(z, w) - \underline{\pi}_B^{EP}(z, w)$, has a very useful and clear economical interpretation in terms of the expected cost deviations and is given by

$$\frac{16\,t^2\,m^2\,l\,\Delta_l + 2\,t\,m\,(3\,l\,\Delta_2 - \Delta_l\,\Delta_1) + (\Delta_E - 3\,\Delta_C)\,(8\,t\,l\,m - \Delta_1)}{24\,t\,m}.$$

Furthermore, for different production costs, the differences between the ex-post profit of firm A, $\underline{\pi}_A^{EP}(z_1, w) - \underline{\pi}_A^{EP}(z_2, w)$, is given by

$$\frac{(c_A^{z_2} - c_A^{z_1})\,(4\,t\,m\,(3\,l + \Delta_l) - \Delta_E + 3\,(c_B^w + E(c_A) - c_A^{z_1} - c_A^{z_2}))}{24\,t\,m}$$

and, similarly, $\underline{\pi}_B^{EP}(z, w_1) - \underline{\pi}_B^{EP}(z, w_2)$ is given by

$$\frac{(c_B^{w_2} - c_B^{w_1})\,(4\,t\,m\,(3\,l - \Delta_l) + \Delta_E + 3\,(c_A^z + E(c_B) - c_B^{w_1} - c_B^{w_2}))}{24\,t\,m}$$

for all $z, z_1, z_2 \in I_A$ and $w, w_1, w_2 \in I_B$.

We observe that the difference between the ex-ante profits of both firms has a very useful and clear economical interpretation in terms of the expected cost deviations.

$$\underline{\pi}_A^{EA}(z) - \underline{\pi}_B^{EA}(w) = \frac{(4\,t\,m\,l - \Delta_1)\,(4\,(t\,m\,\Delta_l - \Delta_E) - 3\,\Delta_2)}{24\,t\,m}.$$

Furthermore, for different production costs, the differences between the ex-ante profit of firm A, $\underline{\pi}_A^{EA}(z_1) - \underline{\pi}_A^{EA}(z_2)$, is given by

$$\frac{(c_A^{z_2} - c_A^{z_1})\,(4\,t\,m\,(3\,l + \Delta_l) - 4\,\Delta_E + 3\,(2\,E(c_A) - c_A^{z_1} - c_A^{z_2}))}{24\,t\,m}$$

and, similarly, $\underline{\pi}_B^{EA}(w_1) - \underline{\pi}_B^{EA}(w_2)$ is given by

$$\frac{(c_B^{w_2} - c_B^{w_1})\,(4\,t\,m\,(3\,l - \Delta_l) + 4\,\Delta_E + 3\,(2\,E(c_B) - c_B^{w_1} - c_B^{w_2}))}{24\,t\,m}$$

for all $z, z_1, z_2 \in I_A$ and $w, w_1, w_2 \in I_B$.

The difference between the ex-post and the ex-ante profit for a firm is the real deviation from the realized gain of the firm and the expected gain of the firm knowing its own production cost but being uncertain about the production cost of the other firm. It is the best measure of the risk involved for the firm given the uncertainty in the production costs of the other firm. The difference between the ex-post profit and the ex-ante profit for firm A is

$$\underline{\pi}_A^{EP}(z,w) - \underline{\pi}_A^{EA}(z) = \frac{\Delta_B}{24\,t\,m}\,\left(2\,t\,m\,(3\,l + \Delta_l) - 2\,\Delta_E - 3\,\Delta_A\right).$$

The difference between the ex-post profit and the ex-ante profit for firm B is

$$\underline{\pi}_B^{EP}(z,w) - \underline{\pi}_B^{EA}(w) = \frac{\Delta_A}{24\,t\,m}\,\left(2\,t\,m\,(3\,l - \Delta_l) + 2\,\Delta_E - 3\,\Delta_B\right).$$

Definition 1.2.10. *The Hotelling model satisfies the* A-bounded uncertain costs and location $(A - BUCL)$ *condition, if for all* $z \in I_A$

$$3\,\Delta_A + 2\,\Delta_E < 2\,t\,m\,(3\,l + \Delta_l).$$

The Hotelling model satisfies the B-bounded uncertain costs and location $(B - BUCL)$ *condition, if for all* $w \in I_B$

$$3\,\Delta_B - 2\,\Delta_E < 2\,t\,m\,(3\,l - \Delta_l).$$

The following corollary tells us that the sign of the risk of a firm has the opposite sign of the deviation of the competitor firm realized production cost from its average. Hence, under incomplete information the sign of the risk of a firm is not accessible to the firm. However, the probability of the sign of the risk of a firm to be positive or negative is accessible to the firm.

Corollary 1.2.3. *Under the A-bounded uncertain costs* $(A - BUCL)$ *condition,*

$$\underline{\pi}_A^{EP}(z, w) < \underline{\pi}_A^{EA}(z) \quad \text{if and only if} \quad \Delta_B < 0. \tag{1.88}$$

Under the B-bounded uncertain costs $(B - BUCL)$ *condition,*

$$\underline{\pi}_B^{EP}(z, w) < \underline{\pi}_B^{EA}(w) \quad \text{if and only if} \quad \Delta_A < 0. \tag{1.89}$$

The proof of the above corollary follows from a simple manipulation of the previous formulas for the ex-post and ex-ante profits.

The expected profit of the firm is the expected gain of the firm. We observe that the ex-ante and the ex-posts profits of both firms are strictly positive with respect to the local optimum price strategy. Hence, the expected profits of both firms are also strictly positive. Since the ex-ante profit $\underline{\pi}_A^{EA}(z)$ of firm A is equal to

$$\underline{\pi}_A^{EA}(z) = \frac{9\,\Delta_A^2 - 12\,\Delta_A\,(t\,m\,(3\,l + \Delta_l) - \Delta_E) + 4\,(t\,m\,(3\,l + \Delta_l) - \Delta_E)^2}{72\,t\,m},$$

from (1.64), we obtain that the expected profit of firm A is given by

$$E(\underline{\pi}_A^{EP}) = \frac{(t\,m\,(3\,l + \Delta_l) - \Delta_E)^2}{18\,t\,m} + \frac{V_A}{8\,t\,m}.$$

Similarly, the expected profit of firm B is given by

$$E(\underline{\pi}_B^{EP}) = \frac{(t\,m\,(3\,l - \Delta_l) + \Delta_E)^2}{18\,t\,m} + \frac{V_B}{8\,t\,m}.$$

The difference between the ex-ante and the expected profit of a firm is the deviation from the expected realized gain of the firm given the realization of its own production cost and the expected gain in average for different realizations of its own production cost, but being in both cases uncertain about the production costs of the competitor firm. It is the best measure of the quality of its realized production cost in terms of the expected profit over its own production costs.

Corollary 1.2.4. *The difference between the ex-ante profit and the expected profit for firm A is*

$$E(\underline{\pi}_A^{EP}) - \underline{\pi}_A^{EA}(z) = \frac{\Delta_A\,(4\,t\,m\,(3\,l + \Delta_l) - 3\,\Delta_A - 4\,\Delta_E) + 3\,V_A}{24\,t\,m}. \tag{1.90}$$

The difference between the ex-ante profit and the expected profit for firm B is

$$E(\underline{\pi}_B^{EP}) - \underline{\pi}_B^{EA}(w) = \frac{\Delta_B\,(4\,t\,m\,(3\,l - \Delta_l) - 3\,\Delta_B + 4\,\Delta_E) + 3\,V_B}{24\,t\,m}. \tag{1.91}$$

Proof. Let $Z = 2\,t\,m\,(3\,l + \Delta_l) - 2\,\Delta_E$. Hence,

$$\begin{aligned} E(\underline{\pi}_A^{EP}) - \underline{\pi}_A^{EA}(z) &= \frac{Z^2 - (Z - 3\,\Delta_A)^2}{72\,t\,m} + \frac{V_A}{8\,t\,m} \\ &= \frac{\Delta_A\,(2\,Z - 3\,\Delta_A) + 3\,V_A}{24\,t\,m}. \end{aligned}$$

and so equality (1.90) holds. The proof of equality (1.91) follows similarly.

□

1.2.7 Comparative consumer surplus and welfare analysis

Consider throughout this subsection that $X = t\,m\,(3\,l + \Delta_l)$.

The ex-post consumer surplus is the realized gain of the consumers community for given outcomes of the production costs of both firms. Under incomplete information, by equation (1.54), the ex-post consumer surplus is

$$\underline{CS}^{EP} = X_1 - \frac{E(c_A) + 2\,E(c_B)}{3}\,l - \frac{\Delta_B}{2}\,l + \frac{(2\,t\,m\,(3\,l + \Delta_l) + \Delta_E - 3\,\Delta_C)^2}{144\,t\,m}.$$

The expected value of the consumer surplus is the expected gain of the consumers community for all possible outcomes of the production costs of both firms. The expected value of the consumer surplus $E(\underline{CS}^{EP})$ is given by

$$\begin{aligned} E(\underline{CS}^{EP}) &= \int_{I_B}\int_{I_A} \underline{CS}^{EP} \mathrm{d}q_A(z)\,\mathrm{d}q_B(w) \\ &= X_1 - \frac{E(c_A) + 2\,E(c_B)}{3}\,l + \frac{4\,(t\,m\,(3\,l + \Delta_l) - \Delta_E)^2 + 9\,(V_A + V_B)}{144\,t\,m}. \end{aligned}$$

We note that, from equalities (1.64) and (1.67), the expected value of

$$\frac{(2\,t\,m\,(3\,l + \Delta_l) + \Delta_E - 3\,\Delta_C)^2}{144\,t\,m}$$

is given by

$$
\begin{aligned}
&\frac{(2\,X + \Delta_E)^2 - 6\,E(\Delta_C)\,(2\,X + \Delta_E) + 9\,E(\Delta_C^2)}{144\,t\,m} \\
&= \frac{(2\,X + \Delta_E)^2 - 6\,\Delta_E\,(2\,X + \Delta_E) + 9\,(V_A + V_B + \Delta_E^2)}{144\,t\,m} \\
&= \frac{4\,(X - \Delta_E)^2 + 9\,(V_A + V_B)}{144\,t\,m}.
\end{aligned}
$$

The difference between the ex-post consumer surplus and the expected value of the consumer surplus measures the difference between the gain of the consumers for the realized outcomes of the production costs of both firms and the expected gain of the consumers for all possible outcomes of the production costs of both firms. Hence, it measures the risk taken by the consumers for different outcomes of the production costs of both firms.

Corollary 1.2.5. *The difference between the ex-post consumer surplus and the expected value of the consumer surplus, $CS^{EP} - E(CS^{EP})$, is*

$$
-\frac{\Delta_A + \Delta_B}{4}\,l + \frac{\Delta_E - \Delta_C}{12}\,\Delta_l + \frac{(\Delta_E - 3\,\Delta_C)^2 - 4\,\Delta_E^2 - 9\,(V_A + V_B)}{144\,t\,m}.
$$

Proof.

$$
\begin{aligned}
&\underline{CS}^{EP} - E(\underline{CS}^{EP}) = \\
&= -\frac{\Delta_B}{2}\,l + \frac{(2\,X + \Delta_E - 3\,\Delta_C)^2 - 4\,(X - \Delta_E)^2 - 9\,(V_A + V_B)}{144\,t\,m} \\
&= -\frac{\Delta_B}{2}\,l + \frac{12\,X\,(\Delta_E - \Delta_C) + (\Delta_E - 3\,\Delta_C)^2 - 4\,\Delta_E^2 - 9\,(V_A + V_B)}{144\,t\,m} \\
&= \frac{\Delta_E - \Delta_C - 2\,\Delta_B}{4}\,l + \frac{\Delta_E - \Delta_C}{12}\,\Delta_l + \frac{(\Delta_E - 3\,\Delta_C)^2 - 4\,\Delta_E^2 - 9\,(V_A + V_B)}{144\,t\,m} \\
&= -\frac{\Delta_A + \Delta_B}{4}\,l + \frac{\Delta_E - \Delta_C}{12}\,\Delta_l + \frac{(\Delta_E - 3\,\Delta_C)^2 - 4\,\Delta_E^2 - 9\,(V_A + V_B)}{144\,t\,m}
\end{aligned}
$$

□

The ex-post welfare is the realized gain of the state that includes the gains of the consumers community and the gains of the firms for a given outcomes of the production costs of both firms. By equation (1.55), the ex-post welfare is

$$\begin{aligned}\underline{W}^{EP} &= \frac{5(\Delta_E - 3\Delta_C) + 3(\Delta_A - \Delta_B)}{36}\,\Delta_l - \frac{\Delta_A + \Delta_B + E(c_A) + E(c_B)}{2} l + \\ &+ X_1 + X_2 + X_3,\end{aligned}$$

where

$$X_3 = \frac{(3\,\Delta_C - \Delta_E)\,(9\,\Delta_C + \Delta_E)}{144\,t\,m}.$$

The expected value of the welfare is the expected gain of the state for all possible outcomes of the production costs of both firms. The expected value of the welfare $E(\underline{W}^{EP})$ is given by

$$\begin{aligned}E(\underline{W}^{EP}) &= \int_{I_B}\int_{I_A} \underline{W}^{EP} \mathrm{d}q_A(z)\,\mathrm{d}q_B(w) \\ &= X_1 + X_2 - \frac{E(c_A) + E(c_B)}{2}\, l - \frac{5\,\Delta_E}{18}\,\Delta_l + U_2\end{aligned}$$

where

$$U_2 = \frac{20\,\Delta_E^2 + 27\,(V_A + V_B)}{144\,t\,m}.$$

We note that, from equalities (1.64) and (1.67), the expected value of X_3 is given by

$$\begin{aligned}U_2 &= \frac{27\,E(\Delta_C^2) - 6\,E(\Delta_C)\,\Delta_E - \Delta_E^2}{144\,t\,m} \\ &= \frac{27\,(\Delta_E^2 + V_A + V_B) - 7\,\Delta_E^2}{144\,t\,m} \\ &= \frac{20\,\Delta_E^2 + 27\,(V_A + V_B)}{144\,t\,m}.\end{aligned}$$

The difference between the ex-post welfare and the expected value of the welfare measures the difference in the gains of the state between the realized outcomes of the production costs of both firms and the expected gain of the state for all possible outcomes of the production costs of both firms. Hence, it measures the risk taken by the state for different outcomes of the production costs of both firms. The difference between the ex-post welfare and the expected value of welfare is

$$\underline{W}^{EP} - E(\underline{W}^{EP}) = \frac{\Delta_A + \Delta_B}{2}\, l + \frac{\Delta_B - \Delta_A)}{3}\, \Delta_l + X_4$$

where

$$X_4 = \frac{9\,(\Delta_C^2 - V_A - V_B) - 2\,\Delta_C\,\Delta_E - 7\,\Delta_E^2}{48\,t\,m}.$$

1.2.8 Complete versus Incomplete information

Let us consider the case where the production costs are revealed to both firms before they choose the prices. In this case, the competition between the firms is under complete information.

A *price strategy* (p_A^{CI}, p_B^{CI}) is given by a pair of functions $p_A^{CI} : I_A \times I_B \to \mathbb{R}_0^+$ and $p_B^{CI} : I_A \times I_B \to \mathbb{R}_0^+$ where $p_A^{CI}(z,w)$ denotes the price of firm A and $p_B^{CI}(z,w)$ denotes the price of firm B when the type of firm A is $z \in I_A$ and the type of firm B is $w \in I_B$.

Under the BC condition, by equations (1.56) and (1.57), the Nash price strategy (p_A^{CI}, p_B^{CI}) is given by

$$\underline{p}_A^{CI}(z,w) = t\,m\,\left(l + \frac{\Delta_l}{3}\right) + c_A - \frac{\Delta_C}{3}$$

and

$$\underline{p}_B^{CI}(z,w) = t\,m\,\left(l - \frac{\Delta_l}{3}\right) + c_B + \frac{\Delta_C}{3}.$$

By equation (1.58), the profit $\pi_A^{CI} : I_A \times I_B \to \mathbb{R}_0^+$ of firm A is given by

$$\underline{\pi}_A^{CI}(z,w) = \frac{(m\,(3\,l + \Delta_l)\,t - \Delta_C)^2}{18\,t\,m}$$

Similarly, by equation (1.59), the profit $\pi_B^{CI} : I_A \times I_B \to \mathbb{R}_0^+$ of firm B is given by

$$\underline{\pi}_B^{CI}(z,w) = \frac{(m\,(3\,l - \Delta_l)\,t + \Delta_C)^2}{18\,t\,m}.$$

Using equality (1.66), the expected profit $E_B(\underline{\pi}_A^{CI})$ for firm A is given by

$$E_B(\underline{\pi}_A^{CI}) = \frac{(m\,t\,(3\,l + \Delta_l) - \Delta_A - \Delta_E)^2 + V_B}{18\,t\,m}$$

Similarly, using equality (1.65), the expected profit $E_A(\underline{\pi}_B^{CI})$ for firm B is given by

$$E_A(\underline{\pi}_B^{CI}) = \frac{(m\,t\,(3\,l - \Delta_l) - \Delta_B + \Delta_E)^2 + V_A}{18\,t\,m}$$

The expected profit $E(\underline{\pi}_A^{CI})$ for firm A is given by

$$E(\underline{\pi}_A^{CI}) = \frac{(m\,t\,(3\,l + \Delta_l) - \Delta_E)^2 + V_A + V_B}{18\,t\,m}$$

Similarly, the expected profit $E(\underline{\pi}_B^{CI})$ for firm B is given by

$$E(\underline{\pi}_B^{CI}) = \frac{(m\,t\,(3\,l - \Delta_l) + \Delta_E)^2 + V_A + V_B}{18\,t\,m}$$

By equation (1.62), the consumer surplus is given by

$$\underline{CS}^{CI}(z,w) = X_1 - \frac{E(c_A) + 2\,E(c_B) + \Delta_A + 2\,\Delta_B}{3}\,l + \frac{(t\,m\,(3\,l + \Delta_l) - \Delta_C)^2}{36\,t\,m},$$

Using equality (1.67), we obtain that the expected value of the consumer surplus $E(\underline{CS}^{CI})$ is

$$E(\underline{CS}^{CI}(z,w)) = X_1 - \frac{E(c_A) + 2\,E(c_B)}{3}\,l + \frac{(t\,m\,(3\,l + \Delta_l) - \Delta_E)^2 + V_A + V_B}{36\,t\,m}.$$

By equation (1.63), the welfare is given by

$$\underline{W}^{CI}(z,w) = X_1 - \frac{E(c_A)+E(c_B)+\Delta_A+\Delta_B}{2}\, l - \frac{5\,\Delta_C}{18}\,\Delta_l + \frac{5\,\Delta_C^2}{36\,t\,m} + X_2.$$

Using equality (1.67), we obtain that the expected value of the welfare $E(\underline{W}^{CI})$ is given by

$$E(\underline{W}^{CI}(z,w)) = X_1 - \frac{E(c_A)+E(c_B)}{2}\, l - \frac{5\,\Delta_E}{18}\,\Delta_l + \frac{5\,(\Delta_E^2+V_A+V_B)}{36\,t\,m} + X_2.$$

Corollary 1.2.6. *The difference between the ex-post profit and the profit, under complete information, for firm A,* $\underline{\pi}_A^{EP}(z,w) - \underline{\pi}_A^{CI}(z,w)$, *is*

$$\frac{(\Delta_A-\Delta_B)(\Delta_A+2\,\Delta_B) - 2\,(t\,m\,(3\,l+\Delta_l)-\Delta_C)\,(2\Delta_A+\Delta_B)}{72\,t\,m}. \tag{1.92}$$

The difference between the ex-post profit and the profit, under complete information, for firm B, $\underline{\pi}_B^{EP}(z,w) - \underline{\pi}_B^{CI}(z,w)$, *is*

$$\frac{(\Delta_B-\Delta_A)(\Delta_B+2\,\Delta_A) - 2\,(t\,m\,(3\,l-\Delta_l)+\Delta_C)\,(2\Delta_B+\Delta_A)}{72\,t\,m}. \tag{1.93}$$

Proof. Let $CI = t\,m\,(3\,l+\Delta_l) - \Delta_C$. Hence,

$$\underline{\pi}_A^{EP}(z,w) - \underline{\pi}_A^{CI}(z,w) = \frac{(2\,IC+\Delta_B-\Delta_A)(2\,CI-\Delta_A-2\,\Delta_B) - 4\,CI^2}{72\,t\,m}$$
$$= \frac{(\Delta_B-\Delta_A)(-\Delta_A-2\,\Delta_B) + 2\,CI(-2\Delta_A-\,\Delta_B)}{72\,t\,m}$$

and so equality (1.92) holds. The proof of equality (1.93) follows similarly. □

Corollary 1.2.7. *The difference between the ex-ante profit* $E_B(\underline{\pi}_A^{EP})$ *and* $E_B(\underline{\pi}_A^{CI})$ *for firm A is*

$$E_B(\underline{\pi}_A^{EP}) - E_B(\underline{\pi}_A^{CI}) = \frac{\Delta_A\ (5\,\Delta_A - 4\,(t\,m\,(3\,l+\Delta_l)-\Delta_E))}{72\,t} - \frac{V_B}{18\,t\,m}.$$

The difference between the ex-ante profit $E_A(\underline{\pi}_B^{EP})$ and $E_A(\underline{\pi}_B^{CI})$ for firm B is

$$E_A(\underline{\pi}_B^{EP}) - E_A(\underline{\pi}_B^{CI}) = \frac{\Delta_B\ (5\,\Delta_B - 4\,(t\,m\,(3\,l - \Delta_l) + \Delta_E))}{72\,t} - \frac{V_A}{18\,t\,m}.$$

The proof of the above corollary follows from a simple manipulation of the previous formulas for the ex-post and ex-ante profits.

The difference between the expected profits of firm A with complete and incomplete information is given by

$$E(\underline{\pi}_A^{EP}) - E(\underline{\pi}_A^{CI}) = \frac{5\,V_A - 4\,V_B}{72\,t\,m}. \tag{1.94}$$

The difference between the expected profits of firm B with complete and incomplete information is given by

$$E(\underline{\pi}_B^{EP}) - E(\underline{\pi}_B^{CI}) = \frac{5\,V_B - 4\,V_A}{72\,t\,m}. \tag{1.95}$$

Corollary 1.2.8. *The difference between the ex-post consumer surplus and the consumer surplus, under complete information,* $\underline{CS}^{EP} - \underline{CS}^{CI}$ *is*

$$\frac{\Delta_A + \Delta_B}{4}\,l + \frac{\Delta_B - \Delta_A}{36}\,\Delta_l + \frac{(\Delta_B - \Delta_A)(\Delta_B - \Delta_A - 4\,\Delta_C)}{144\,t\,m}. \tag{1.96}$$

Therefore, equation (1.96) determines in which cases it is better to have uncertainty in the production costs instead of complete information in terms of consumer surplus $\underline{CS}^{EP} > \underline{CS}^{CI}$.

Proof. Let $X = t\,m\,(3\,l + \Delta_l)$. The difference between the ex-post consumer surplus and the consumer surplus, under complete information, is

$$\underline{CS}^{EP} - \underline{CS}^{CI} = \frac{\Delta_A + 2\,\Delta_B}{3}\,l - \frac{\Delta_B}{2}\,l + \frac{(2\,X + \Delta_E - 3\,\Delta_C)^2}{144\,t\,m} - \frac{(X - \Delta_C)^2}{36\,t\,m}$$
$$= \frac{2\,\Delta_A + \Delta_B}{6}\,l + \frac{(2\,X - 2\,\Delta_C + \Delta_E - \Delta_C)^2 - (2\,X - 2\,\Delta_C)^2}{144\,t\,m}$$
$$= \frac{2\,\Delta_A + \Delta_B}{6}\,l + \frac{X\,(\Delta_B - \Delta_A)}{36\,t\,m} + \frac{(\Delta_B - \Delta_A)\,(\Delta_B - \Delta_A - 4\,\Delta_C)}{144\,t\,m}$$
$$= \frac{2\,\Delta_A + \Delta_B}{6}\,l + \frac{(3\,l + \Delta_l)\,(\Delta_B - \Delta_A)}{36} + \frac{(\Delta_B - \Delta_A)\,(\Delta_B - \Delta_A - 4\,\Delta_C)}{144\,t\,m}$$
$$= \frac{\Delta_A + \Delta_B}{4}\,l + \frac{\Delta_B - \Delta_A}{36\,t\,m}\,\Delta_l + \frac{(\Delta_B - \Delta_A)\,(\Delta_B - \Delta_A - 4\,\Delta_C)}{144\,t\,m}$$

□

The difference between expected value of the consumer surplus and the expected value of the consumer surplus under complete information, is

$$E(\underline{CS}^{EP}) - E(\underline{CS}^{CI}) = \frac{5\,(V_A + V_B)}{144\,t\,m}. \tag{1.97}$$

Therefore, in expected value the consumer surplus is greater with incomplete information than with complete information.

The difference between the ex-post welfare and the welfare, under complete information, is

$$\underline{W}^{EP} - \underline{W}^{CI} = \frac{\Delta_B - \Delta_A}{18}\,\Delta_l + \frac{7\,\Delta_C^2 - 6\,\Delta_C\,\Delta_E - \Delta_E^2}{144\,t\,m} \tag{1.98}$$

Therefore, equation (1.98) determines in which cases it is better to have uncertainty in the production costs instead of complete information in terms of welfare $\underline{W}^{EP} > \underline{W}^{CI}$.

The difference between expected value of the welfare and the expected value of the welfare under complete information, is

$$E(\underline{W}^{EP}) - E(\underline{W}^{CI}) = \frac{7\,(V_A + V_B)}{144\,t\,m}. \tag{1.99}$$

Therefore, in expected value the welfare is greater with incomplete information than with complete information.

1.2.9 Example: Symmetric Hotelling

A Hotelling game is *symmetric*, if $(I_A, \Omega_A, q_A) = (I_B, \Omega_B, q_B)$ and $c = c_A = c_B$. Hence, we observe that all the formulas of this section hold with the following simplifications

$$\Delta_E = 0; \quad E(c) = E(c_A) = E(c_B) \text{ and } V = V_A = V_B.$$

The bounded uncertain costs in the symmetric case can be written in the following simple way.

Definition 1.2.11. *The symmetric Hotelling model satisfies the* bounded uncertain costs $(BUCL1)$ *condition, if*

$$|2\,\Delta_l\,t\,m - 3\,\Delta_C| < 6\,t\,m\,l.$$

for all $z \in I_A$ and for all $w \in I_B$.

The Hotelling model with incomplete symmetric information satisfies the *bounded uncertain costs $(BUCL2)$* condition, if

$$6\,(c^M - c^m) + \frac{\Delta_l\,(c^M - E(c))}{l} \leq \frac{t\,m\,(3\,l - \Delta_l)^2}{3\,l} + \frac{3\,(c^M - E(c))^2}{4\,t\,m\,l}$$

and

$$6\,(c^M - c^m) - \frac{\Delta_l\,(c^M - E(c))}{l} \leq \frac{t\,m\,(3\,l + \Delta_l)^2}{3\,l} + \frac{3\,(c^M - E(c))^2}{4\,t\,m\,l}.$$

Under the $BUC1$ condition, the expected prices of the local optimum price strategy have the simple expression

$$E(\underline{p}_A) = t\,m\left(l + \frac{\Delta_l}{3}\right) + E(c); E(\underline{p}_B) = t\,m\left(l - \frac{\Delta_l}{3}\right) + E(c)$$

By Proposition 1.2.1, for the Hotelling game with incomplete symmetric information, the local optimum price strategy (p_A, p_B) has the form

$$p_A^z = E(\underline{p}_A) + \frac{\Delta_A}{2}; \quad p_B^w = E(\underline{p}_B) + \frac{\Delta_B}{2}.$$

The ex-post profit of firm A and firm B are, respectively

$$\underline{\pi}_A^{EP}(z,w) = \frac{(2\,t\,m\,(3\,l + \Delta_l) - 3\,\Delta_A)\,(2\,t\,m\,(3\,l + \Delta_l) - 3\,\Delta_C)}{72\,t\,m}$$

and

$$\underline{\pi}_B^{EP}(z,w) = \frac{(2\,t\,m\,(3\,l - \Delta_l) - 3\,\Delta_B)\,(2\,t\,m\,(3\,l - \Delta_l) + 3\,\Delta_C)}{72\,t\,m}.$$

The difference between the ex-post profits, $\underline{\pi}_A^{EP}(z,w) - \underline{\pi}_B^{EP}(z,w)$, of both firms is given by

$$\frac{16\,t^2\,m^2\,l\,\Delta_l + 2\,t\,m\,(3\,l\,\Delta_C - \Delta_l\,(\Delta_A + \Delta_B)) - 3\,\Delta_C\,(8\,t\,l\,m - \Delta_A - \Delta_B)}{24\,t\,m}.$$

Furthermore, for different production costs, the difference between the ex-post profit of firm A, $\underline{\pi}_A^{EP}(z_1,w) - \underline{\pi}_A^{EP}(z_2,w)$, is given by

$$\frac{(c_A^{z_2} - c_A^{z_1})\,(4\,t\,m\,(3\,l + \Delta_l) + 3\,(c_B^w + E(c_A) - c_A^{z_1} - c_A^{z_2}))}{24\,t\,m}$$

and, for different production costs, the difference between the ex-post profit of firm B, $\underline{\pi}_B^{EP}(z, w_1) - \underline{\pi}_B^{EP}(z, w_2)$, is given by

$$\frac{(c_B^{w_2} - c_B^{w_1})\,(4\,t\,m\,(3\,l - \Delta_l) + 3\,(c_A^z + E(c_B) - c_B^{w_1} - c_B^{w_2}))}{24\,t\,m}$$

for all $z, z_1, z_2 \in I_A$ and $w, w_1, w_2 \in I_B$. The ex-ante profit profit of firm A and firm B are, respectively

$$\underline{\pi}_A^{EA}(z) = \frac{(2\,t\,m\,(3\,l + \Delta_l) - 3\,\Delta_A)^2}{72\,t\,m}$$

and

$$\underline{\pi}_B^{EA}(w) = \frac{(2\,t\,m\,(3\,l - \Delta_l) - 3\,\Delta_B)^2}{72\,t\,m}.$$

The difference between the ex-ante profits of both firms is given by

$$\underline{\pi}_A^{EA}(z) - \underline{\pi}_B^{EA}(w) = \frac{(4\,t\,m\,l - \Delta_A - \Delta_B)\,(4\,t\,m\,\Delta_l - 3\,\Delta_C)}{24\,t\,m}$$

Furthermore, for different production costs, the differences between the ex-ante profits of a firm are given by

$$\underline{\pi}_A^{EA}(z_1) - \underline{\pi}_A^{EA}(z_2) = \frac{(c_A^{z_2} - c_A^{z_1})\,(4\,t\,m\,(3\,l + \Delta_l) + 3\,(2\,E(c) - c_A^{z_1} - c_A^{z_2}))}{24\,t\,m}$$

and

$$\underline{\pi}_B^{EA}(w_1) - \underline{\pi}_B^{EA}(w_2) = \frac{(c_B^{w_2} - c_B^{w_1})\,(4\,t\,m\,(3\,l - \Delta_l) + 3\,(2\,E(c) - c_B^{w_1} - c_B^{w_2}))}{24\,t\,m}$$

for all $z, z_1, z_2 \in I_A$ and $w, w_1, w_2 \in I_B$. The difference between the ex-post profit and the ex-ante profit for firm A is

$$\underline{\pi}_A^{EP}(z, w) - \underline{\pi}_A^{EA}(z) = \frac{\Delta_B}{24\,t\,m}\;(2\,t\,m\,(3\,l + \Delta_l) - 3\,\Delta_A)\,.$$

The difference between the ex-post profit and the ex-ante profit for firm B is

$$\underline{\pi}_B^{EP}(z,w) - \underline{\pi}_B^{EA}(w) = \frac{\Delta_A}{24\,t\,m}\,(2\,t\,m\,(3\,l-\Delta_l) - 3\,\Delta_B)\,.$$

We observe that that the $A-BUCL$ and $B-BUCL$ conditions are implied by the $BUCL1$ condition. Hence, Corollary 1.2.3 can be rewritten without any restriction, i.e.

$$\underline{\pi}_A^{EP}(z,w) < \underline{\pi}_A^{EA}(z) \quad \text{if and only if} \quad \Delta_B < 0;$$

and

$$\underline{\pi}_B^{EP}(z,w) < \underline{\pi}_B^{EA}(w) \quad \text{if and only if} \quad \Delta_A < 0.$$

The expected profit of firm A and firm B are

$$E(\underline{\pi}_A^{EP}) = \frac{t\,m\,(3\,l+\Delta_l)^2}{18} + \frac{V}{8\,t\,m}; E(\underline{\pi}_B^{EP}) = \frac{t\,m\,(3\,l-\Delta_l)^2}{18} + \frac{V}{8\,t\,m}.$$

The difference between the ex-ante profit and the expected profit for firm A is

$$E(\underline{\pi}_A^{EP}) - \underline{\pi}_A^{EA}(z) = \frac{\Delta_A\,(4\,t\,m\,(3\,l+\Delta_l) - 3\,\Delta_A) + 3\,V}{24\,t\,m}.$$

The difference between the ex-ante profit and the expected profit for firm B is

$$E(\underline{\pi}_B^{EP}) - \underline{\pi}_B^{EA}(w) = \frac{\Delta_B\,(4\,t\,m\,(3\,l-\Delta_l) - 3\,\Delta_B) + 3\,V}{24\,t\,m}.$$

The ex-post consumer surplus is

$$\underline{CS}^{EP} = X_1 - E(c)\,l - \frac{\Delta_B}{2}\,l + \frac{(2\,t\,m\,(3\,l+\Delta_l) - 3\,\Delta_C)^2}{144\,t\,m}.$$

The expected value of the consumer surplus is

$$E(\underline{CS}^{EP}) = X_1 - E(c)\,l + \frac{4\,t^2\,m^2\,(3\,l+\Delta_l)^2 + 18\,V}{144\,t\,m}.$$

The difference between the ex-post consumer surplus and the expected value of the consumer surplus is

$$\underline{CS}^{EP} - E(\underline{CS}^{EP}) = -\frac{\Delta_A + \Delta_B}{4}\, l - \frac{\Delta_C}{12}\, \Delta_l + \frac{9\,\Delta_C^2 - 18\,V}{144\,t\,m}$$

The ex-post welfare is

$$\underline{W}^{EP} = X_1 + X_2 - E(c)\, l - \frac{\Delta_C}{3}\, \Delta_l + \frac{27\,\Delta_C^2}{144\,t\,m},$$

The expected value of the welfare $E(\underline{W}^{EP})$ is given by

$$E(\underline{W}^{EP}) = X_1 + X_2 - E(c)\, l - \frac{5\,\Delta_E}{18}\, \Delta_l + \frac{27\,(V_A + V_B)}{144\,t\,m}.$$

The difference between the ex-post welfare and the expected value of welfare is

$$\underline{W}^{EP} - E(\underline{W}^{EP}) = \frac{\Delta_A + \Delta_B}{2}\, l - \frac{\Delta_C)}{3}\, \Delta_l + \frac{9\,(\Delta_C^2 - 2\,V)}{48\,t\,m}.$$

The expected profit $E_B(\underline{\pi}_A^{CI})$ for firm A is given by

$$E_B(\underline{\pi}_A^{CI}) = \frac{(m\,t\,(3\,l + \Delta_l) - \Delta_A)^2 + V}{18\,t\,m}$$

and the expected profit $E_A(\underline{\pi}_B^{CI})$ for firm B is given by

$$E_A(\underline{\pi}_B^{CI}) = \frac{(m\,t\,(3\,l - \Delta_l) - \Delta_B)^2 + V}{18\,t\,m}$$

The expected profits for firm A and B are given by

$$E(\underline{\pi}_A^{CI}) = \frac{m^2\,t^2\,(3\,l + \Delta_l)^2 + 2\,V}{18\,t\,m} \quad \text{and} \quad E(\underline{\pi}_B^{CI}) = \frac{m^2\,t^2\,(3\,l - \Delta_l)^2 + 2\,V}{18\,t\,m}.$$

The expected value of the consumer surplus $E(\underline{CS}^{CI})$ is

$$E(\underline{CS}^{CI}(z,w)) = X_1 - E(c_B)\,l + \frac{t^2\,m^2\,(3\,l+\Delta_l)^2 + 2\,V}{36\,t\,m}.$$

The expected value of the welfare $E(\underline{W}^{CI})$ is given by

$$E(\underline{W}^{CI}(z,w)) = X_1 - E(c_B)\,l + \frac{10\,V}{36\,t\,m} + \frac{m\,t}{36}\,(45\,l^2 + 6\,l\,\Delta_l + 5\,\Delta_l^2).$$

The difference between the ex-post profit and the profit, under complete information, for firm A, is

$$\underline{\pi}_A^{EP}(z,w) - \underline{\pi}_A^{CI}(z,w) = \frac{\Delta_C\,(5\,\Delta_A + 4\,\Delta_B) - 2\,t\,m\,(3\,l+\Delta_l)\,(2\Delta_A + \Delta_B)}{72\,t\,m}.$$

The difference between the ex-post profit and the profit, under complete information, for firm B, is

$$\underline{\pi}_B^{EP}(z,w) - \underline{\pi}_B^{CI}(z,w) = \frac{-\Delta_C\,(5\,\Delta_B + 4\,\Delta_A) - 2\,t\,m\,(3\,l-\Delta_l)\,(2\Delta_B + \Delta_A)}{72\,t\,m}.$$

The difference between the ex-ante profit and the expected profit, under complete information, for firm A is

$$E_B(\underline{\pi}_A^{EP}) - E_B(\underline{\pi}_A^{CI}) = \frac{\Delta_A\,\left(5\,\Delta_A - 4\,t\,m\,(3\,l+\Delta_l)\right)}{72\,t} - \frac{V}{18\,t\,m}.$$

The difference between the ex-ante profit and the expected profit, under complete information, for firm B is

$$E_A(\underline{\pi}_B^{EP}) - E_A(\underline{\pi}_B^{CI}) = \frac{\Delta_B\,\left(5\,\Delta_B - 4\,t\,m\,(3\,l-\Delta_l)\right)}{72\,t} - \frac{V}{18\,t\,m}.$$

The differences between the expected profits with complete and incomplete information for firm A and firm B are given by

$$E(\underline{\pi}_A^{EP}) - E(\underline{\pi}_A^{CI}) = E(\underline{\pi}_B^{EP}) - E(\underline{\pi}_B^{CI}) = \frac{V}{72\,t\,m}.$$

The difference between the ex-post consumer surplus and the consumer surplus, under complete information, is

$$\underline{CS}^{EP} - \underline{CS}^{CI} = \frac{\Delta_A + \Delta_B}{4}\,l - \frac{\Delta_C}{36}\,\Delta_l + \frac{5\Delta_C^2}{144\,t\,m}.$$

The difference between expected value of the consumer surplus and the expected value of the consumer surplus under complete information, is

$$E(\underline{CS}^{EP}) - E(\underline{CS}^{CI}) = \frac{10\,V}{144\,t\,m}.$$

The difference between the ex-post welfare and the welfare, under complete information, is

$$\underline{W}^{EP} - \underline{W}^{CI} = -\frac{\Delta_C}{18}\,\Delta_l + \frac{7\,\Delta_C^2}{144\,t\,m}$$

The difference between expected value of the welfare and the expected value of the welfare under complete information, is

$$E(\underline{W}^{EP}) - E(\underline{W}^{CI}) = \frac{7\,V}{72\,t\,m}.$$

Chapter 2

Hotelling Network

The *Hotelling town* model consists of a network of *consumers* and *firms*. The consumers (*buyers*) are located along the *edges* (*roads*) of the network and the firms (*shops*) are located at neighborhoods of the *vertices* (*nodes*) of the network. Every road has two vertices and in a neighborhood of every vertex is located a single firm. The *degree* k of the vertex is given by the number of incident edges. If the degree k is greater that 2 then the vertex is a crossroad of k roads; if the degree k is equal to 2 then the vertex is a junction between two roads; and if k is equal to 1 the vertex is in the end of a road with no exit. Every consumer will buy one unit of the commodity from only one firm in the network and each firm will charge its customers the same price for the commodity.

A Hotelling town *price strategy* $\mathbf{P}$ consists of a vector whose coordinates are the prices p_i of each firm F_i. Every firm F_i is located at a position y_i in a neighborhood of a vertex $i \in V$, where V is the set of all vertices of the Hotelling town. A consumer located at a point x of the network who decides to buy at firm F_i *spends*

$$E(x; i, \mathbf{P}) = p_i + T(t_i, d(x, y_i))$$

the price p_i charged by the firm F_i plus a value, T, that depends on the *transportation cost* t_i and on minimal distance measured in the network between the position y_i of the firm F_i and the position x of the consumer. Given a price strategy $\mathbf{P}$, the consumer will choose to buy in the firm $F_{v(x,\mathbf{P})}$ that minimizes his expenditure

$$v(x, \mathbf{P}) = argmin_{i \in V} E(x; i, \mathbf{P}).$$

Hence, for every firm F_i, the *market*

$$M(i, \mathbf{P}) = \{x : v(x, \mathbf{P}) = i\}$$

consists of all consumers who minimize their expenditures by opting to buy in firm F_i. The *road market size* $l_{i,j}$ of a road $R_{i,j}$ is the Lebesgue measure (or length) of the road $R_{i,j}$, because the consumers are uniformly distributed along the roads. The *market size* $S(i, \mathbf{P})$ of the firm F_i is the Lebesgue measure of $M(i, \mathbf{P})$. The Hotelling town *production cost* $\mathbf{C}$ is the vector whose coordinates are the production costs c_i of the firms F_i. The Hotelling town *profit* $\Pi(\mathbf{P}, \mathbf{C})$ is the vector whose coordinates

$$\pi_i(\mathbf{P}, \mathbf{C}) = (p_i - c_i)\, S(i, \mathbf{P})$$

are the *profits* of the firms F_i. The *local firms* of a consumer located at a point x in a road $R_{i,j}$ with vertices i and j are the firms F_i and F_j. For every vertex i let N_i be the set of all neighboring vertices j for which there is a road $R_{i,j}$ connecting the vertices. A price strategy $\mathbf{P}$ determines a *local market structure* if every consumer buys from one of his local firms, i.e.

$$M(i, \mathbf{P}) \subset \bigcup_{j \in N_i} R_{i,j}.$$

If a price strategy $\mathbf{P}$ determines a local market structure then for every road

$R_{i,j}$ there is one consumer located at a point $\mathbf{x}_{i,j} \in R_{i,j}$ who is *indifferent* to the local firm from which he going to buy his commodity, i.e. $E(x; i, \mathbf{P}) = E(x; j, \mathbf{P})$.

We denote by c_M (resp. c_m) the maximum (resp. minimum) production cost of the Hotelling town

$$c_M = \max\{c_i : i \in V\} \quad \text{and} \quad c_m = \min\{c_i : i \in V\}.$$

We denote by l_M (resp. l_m) the maximum (resp. minimum) road length of the Hotelling town

$$l_M = \max\{l_e : e \in E\} \quad \text{and} \quad l_m = \min\{l_e : e \in E\},$$

where E is the set of all edges of the Hotelling town. Let $\Delta(c)$ be the maximal difference between the firm's production cost of the commodity, $\Delta(l)$ be the maximal difference between the road lengths in the network and $\Delta_2(l)$ be the maximal difference between the square road lengths in the network

$$\Delta(c) = c_M - c_m \quad , \quad \Delta(l) = l_M - l_m \quad \text{and} \quad \Delta_2(l) = l_M^2 - l_m^2.$$

We introduce the *weak-bound* WB condition that defines a bound for the $\Delta(c)$ and $\Delta(l)$ ($\Delta(c)$ and $\Delta_2(l)$, in the quadratic transportation cost case) in terms of the transportation cost t and the minimal road length l_m of the network (see sections 2.1.1 and 2.2.1). We prove that a Hotelling town network satisfying the WB condition has a unique *local optimum price strategy* $\mathbf{P}^L$, i.e. the profit of every firm is optimal for small perturbations of its own price. We prove that if a Hotelling town network satisfying the WB condition the local optimum price strategy $\mathbf{P}^L$ determines a local market structure. Furthermore, if there is a Nash price equilibrium $\mathbf{P}^*$ then the Nash price equilibrium is the local optimum price strategy $\mathbf{P}^L$. However, in sections 2.1.2 and 2.2.2, we exhibit simple Hotelling town networks that satisfy the WB condition but the local optimum price strategy is not a Nash price equilibrium.

We denote by k_M (resp. k_m) the maximum (resp. minimum) node degree of the Hotelling town

$$k_M = \max\{k_i : i \in V\} \quad \text{and} \quad k_m = \min\{k_i : i \in V\}.$$

We introduce the *strong-bound* SB condition that defines a bound for $\Delta(c)$ and $\Delta(l)$ ($\Delta(c)$ and $\Delta_2(l)$, in the quadratic transportation cost case) in terms of the transportation cost t, the minimal road length l_m and also on the maximum node degree k_M (see Subsections 2.1.2 and 2.2.2). We prove that a Hotelling town network satisfying the SB condition has a unique Nash price equilibrium $\mathbf{P}^*$. Since the SB condition implies the WB condition, the Nash price equilibrium $\mathbf{P}^*$ is equal to the local optimum price strategy $\mathbf{P}^L$. We give an explicit series expansion formula for the Nash price equilibrium $\mathbf{P}^*$. This formula has the feature to show explicitly how the Nash price equilibrium of a firm depends on the production costs, road market sizes and firms locations of its local neighborhood network structure. Furthermore, the influence of a firm in the Nash price equilibrium of other firm decreases exponentially with the distance between the firms.

We say that a firm has *n-space bounded information*, if the firm knows the production costs of the other firms and the road lengths of the network up to n consecutive nodes of distance. Given a Hotelling town network satisfying the WB condition, every firm with n-space bounded information can readily compute a price $p_i(n)$ that estimates its own local optimum price p_i^L, with exponential precision depending upon n. In addition, the firm can then easily estimate the profit obtained with the local optimum price strategy, also with exponential precision depending upon n.

A localization strategy for the firms in the network consists in every firm F_i to choose its position in the neighborhood of its vertex i. For every given localization strategy, we assume the firms opt for their Nash price strategy.

A *local optimal localization strategy* is achieved when for every firm F_i small perturbations in its location no longer result in improved profits for the firm F_i. In Subsection 2.1.3, we prove that a Hotelling town network with linear transportation costs satisfying the SB condition and with $k_m \geq 3$ has a local optimal localization strategy, whereby every firm F_i is located at the corresponding node i. Furthermore, the network can also have nodes with degree 2 under appropriate symmetric assumptions.

We say that a price strategy has the *profit degree growth* property if the profits of the firms increase with the degree of the nodes in the neighborhoods in which they are located. In Subsection 2.1.1 we introduce the *degree-bound* DB condition that gives a new bound for $\Delta(c)$ and $\Delta(l)$ and we prove that for a Hotelling town network with linear transportation costs satisfying the WB and DB conditions the Nash price strategy $\mathbf{P}^*$ has the profit degree growth property.

For example, the Hotelling town networks, where all firms have the same production costs and all roads have the same length, satisfy the SB and DB conditions. Therefore, these networks have a Nash price equilibrium satisfying the profit degree growth property. Furthermore, if $k_m \geq 3$ the firms have a local optimal localization strategy whereby they are located at the corresponding nodes. The original idea of the Hotelling town model was presented in [30].

2.1 Linear transportation costs

This section extends the Hotelling model with linear transportation costs to networks.

A consumer located at a point x of the network who decides to buy at firm F_i *spends*

$$E(x; i, \mathbf{P}) = p_i + t\, d(x, y_i)$$

the price p_i charged by the firm F_i plus the *transportation cost* that is proportional t to the minimal distance measured in the network between the position y_i of the firm F_i and the position x of the consumer.

2.1.1 Local optimal equilibrium price strategy

For every $v \in V$, let $\epsilon_v = d(v, y_v)$ and $j(v)$ be the node with the property that y_v is at the road $R_{v,j(v)}$. The *shift location matrix* $\mathbf{S}(v)$ associated to node v is defined by

$$s_{i,j}(v) = \begin{cases} \epsilon_v & \text{if } i = v \text{ and } j \in N_v \setminus \{j(v)\} \ ; \\ -\epsilon_v & \text{if } i = v \text{ and } j = j(v) \ ; \\ \epsilon_v & \text{if } j = v \text{ and } i \in N_v \setminus \{j(v)\} \ ; \\ -\epsilon_v & \text{if } j = v \text{ and } i = j(v) \ ; \\ 0 & \text{otherwise.} \end{cases}$$

The distance $\tilde{l}_{i,j} = d(y_i, y_j)$ between the location of firms F_i and F_j is given by

$$\tilde{l}_{i,j} = l_{i,j} + \sum_{v \in \{i,j\}} s_{i,j}(v). \tag{2.1}$$

Let $\epsilon = \max_{v \in V} \epsilon_v$. Hence, for every $i, j \in V$ we have

$$l_{i,j} - 2\,\epsilon \leq \tilde{l}_{i,j} \leq l_{i,j} + 2\,\epsilon.$$

We observe that, for every road $R_{i,j}$ there is an *indifferent consumer* located at a distance

$$0 < x_{i,j} = (2\,t)^{-1}(p_j - p_i + t\,\tilde{l}_{i,j}) < \tilde{l}_{i,j} \tag{2.2}$$

of firm F_i if and only if $|p_i - p_j| < t\,\tilde{l}_{i,j}$. Thus, a price strategy $\mathbf{P}$ determines a local market structure if and only if $|p_i - p_j| < t\,\tilde{l}_{i,j}$ for every road $R_{i,j}$. Hence, if

$$|p_i - p_j| < t\,l_{i,j} - 2\,t\,\epsilon \tag{2.3}$$

then condition (2.2) is satisfied. Therefore, if condition (2.3) holds then the price strategy $\mathbf{P}$ determines a local market structure.

Let k_i denote is the cardinality of the set N_i that is equal to the degree of the vertex i. If the price strategy determines a local market structure then

$$S(i, \mathbf{P}) = (2 - k_i)\,\epsilon_i + \sum_{j \in N_i} x_{i,j}$$

and

$$\begin{aligned} \pi_i(\mathbf{P}, \mathbf{C}) &= (p_i - c_i)\, S(i, \mathbf{P}) \\ &= (2\,t)^{-1}(p_i - c_i) \left(2\,t\,(2 - k_i)\,\epsilon_i + \sum_{j \in N_i} p_j - p_i + t\,\tilde{l}_{i,j} \right). \end{aligned} \quad (2.4)$$

Given a pair of price strategies $\mathbf{P}$ and $\mathbf{P}^*$ and a firm F_i, we define the price vector $\tilde{\mathbf{P}}(i, \mathbf{P}, \mathbf{P}^*)$ whose coordinates are $\tilde{p}_i = p_i^*$ and $\tilde{p}_j = p_j$, for every $j \in V \setminus \{i\}$. Let $\mathbf{P}$ and $\mathbf{P}^*$ be price strategies that determine local market structures. The price strategy $\mathbf{P}^*$ is a *local best response* to the price strategy $\mathbf{P}$, if for every $i \in V$ the price strategy $\tilde{\mathbf{P}}(i, \mathbf{P}, \mathbf{P}^*)$ determines a local market structure and

$$\frac{\partial \pi_i(\tilde{\mathbf{P}}(i, \mathbf{P}, \mathbf{P}^*), \mathbf{C})}{\partial \tilde{p}_i} = 0 \quad \text{and} \quad \frac{\partial^2 \pi_i(\tilde{\mathbf{P}}(i, \mathbf{P}, \mathbf{P}^*), \mathbf{C})}{\partial \tilde{p}_i^2} < 0.$$

The Hotelling town *admissible market size* $\mathbf{L}$ is the vector whose coordinates are the *admissible local firm market sizes*

$$L_i = \frac{1}{k_i} \sum_{j \in N_i} l_{i,j}.$$

The Hotelling town *neighboring market structure* $\mathbf{K}$ is the matrix whose elements are (i) $k_{i,j} = k_i^{-1}$, if there is a road $R_{i,j}$ between the firms F_i and F_j; and (ii) $k_{i,j} = 0$, if there is not a road $R_{i,j}$ between the firms F_i and F_j.

The Hotelling town *firm deviation* is the vector $\mathbf{Y}$ whose coordinates are

$$Y_i = k_i^{-1}\left((2-k_i)\,\epsilon_i + \sum_{j\in N_i} s_{i,j}(j)\right).$$

Let $\mathbf{1}$ denote the identity matrix.

Lemma 2.1.1. *Let $\boldsymbol{P}$ and $\boldsymbol{P}^*$ be price strategies that determine local market structures. The price strategy $\boldsymbol{P}^*$ is the local best response to price strategy $\boldsymbol{P}$ if and only if*

$$\boldsymbol{P}^* = \frac{1}{2}\,(\boldsymbol{C} + t\,(\boldsymbol{L} + \boldsymbol{Y})) + \frac{1}{2}\,\boldsymbol{K}\,\boldsymbol{P} \tag{2.5}$$

and the price strategies $\tilde{\boldsymbol{P}}(i, \boldsymbol{P}, \boldsymbol{P}^*)$ determine local market structures for all $i \in V$.

Proof. By (2.4), the *profit function* $\pi_i(\mathbf{P}, \mathbf{C})$ of firm F_i, in a local market structure, is given by

$$\pi_i(\mathbf{P}, \mathbf{C}) = (2\,t)^{-1}(p_i - c_i)\left(2\,t\,(2-k_i)\,\epsilon_i + \sum_{j\in N_i} p_j - p_i + t\,\tilde{l}_{i,j}\right).$$

Let $\tilde{\mathbf{P}}(i, \mathbf{P}, \mathbf{P}^*)$ be the price vector whose coordinates are $\tilde{p}_i = p_i^*$ and $\tilde{p}_j = p_j$, for every $j \in V \setminus \{i\}$. Since $\mathbf{P}$ and $\mathbf{P}^*$ are local price strategies, the local best response of firm F_i to the price strategy $\mathbf{P}$, is given by computing $\partial\pi_i(\tilde{\mathbf{P}}(i, \mathbf{P}, \mathbf{P}^*), \mathbf{C})/\partial\tilde{p}_i = 0$. Hence,

$$p_i^* = \frac{1}{2}\left(c_i + \frac{2\,t\,(2-k_i)}{k_i}\,\epsilon_i + \frac{1}{k_i}\sum_{j\in N_i} t\,\tilde{l}_{i,j} + p_j\right). \tag{2.6}$$

By (2.1), we obtain

$$p_i^* = \frac{1}{2}\left(c_i + \frac{2\,t\,(2-k_i)}{k_i}\,\epsilon_i + \frac{t}{k_i}\sum_{j\in N_i}\sum_{v\in\{i,j\}} s_{i,j}(v) + \frac{1}{k_i}\sum_{j\in N_i} t\,l_{i,j} + p_j\right).$$

We note that

$$\sum_{j\in N_i}\sum_{v\in\{i,j\}} s_{i,j}(v) = \sum_{j\in N_i} s_{i,j}(i) + \sum_{j\in N_i} s_{i,j}(j) = (k_i - 2)\,\epsilon_i + \sum_{j\in N_i} s_{i,j}(j).$$

Hence,

$$p_i^* = \frac{1}{2}\left(c_i + \frac{t}{k_i}\left((2-k_i)\,\epsilon_i + \sum_{j\in N_i} s_{i,j}(j)\right) + \frac{1}{k_i}\sum_{j\in N_i} t\, l_{i,j} + p_j\right).$$

Therefore, since $\partial^2\pi_i(\tilde{\mathbf{P}}(i,\mathbf{P},\mathbf{P}^*),\mathbf{C})/\partial\tilde{p}_i^2 = -k_i/t < 0$, the local best response strategy prices $\mathbf{P}^*$ is given by

$$\mathbf{P}^* = \frac{1}{2}\,(\mathbf{C} + t\,(\mathbf{Y}+\mathbf{L}) + \mathbf{K}\,\mathbf{P})\,.$$

□

Definition 2.1.1. *A Hotelling town satisfies the* weak bounded length and costs (WB) *condition, if*

$$\Delta(c) + t\Delta(l) < t\,l_m - 6\,t\,\epsilon.$$

Hence, the WB condition implies $\epsilon < l_m/6$.

Let $\mathbf{P}$ and $\mathbf{P}^*$ be price strategies that determine local market structures. A price strategy $\mathbf{P}^*$ is a *local optimum price strategy* if $\mathbf{P}^*$ is the local best response to $\mathbf{P}^*$.

Proposition 2.1.1. *If the Hotelling town satisfies the WB condition, then there is unique local optimum price strategy given by*

$$\begin{aligned} \boldsymbol{P}^L &= \frac{1}{2}\left(\boldsymbol{1} - \frac{1}{2}\boldsymbol{K}\right)^{-1}(\boldsymbol{C} + t\,(\boldsymbol{L}+\boldsymbol{Y})) \\ &= \sum_{m=0}^{\infty} 2^{-(m+1)}\boldsymbol{K}^m\,(\boldsymbol{C} + t\,(\boldsymbol{L}+\boldsymbol{Y}))\,. \end{aligned} \tag{2.7}$$

The local optimum price strategy $\boldsymbol{P}^L$ *determines a local market structure. Furthermore, the local optimal equilibrium prices* p_i^L *are bounded by*

$$t\,l_m + \frac{1}{2}\,(c_i + c_m) - 2\,t\,\epsilon \le p_i^L \le t\,l_M + \frac{1}{2}\,(c_i + c_M) + 2\,t\,\epsilon. \qquad (2.8)$$

The local optimal profit $\pi_i^L = \pi_i^L(\boldsymbol{P}, \boldsymbol{C})$ *of firm* F_i *is given by*

$$\pi_i^L(\boldsymbol{P}, \boldsymbol{C}) = (2t)^{-1}\,k_i\,(p_i^L - c_i)^2$$

and it is bounded by

$$(8t)^{-1}\,k_i\,(2t\,l_m - \Delta(c) - 4\,t\,\epsilon)^2 \le \pi_i^L(\boldsymbol{P}, \boldsymbol{C}) \le (8t)^{-1}\,k_i\,(2\,t\,l_M + \Delta(c) + 4\,t\,\epsilon)^2.$$

Corollary 2.1.1. *Consider a Hotelling town where all firms are located at the nodes. If* $\Delta(c) + t\Delta(l) < t\,l_m$*, then there is unique local optimum price strategy given by*

$$\boldsymbol{P}^L = \sum_{m=0}^{\infty} 2^{-(m+1)}\boldsymbol{K}^m\,(\boldsymbol{C} + t\,\boldsymbol{L})\,.$$

The local optimum price strategy $\boldsymbol{P}^L$ *determines a local market structure. Furthermore, the local optimal equilibrium prices* p_i^L *are bounded by*

$$t\,l_m + \frac{1}{2}\,(c_i + c_m) \le p_i^L \le t\,l_M + \frac{1}{2}\,(c_i + c_M).$$

The local optimal profit $\pi_i^L = \pi_i^L(\boldsymbol{P}, \boldsymbol{C})$ *of firm* F_i *is given by*

$$\pi_i^L(\boldsymbol{P}, \boldsymbol{C}) = (2t)^{-1}\,k_i\,(p_i^L - c_i)^2$$

and it is bounded by

$$(8t)^{-1}\,k_i\,(2t\,l_m - \Delta(c))^2 \le \pi_i^L(\boldsymbol{P}, \boldsymbol{C}) \le (8t)^{-1}\,k_i\,(2\,t\,l_M + \Delta(c))^2.$$

Proof of Proposition 2.1.1.
The matrix $\mathbf{K}$ is a stochastic matrix (i.e., $\sum_{j\in V} k_{i,j} = 1$, for every $i \in V$). Thus, we have $\|\mathbf{K}\| = 1$. Hence, the matrix Q is well-defined by

$$\mathbf{Q} = \frac{1}{2}\left(\mathbf{1} - \frac{1}{2}\,\mathbf{K}\right)^{-1} = \sum_{m=0}^{\infty} 2^{-(m+1)}\,\mathbf{K}^m$$

and Q is also a non-negative and stochastic matrix. By Lemma 2.1.1, a local optimum price strategy satisfy equality (2.5). Therefore,

$$\begin{aligned}\mathbf{P}^L &= \frac{1}{2}\left(\mathbf{1} - \frac{1}{2}\,\mathbf{K}\right)^{-1} (\mathbf{C} + t\,(\mathbf{L} + \mathbf{Y})) \\ &= \sum_{m=0}^{\infty} 2^{-(m+1)}\,\mathbf{K}^m\,\left(\mathbf{C} + t\,(\mathbf{L} + \mathbf{Y})\right),\end{aligned}$$

and so $\mathbf{P}^L$ satisfies (2.7). By construction,

$$p_i^L = \sum_{v\in V} Q_{i,v}(c_v + t\,(L_v + Y_v)). \tag{2.9}$$

Let us prove that the price strategy $\mathbf{P}^L$ is local, i.e., the indifferent consumer $x_{i,j}$ satisfies $0 < x_{i,j} < \tilde{l}_{i,j}$ for every $R_{i,j} \in E$. We note that

$$l_m \le L_v = k_v^{-1} \sum_{j\in N_v} l_{v,j} \le l_M. \tag{2.10}$$

We note that

$$-k_v\,\epsilon \le \sum_{j\in N_v} s_{v,j}(j) \le k_v\,\epsilon$$

Hence, if $k_v = 1$ then

$$-\epsilon \leq \epsilon_v - \epsilon \leq Y_v = k_v^{-1}\left(\epsilon_v + \sum_{j \in N_v} s_{v,j}(j)\right) \leq \epsilon_v + \epsilon \leq 2\,\epsilon; \qquad (2.11)$$

if $k_v = 2$ then

$$-\epsilon \leq Y_v = k_v^{-1} \sum_{j \in N_v} s_{v,j}(j) \leq \epsilon; \qquad (2.12)$$

and if $k_v \geq 3$ then

$$\frac{2-k_v}{k_v}\,\epsilon_v - \epsilon \leq Y_v = k_v^{-1}\left((2-k_v)\,\epsilon_v + \sum_{j \in N_v} s_{v,j}(j)\right) \leq \frac{2-k_v}{k_v}\epsilon_v + \epsilon.$$

Hence,

$$-2\,\epsilon \leq -\epsilon_v - \epsilon \leq Y_v = k_v^{-1}\left((2-k_v)\,\epsilon_v + \sum_{j \in N_v} s_{v,j}(j)\right) \leq \epsilon. \qquad (2.13)$$

Therefore, from (2.11), (2.12) and (2.13), we have

$$-2\,\epsilon \leq Y_v = k_v^{-1}\left((2-k_v)\,\epsilon_v + \sum_{j \in N_v} s_{v,j}(j)\right) \leq 2\,\epsilon. \qquad (2.14)$$

Since $\mathbf{Q}$ is a nonnegative and stochastic matrix, we obtain

$$\sum_{v \in V} Q_{i,v}(c_m + t\,l_m - 2\,t\,\epsilon) = c_m + t\,l_m - 2\,t\,\epsilon$$

and

$$\sum_{v \in V} Q_{i,v}(c_M + t\,l_M + 2\,t\,\epsilon) = c_M + t\,l_M + 2\,t\,\epsilon.$$

Hence, putting (2.9), (2.10) and (2.14) together we obtain that

$$c_m + t\,l_m - 2\,t\,\epsilon \leq p_i^L \leq c_M + t\,l_M + 2\,t\,\epsilon.$$

Since the last relation is satisfied for every firm, we obtain

$$-(c_M - c_m + t(l_M - l_m) + 4\,t\,\epsilon) \le p_i^L - p_j^L \le c_M - c_m + t(l_M - l_m) + 4\,t\,\epsilon.$$

Therefore,

$$|p_i^L - p_j^L| \le \Delta(c) + t\,\Delta(l) + 4\,t\,\epsilon.$$

Hence, by the WB condition, we conclude that

$$|p_i^L - p_j^L| < t\,l_m - 2\,t\,\epsilon.$$

Thus, by equation (2.3), we obtain that the indifferent consumer is located at $0 < x_{i,j} < \tilde{l}_{i,j}$ for every road $R_{i,j} \in E$. Hence, the price strategy $\mathbf{P}^L$ is local and is the unique local optimum price strategy.

From (2.9), (2.10) and (2.14), we obtain

$$p_i^L \ge \sum_{v \in V} Q_{i,v}(t\,l_m - 2\,t\,\epsilon) + \sum_{v \in V \setminus \{i\}} Q_{i,v}\,c_m + Q_{i,i}\,c_i.$$

By construction of matrix $\mathbf{Q}$, we have $Q_{i,i} > 1/2$. Furthermore, since $\mathbf{Q}$ is stochastic,

$$\sum_{v \in V \setminus \{i\}} Q_{i,v} < 1/2,$$

$\sum_{v \in V} Q_{i,v} t\,l_m = t\,l_m$ and $\sum_{v \in V} Q_{i,v} 2\,t\,\epsilon = 2\,t\,\epsilon$. Hence,

$$p_i^L \ge t\,l_m - 2\,t\,\epsilon + \frac{1}{2}\,(c_i + c_m).$$

Similarly, we obtain

$$p_i^L \le t\,l_M + 2\,t\,\epsilon + \frac{1}{2}\,(c_i + c_M),$$

and so the local optimal equilibrium prices p_i^L are bounded and satisfy (2.8).

We can write the the profit function (2.4) of firm F_i for the price strategy

P^L as

$$\pi_i^L = \pi_i(\mathbf{P}^L, \mathbf{C}) = (2t)^{-1}(p_i^L - c_i)\left(2\,t\,(2-k_i)\,\epsilon_i - k_i\,p_i^L + \sum_{j\in N_i}(p_j^L + t\,\tilde{l}_{i,j})\right) \tag{2.15}$$

Since $\mathbf{P}^L$ satisfies the best response function (2.6), we have

$$2\,p_i^L = c_i + \frac{2\,t\,(2-k_i)}{k_i}\,\epsilon_i + \frac{1}{k_i}\sum_{j\in N_i}\left(t\,\tilde{l}_{i,j} + p_j^L\right).$$

Therefore, $\sum_{j\in N_i}\left(t\,\tilde{l}_{i,j} + p_j^L\right) = 2\,k_i\,p_i^L - k_i\,c_i + 2\,t\,(k_i-2)\,\epsilon_i$, and replacing this sum in the profit function (2.15), we obtain

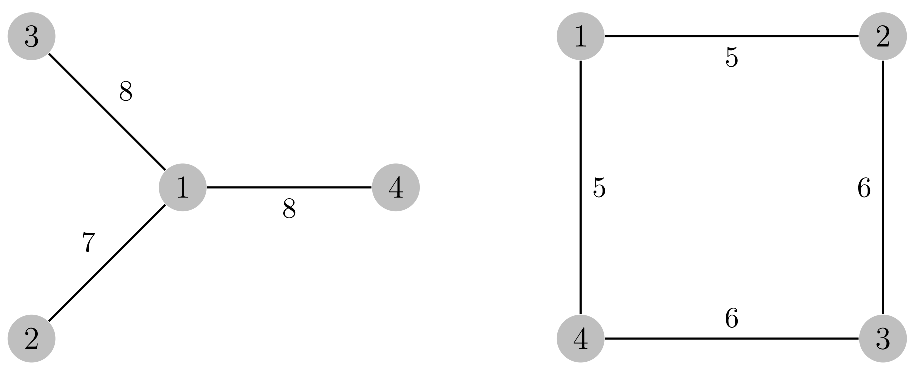
$$\pi_i^L = (2t)^{-1}(p_i^L - c_i)\left(-k_i\,p_i^L + 2\,k_i\,p_i^L - k_i\,c_i\right) = (2t)^{-1}\,k_i\,(p_i^L - c_i)^2.$$

Hence, using the price bounds (2.8), we conclude

$$(2t)^{-1}\,k_i\,(t\,l_m - \Delta(c)/2 - 2\,t\,\epsilon)^2 \le \pi_i^L \le (2t)^{-1}\,k_i\,(t\,l_M + \Delta(c)/2 + 2\,t\,\epsilon)^2.$$

□

Consider the two networks presented in Figure 2.1.



(a) Star Network (b) Regular Network

Figure 2.1: Hotelling networks satisfying WB condition

For network 2.1a the parameter values are $\epsilon_i = 0$, $c_i = 0$, $l_m = 7$, $l_M = 8$, $\Delta(l) = 1$ and $k_M = 3$. For network 2.1b the parameter values are $\epsilon_i = 0$, $c_i = 0$, $l_m = 5$, $l_M = 6$, $\Delta(l) = 1$ and $k_M = 2$. Both networks satisfies the WB condition. Hence, by Proposition 2.1.1, there is a local optimum price strategy P^L. The local optimal prices for network 2.1a are given by

$$p_i^L = t\left(\frac{23}{3}, \frac{22}{3}, \frac{47}{6}, \frac{47}{6}\right)$$

and the correspondent profits are given by

$$\pi^L = t\left(\frac{529}{6}, \frac{242}{9}, \frac{2209}{72}, \frac{2209}{72}\right).$$

The local optimal prices for network 2.1b are given by

$$p_i^L = t\left(\frac{21}{4}, \frac{11}{2}, \frac{23}{4}, \frac{11}{2}\right)$$

and the correspondent profits are given by

$$\pi^L = t\left(\frac{441}{16}, \frac{121}{4}, \frac{529}{16}, \frac{121}{4}\right).$$

We say that a price strategy $\mathbf{P}$ has the *profit degree growth* property if

$$k_i > k_j \Rightarrow \pi_i(\mathbf{P}, \mathbf{C}) > \pi_j(\mathbf{P}, \mathbf{C})$$

for every $i, j \in V$.

Lemma 2.1.2. *Let F_i be a firm located in a node of degree k_i and F_j a firm located in a node of degree k_j. Then, $\pi_i^L > \pi_j^L$ if and only if*

$$\frac{k_i - k_j}{k_j} > \frac{(p_j^L - c_j)^2 - (p_i^L - c_i)^2}{(p_i^L - c_i)^2}.$$

Let $\bar{p}_i = p_i^L - c_i$ and $\bar{p}_j = p_j^L - c_j$ represent the unit profit of firms F_i and F_j located at nodes of degree k_i and k_j, respectively. Let $\theta(p) = p_i^L - p_j^L$, $\theta(k) = k_i - k_j$ and $\theta(\bar{p}) = \bar{p}_i - \bar{p}_j$.

Proof of Lemma 2.1.2.
If F_j is a firm located in a node of degree k_j, then

$$\pi_j^L = (2t)^{-1}\, k_j\, (p_j^L - c_j)^2 = (2t)^{-1}\, k_j\, \bar{p}_j^2.$$

Similarly, if F_i is a firm located in a node of degree k_i, then

$$\pi_i^L = (2t)^{-1}\, k_i\, (p_i^L - c_i)^2 = (2t)^{-1}\, k_i\, \bar{p}_i^2 = (2t)^{-1}\, (k_j + \theta(k))\, (\bar{p}_j + \theta(\bar{p}))^2\,.$$

Hence,

$$\begin{aligned} 2\,t\,\pi_i^L &= k_j\,\bar{p}_j^2 + k_j\,\theta(\bar{p})\,(2\,\bar{p}_j + \theta(\bar{p})) + \theta(k)\,(\bar{p}_j + \theta(\bar{p}))^2 \\ &= 2\,t\,\pi_j^L + k_j\,\theta(\bar{p})\,(\bar{p}_j + \bar{p}_i) + \theta(k)\,\bar{p}_i^2, \end{aligned}$$

and so

$$2\,t\,(\pi_i^L - \pi_j^L) = k_j\,(\bar{p}_i - \bar{p}_j)\,(\bar{p}_j + \bar{p}_i) + \theta(k)\,\bar{p}_i^2 = k_j\,(\bar{p}_i^2 - \bar{p}_j^2) + (k_i - k_j)\,\bar{p}_i^2.$$

Therefore,

$$\pi_i^L > \pi_j^L \quad \text{if and only if} \quad \frac{k_i - k_j}{k_j} > \frac{\bar{p}_j^2 - \bar{p}_i^2}{\bar{p}_i^2}.$$

□

Definition 2.1.2. *A Hotelling town network satisfies the* degree bounded lengths and costs *(DB) condition if*

$$\Delta(c) + t\,\Delta(l) < \left(\sqrt{1 + 1/k_M} - 1\right)\,(t\,l_m - \Delta(c)/2 - 2\,t\,\epsilon) - 4\,t\,\epsilon.$$

Theorem 2.1.1. *A Hotelling town network satisfying the WB and DB conditions has the profit degree growth property.*

Proof. Let F_i and F_j be firms in the Hotelling town network such that $k_i > k_j$. We need to prove that $\pi_i^L > \pi_j^L$. From Lemma 2.1.2 we say that $\pi_i^L > \pi_j^L$ if and only if

$$k_j\,\theta(\bar{p})\,(\bar{p}_j + \bar{p}_i) + \theta(k)\,\bar{p}_i^2 > 0. \tag{2.16}$$

Since $k_i > k_j$, then $\theta(k) > 0$. Hence, if $\theta(\bar{p}) > 0$, i.e. $\bar{p}_i > \bar{p}_j$, then condition (2.16) is satisfied.

Let us now consider the case where $\theta(\bar{p}) < 0$. Condition (2.16) is equivalent to

$$k_j\,\theta(\bar{p})^2 - 2\,k_j\,\bar{p}_i\,\theta(\bar{p}) - \theta(k)\,\bar{p}_i^2 < 0. \tag{2.17}$$

Solving the second degree equation $k_j\,\theta(\bar{p})^2 - 2\,k_j\,\bar{p}_i\,\theta(\bar{p}) - \theta(k)\,\bar{p}_i^2 = 0$, we obtain

$$\theta(\bar{p})_{\pm} = \bar{p}_i\,\left(1 \pm \sqrt{1 + \theta(k)/k_j}\right).$$

Let $f(\theta(k), k_j)$ be the function given by

$$f(\theta(k), k_j) = \sqrt{1 + \theta(k)/k_j} - 1.$$

We note that $f(\theta(k), k_j) > 0$ and $\theta(\bar{p})_- = -f(\theta(k), k_j)\,\bar{p}_i$. If $\theta(\bar{p})_- < \theta(\bar{p}) < 0$ then condition (2.17) is satisfied. By hypothesis $\theta(\bar{p}) < 0$ and, so, if

$$f(\theta(k), k_j)\,\bar{p}_i > -\theta(\bar{p}) \tag{2.18}$$

then (2.17) is satisfied.

Since $\theta(\bar{p}) = \bar{p}_i - \bar{p}_j$, from (2.8) we have $|\theta(\bar{p})| < \Delta(c) + t\,\Delta(l) + 4\,t\,\epsilon$. Hence, if

$$f(\theta(k), k_j)\,\bar{p}_i > \Delta(c) + t\,\Delta(l) + 4\,t\,\epsilon \tag{2.19}$$

then (2.18) is satisfied. Noting that $f(\theta(k), k_j) > f(1, k_M) = \sqrt{1 + 1/k_M} - 1$, if

$$\Delta(c) + t\,\Delta(l) + 4\,t\,\epsilon < \left(\sqrt{1 + 1/k_M} - 1\right)\,\bar{p}_i \tag{2.20}$$

then (2.19) is satisfied. By (2.8), we have $\bar{p}_i \geq t\,l_m - \Delta(c)/2 - 2\,t\,\epsilon$. Hence, if

$$\Delta(c) + t\,\Delta(l) + 4\,t\,\epsilon < \left(\sqrt{1 + 1/k_M} - 1\right)\,(t\,l_m - \Delta(c)/2 - 2\,t\,\epsilon) \tag{2.21}$$

then (2.20) is satisfied. Hence, if condition (2.21) is satisfied, then (2.16) is satisfied, $\pi_i^L > \pi_j^L$ for every firms F_i and F_j such that $k_i > k_j$, and, so, the network has the profit degree growth property. □

We are going to present an example satisfying the WB condition but not the DB condition. Furthermore, we will show that in this example does not has the profit degree growth property. Consider the Hotelling town network presented in Figure 2.2. The parameter values are $\epsilon_i = 0$, $c_i = 0$, $l_m = 5$,

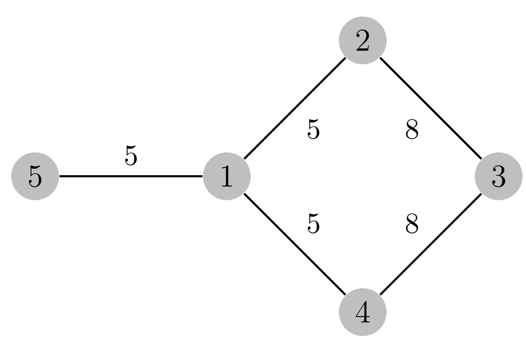


Figure 2.2: Network not satisfying the DB condition

$l_M = 8$, $\Delta(l) = 3$ and $k_M = 3$.

Hence, Network 2.2 satisfies the WB condition. Hence, by Proposition 2.1.1, there is a local optimum price strategy P^L. The profits valued at the local optimal prices are given by

$$\pi^L = t\,\left(\frac{48387}{1058}, \frac{21904}{529}, \frac{27556}{529}, \frac{21904}{529}, \frac{14641}{1058}\right).$$

Network 2.2 does not satisfy the DB condition and does not has the profit degree growth property, since $k_1 > k_3$ and $\pi_3^L > \pi_1^L$.

The two networks presented in Figure 2.1 satisfies the DB condition. Hence, both networks have the profit degree growth property.

2.1.2 Nash equilibrium price strategy

The price strategy $\mathbf{P}^*$ is a *best response* to the price strategy $\mathbf{P}$, if

$$(\tilde{p}_i - c_i)\, S(i, \tilde{\mathbf{P}}(i, \mathbf{P}, \mathbf{P}^*)) \geq (p_i' - c_i)\, S(i, \mathbf{P}_i'),$$

for all $i \in V$ and for all price strategies $\mathbf{P}_i'$ whose coordinates satisfy $p_i' \geq c_i$ and $p_j' = p_j$ for all $j \in V \setminus \{i\}$. A price strategy $\mathbf{P}^*$ is a Hotelling town *Nash equilibrium* if $\mathbf{P}^*$ is the best response to $\mathbf{P}^*$.

Lemma 2.1.3. *In a Hotelling town satisfying the WB condition, if there is a Nash price $\boldsymbol{P}^*$ then $\boldsymbol{P}^*$ is unique and $\boldsymbol{P}^* = \boldsymbol{P}^L$.*

Hence, the local optimum price strategy $\mathbf{P}^L$ is the only candidate to be a Nash equilibrium price strategy. However, $\mathbf{P}^L$ might not be a Nash equilibrium price strategy because there can be a firm F_i that by decreasing his price is able to absorb markets of other firms in such a way that increases its own profit. Therefore, the best response price strategy $\mathbf{P}^{L,*}$ to the local optimum price strategy $\mathbf{P}^L$ might be different from $\mathbf{P}^L$.

Proof of Lemma 2.1.3.
Suppose that P^* is a Nash price strategy and that $\mathbf{P}^* \neq \mathbf{P}^L$. Hence, $\mathbf{P}^*$ does not determine a local market structure, i.e., there exists $i \in V$ such that

$$M(i, \mathbf{P}^*) \not\subset \cup_{j \in N_i} R_{i,j}.$$

Hence, there exists $j \in N_i$ such that $M(j, \mathbf{P}^*) = 0$ and, therefore, $\pi_j^* = 0$.

Moreover, in this case, we have that

$$p_j^* > p_i^* + t\,\tilde{l}_{i,j}.$$

Consider, now, that F_j changes his price to $p_j = c_j + t\,\Delta(l) + 4\,t\,\epsilon$. Since $p_i^* > c_i$ and $c_j - c_i \leq \Delta(c)$ we have that

$$p_j - p_i^* < p_j - c_i = c_j + t\,\Delta(l) + 4\,t\,\epsilon - c_i \leq \Delta(c) + t\,\Delta(l) + 4\,t\,\epsilon$$

Since the Hotelling town satisfies the WB condition, we obtain

$$p_j - p_i^* < t\,l_m - 2\,t\,\epsilon \leq t\,l_{i,j} - 2\,t\,\epsilon \leq t\,\tilde{l}_{i,j}.$$

Hence, $M(j, \tilde{\mathbf{P}}(j, \mathbf{P}^*, \mathbf{P})) > 0$ and $\pi_j = (t\,\Delta(l) + 4\,t\,\epsilon)\,S(j, \tilde{\mathbf{P}}(j, \mathbf{P}^*, \mathbf{P})) > 0$. Therefore, F_j will change its price and so $\mathbf{P}^*$ is not a Nash equilibrium price strategy. Hence, if there is a Nash price $\mathbf{P}^*$ then $\mathbf{P}^* = \mathbf{P}^L$. □

Let $\cup_{j\in N_i} R_{i,j}$ be the *1-neighbourhood* $\mathcal{N}(i,1)$ of a firm $i \in V$. Let $\cup_{j\in N_i} \cup_{k\in N_j} R_{j,k}$ be the *2-neighbourhood* $\mathcal{N}(i,2)$ of a firm $i \in V$.

Lemma 2.1.4. *In a Hotelling town satisfying the WB condition,*

$$M(i, \tilde{\boldsymbol{P}}(i, \boldsymbol{P}^L, \boldsymbol{P}^{L,*})) \subset \mathcal{N}(i,2)$$

for every $i \in V$.

Hence, a consumer $x \in R_{j,k}$ might not buy in its local firms F_j and F_k. However, the consumer $x \in R_{j,k}$ still has to buy in a firm F_i that is a neighboring firm of its local firms F_j and F_k, i.e. $i \in N_j \cup N_k$.

Proof of Lemma 2.1.4.
By contradiction, let us consider a consumer $z \in M(i, \tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*}))$ and $z \notin \mathcal{N}(i,2)$. The price that consumer z pays to buy in firm F_i is given by

$$e = p_i + t\,\left(\tilde{l}_{i_1,i_2} + \tilde{l}_{i_2,i_3} + d\,(y_{i_3}, z)\right) \geq p_i + t\,\left(l_{i_1,i_2} + l_{i_2,i_3} - 2\,\epsilon + d\,(y_{i_3}, z)\right)$$

where $p_i = p_i^{L,*}$ is the coordinate of the vector $\tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*})$ and for the 2-path $(R_{i_1,i_2}, R_{i_2,i_3})$ with $i_1 = i$. If the consumer z buys at firm F_{i_3}, then the price that has to pay is

$$\tilde{e} = p_{i_3}^L + t\, d\,(y_{i_3}, z).$$

Since, by hypothesis, $z \in M(i, \tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*}))$, we have $e < \tilde{e}$. Therefore

$$p_i < p_{i_3}^L - t\; (l_{i_1,i_2} + l_{i_2,i_3} - 2\,\epsilon)\,.$$

By (2.8), $p_i^L \leq t\, l_M + 2\, t\, \epsilon + \frac{1}{2}(c_i + c_M)$ for all $i \in V$. Since $l_{i,j} \geq l_m$ for all $R_{i,j} \in E$,

$$p_i < t\, l_M + \frac{1}{2}(c_M + c_{i_3}) - 2\, t\, l_m + 4\, t\, \epsilon \leq c_M + t\, \Delta(l) - t\, l_m + 4\, t\, \epsilon.$$

Furthermore,

$$p_i - c_i < \Delta(c) + t\, \Delta(l) - t\, l_m + 4\, t\, \epsilon.$$

By the WB condition, $p_i - c_i < 0$. Hence, $\pi_i^{L,*} < 0$ which contradicts the fact that p_i is the best response to $\mathbf{P}^L$ (since $\pi_i^L > 0$). Therefore, $z \in \mathcal{N}(i, 2)$ and $M(i, \tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*})) \subset \mathcal{N}(i, 2)$. □

Definition 2.1.3. *A Hotelling town satisfies the* strong bounded length and costs (SB) *condition, if*

$$\Delta(c) + t\Delta(l) \leq \frac{(2\, t\, l_m - \Delta(c) - 4\, t\, \epsilon)^2}{8\, t\, k_M\, (l_M + \epsilon)} - 3\, t\, \epsilon.$$

The SB condition implies the WB condition, and so under the SB condition the only candidate to be a Nash equilibrium price strategy is the local optimum strategy price $\mathbf{P}^L$. On the other hand, the condition

$$\Delta(c) + t\, \Delta(l) \leq \frac{t\, l_M^2}{8\, k_M\, (l_M + \epsilon)} - 3\, t\, \epsilon.$$

together with the

$$WB$$

condition implies the SB condition. Hence, we note that the condition

$$\Delta(c) + t\,\Delta(l) \leq \frac{t\,l_m^2}{8\,k_M\,(l_M + \epsilon)} - 6\,t\,\epsilon.$$

implies the WB and SB conditions.

Theorem 2.1.2. *If a Hotelling town satisfies the SB condition then there is a unique Hotelling town Nash equilibrium price strategy $\boldsymbol{P}^* = \boldsymbol{P}^L$.*

Hence, the Nash equilibrium price strategy for the Hotelling town satisfying the SB condition determines a local market structure, i.e. every consumer located at $x \in R_{i,j}$ spends less by shopping at his local firms F_i or F_j than in any other firm in the town and so the consumer at x will buy either at his local firm F_i or at his local firm F_j.

For ϵ small enough, a *cost and length uniform* Hotelling town, i.e. $c_m = c_M$ and $l_m = l_M$, has a unique pure network Nash price strategy which satisfies the profit degree growth property.

Corollary 2.1.2. *Consider a Hotelling town where all firms are located at the nodes. If*

$$\Delta(c) + t\Delta(l) \leq \frac{(2\,t\,l_m - \Delta(c))^2}{8\,t\,k_M\,l_M}$$

then there is a unique Hotelling town Nash equilibrium price strategy $\boldsymbol{P}^* = \boldsymbol{P}^L$.

Proof of Theorem 2.1.2.
By Proposition 2.1.1 and Lemma 2.1.3, if there is a Nash equilibrium price strategy $\mathbf{P}^*$ then $\mathbf{P}^*$ is unique and $\mathbf{P}^* = \mathbf{P}^L$.

We note that if $M(i, \tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*})) \subset \mathcal{N}(i,1)$ for every $i \in V$ then $\tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*}) = p_i^L$ and so $\mathbf{P}^L$ is a Nash equilibrium.

By Lemma 2.1.4, we have that $M(i, \tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*})) \subset \mathcal{N}(i, 2)$ for every $i \in V$. Now, we will prove that the SB condition implies that firm F_i earns more competing only in the 1-neighborhood than competing in a 2-neighborhood. By Proposition 2.1.1,

$$\pi_i^L \geq (2\,t)^{-1}\, k_i\, (t\, l_m - \Delta(c)/2 - 2\, t\, \epsilon)^2 \tag{2.22}$$

By Lemma 2.1.4,

$$\begin{aligned} \pi_i(\tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*}), \mathbf{C}) &\leq (p_i - c_i) \sum_{j \in N_i} \left(\tilde{l}_{i,j} + \sum_{k \in N_j \setminus \{i\}} \tilde{l}_{j,k} \right) \\ &\leq (p_i - c_i) \sum_{j \in N_i} \sum_{k \in N_j} \tilde{l}_{j,k}, \end{aligned}$$

where $p_i = p_i^{L,*}$ is the coordinate of the vector $\tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*})$. Hence,

$$\pi_i(\tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*}), \mathbf{C}) \leq (p_i - c_i) \sum_{j \in N_i} \sum_{k \in N_j} (l_{j,k} + \epsilon) \leq (p_i - c_i) k_i\, k_M\, (l_M + \epsilon). \tag{2.23}$$

By contradiction, let us consider a consumer $z \in M(i, \tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*}))$ and $z \notin \mathcal{N}(i, 1)$. Let $i_2 \in N_i$ be the vertex such that $z \in \mathcal{N}(i_2, 1)$. The price that consumer z pays to buy in firm F_i is given by

$$e = p_i + t\, \tilde{l}_{i,i_2} + t\, d\,(y_{i_2}, z) \geq p_i + t\, l_{i,i_2} + t\, d\,(y_{i_2}, z) - t\, \epsilon.$$

If the consumer y buys at firm F_{i_2}, then the price that has to pay is

$$\tilde{e} = p_{i_2}^L + t\, d\,(y_{i_2}, z).$$

Since, by hypothesis, $z \in M(i, \tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*}))$, we have $e < \tilde{e}$. Therefore

$$p_i < p_{i_2}^L - t\, l_{i,i_2} + t\, \epsilon.$$

By (2.8), $p_{i_2}^L \leq t\,l_M + 2\,t\,\epsilon + \dfrac{1}{2}(c_M + c_{i_2})$. Since $l_{i,i_2} \geq l_m$, we have

$$p_i < t\,l_M + \frac{1}{2}(c_M + c_{i_2}) + 2\,t\,\epsilon - t\,l_m + t\,\epsilon \leq c_M + t\,\Delta(l) + 3\,t\,\epsilon.$$

Thus,

$$p_i - c_i < \Delta(c) + t\,\Delta(l) + 3\,t\,\epsilon.$$

Hence, from (2.23) we obtain

$$\pi_i(\tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*}), \mathbf{C}) < k_i\,k_M\,(l_M + \epsilon)\;(\Delta(c) + t\,\Delta(l) + 3\,t\,\epsilon)\,.$$

By the SB condition,

$$\pi_i(\tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*}), \mathbf{C}) < (2\,t)^{-1}\,k_i\,(t\,l_m - \Delta(c)/2 - 2\,t\,\epsilon)^2. \tag{2.24}$$

Hence, by inequalities (2.22) and (2.24), $\pi_i^L > \pi_i(\tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*}), \mathbf{C})$, which contradicts the fact that p_i is the best response to $\mathbf{P}^L$. Therefore, $z \in \mathcal{N}(i, 1)$ and $M(i, \tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*})) \subset \mathcal{N}(i, 1)$. Hence, $\tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*}) = p_i^L$ and so $\mathbf{P}^L$ is a Nash equilibrium. □

We are going to present an example satisfying the WB condition but not the SB condition. Furthermore, we will show that in this example the local optimum price strategy do not form a Nash price equilibrium. Consider the Hotelling town network presented in figure 2.3.

The parameter values are $\epsilon_i = 0$, $c_i = 0$, $l_m = 4$, $l_M = 7$, $\Delta(l) = 3$ and $k_M = 3$. Hence, Network 2.3 satisfies the WB condition. By Proposition 2.1.1, the local optimal equilibrium prices and the correspondent profits are

$$\mathbf{P}^L = t\,\left(\frac{16}{3}, \frac{14}{3}, \frac{31}{6}, \frac{37}{6}\right); \quad \pi^L = t\,\left(\frac{128}{3}, \frac{98}{9}, \frac{961}{72}, \frac{1369}{72}\right).$$

We will show that the local optimum price strategy is not a Nash equilibrium. The profits of the firms are given by $\pi_i^L = p_i\,S(i, \mathbf{P}^L)$, and the local market

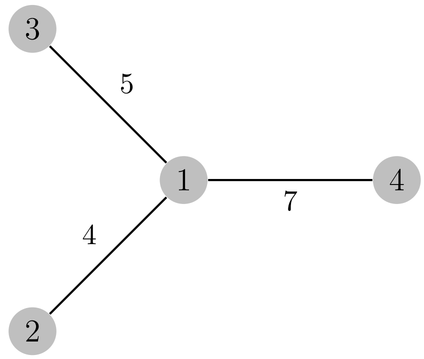


Figure 2.3: Star Network not satisfying the SB condition

sizes $S(i, \mathbf{P}^L)$ are

$$S(i, \mathbf{P}^L) = \frac{\pi_i^L}{p_i^L} = \frac{k_i\, p_i^L}{2\, t}$$

Hence, the local market sizes are

$$S(1, \mathbf{P}^L) = 8; \quad S(2, \mathbf{P}^L) = \frac{14}{6}; \quad S(3, \mathbf{P}^L) = \frac{31}{12}; \quad S(4, \mathbf{P}^L) = \frac{37}{12}.$$

Suppose that firm F_2 decides to lower its price in order to capture the market of firm F_1. The firm F_2 captures the market of F_1, excluding F_1 from the game, if the firm F_2 charges a price p_2 such that $p_2 + 4\,t < p_1^L$ or, equivalently $p_2 < 4/3\,t$. Let us consider $p_2 = 4/3\,t - \delta$, where δ is sufficiently small. Hence, for this new price, firm F_2 keeps the market $M(2, \mathbf{P}^L)$ and, since the price of F_2 at location of F_1 is less that p_1^L, firm F_2 gains at least the market of firm F_1. Thus, the new market $M(2, \mathbf{P})$ of firm F_2 is such that $S(2, \mathbf{P}) > S(1, \mathbf{P}^L) + S(2, \mathbf{P}^L)$. Therefore, $S(2, \mathbf{P}) > 31/3$ and so

$$\pi_2 > p_2\, S(2, \mathbf{P}) = \left(\frac{4}{3}\,t - \delta\right)\frac{31}{3} = \frac{124}{9}\,t - \frac{31}{3}\,\delta.$$

Thus $\pi_2 > 98\,t/9 = \pi_2^L$, and so firm F_2 prefers to alter its price p_2^L. Therefore, $\mathbf{P}^L$ is not a Nash equilibrium price.

The two networks presented in figure 2.1 satisfies the SB condition.

Hence, the local optimum price strategy $\mathbf{P}^L$ is also a Nash equilibrium price strategy.

2.1.3 Strategic optimal location

The marginal rate of the price of a firm F_i located at y_i with respect to the deviation of the localization of the firm is given by

$$\begin{aligned}\partial p_i^L/\partial \epsilon_i &= t\left(Q_{i,i}\,\partial Y_i/\partial \epsilon_i + \sum_{j\in N_i} Q_{i,j}\,\partial Y_j/\partial \epsilon_i\right)\\ &= t\left(Q_{i,i}\,\frac{2-k_i}{k_i} - \frac{2\,Q_{i,j(i)}}{k_{j(i)}} + \sum_{j\in N_i}\frac{Q_{i,j}}{k_j}\right).\end{aligned}$$

The marginal rate of the profit of a firm F_i located at y_i with respect to the deviation of the localization of the firm is given by

$$\partial \pi_i^L/\partial \epsilon_i = \frac{k_i\,(p_i^L - c_i)}{t}\cdot \partial p_i^L/\partial \epsilon_i.$$

Definition 2.1.4. *Let us explicit $\pi(\epsilon_i)$ the dependence of π_i on the parameter ϵ_i. We say that a firm F_i is* node local stable *if there is $\epsilon > 0$ such that $\pi_i(0) > \pi_i(\epsilon_i)$ for every $0 < \epsilon_i < \epsilon$, with respect to the local optimum price strategy. A Hotelling network is* firm node local stable *if every firm in the network is node stable.*

If node i has degree $k_i = 2$, let us define

$$U_i = \frac{Q_{i,v}}{k_v} - \frac{Q_{i,j(i)}}{k_{j(i)}}$$

where v is uniquely determined by $\{v\} = N_i \setminus \{j(i)\}$ and $j(i)$ is the node with the property that y_i is at the road $R_{i,j(i)}$.

Theorem 2.1.3. *The marginal rate of the profit of a firm F_i located at y_i with* *respect to the deviation of the localization of the firm satisfies the following inequalities:*

(i) *Case* $k_i \geq 1$. *Then* $\partial \pi_i^L / \partial \epsilon_i > 0$.

(ii) *Case* $k_i = 2$.

(a) *If* $U_i > 0$ *then* $\partial \pi_i^L / \partial \epsilon_i > 0$;

(b) *if* $U_i < 0$ *then* $\partial \pi_i^L / \partial \epsilon_i < 0$;

(c) *if* $U_i = 0$ *then* $\partial \pi_i^L / \partial \epsilon_i = 0$.

(iii) *Case* $k_i \geq 3$ *and* $k_v \geq 3$, *for every* $v \in N_i$. *Then* $\partial \pi_i^L / \partial \epsilon_i < 0$.

(iv) *Case* $k_i \geq 4$ *and* $k_v \geq 2$, *for every* $v \in N_i$. *Then* $\partial \pi_i^L / \partial \epsilon_i < 0$.

Hence, a Hotelling town network satisfying the WB condition and with $k_m \geq 3$ has a local optimal localization strategy, whereby every firm F_i is located at the corresponding node i.

We observe that firms F_i with node degree $k_i = 1$ are node local unstable. Firms F_i with $k_i = 2$ are node local unstable, except for networks satisfying special symmetric properties. Firms F_i with $k_i = 3$ whose neighboring firms have nodes degree greater or equal to 3 are node local stable. Furthermore, firms F_i with $k_i \geq 4$ whose neighboring firms have nodes degree greater or equal to 2 are node local stable.

Proof of Theorem 2.1.3.
From Theorem 2.1.2, we have

$$p_i^L = \sum_{v \in V} Q_{i,v} (c_v + t\, L_v + t\, Y_v), \tag{2.25}$$

and

$$\pi_i^L = (2\, t)^{-1}\, k_i\, (p_i^L - c_i)^2.$$

Hence,

$$\partial \pi_i^L / \partial \epsilon_i = \frac{k_i \left(p_i^L - c_i\right)}{t} \cdot \partial p_i^L / \partial \epsilon_i.$$

Hence, to study the influence of ϵ_i in the profit π_i^L, we only have to study the signal of $\partial p_i^L / \partial \epsilon_i$. By (2.25), we have

$$\partial p_i^L / \partial \epsilon_i = \sum_{v \in V} \partial p_i^L / \partial Y_v \cdot \partial Y_v / \partial \epsilon_i.$$

Since, for every $v \in V$, $\partial p_i^L / \partial Y_v = t\, Q_{i,v}$, we have

$$\partial p_i^L / \partial \epsilon_i = t \sum_{v \in V} Q_{i,v}\, \partial Y_v / \partial \epsilon_i.$$

Recall that

$$Y_v = \frac{1}{k_v} \left(\sum_{j \in N_v} s_{v,j}(j) - \epsilon_v \left(k_v - 2\right) \right)$$

Hence, for $v = i$, we have

$$\partial Y_i / \partial \epsilon_i = \frac{2 - k_i}{k_i};$$

for $v \in N_i$, we have

$$\partial Y_v / \partial \epsilon_i = \partial / \partial \epsilon_i \left(\frac{1}{k_v} s_{v,i}(i) \right) = \pm \frac{1}{k_v};$$

and for $v \notin N_i$, we have $\partial Y_i / \partial \epsilon_i = 0$. Therefore,

$$\partial p_i^L / \partial \epsilon_i = t \left(Q_{i,i} \frac{2 - k_i}{k_i} - \frac{2\, Q_{i,j(i)}}{k_{j(i)}} + \sum_{j \in N_i} \frac{Q_{i,j}}{k_j} \right)$$

If $k_i = 1$, then

$$\partial p_i^L / \partial \epsilon_i = t\, Q_{i,i} > 0.$$

If $k_i = 2$, then

$$\partial p_i^L / \partial \epsilon_i = t \left(\frac{Q_{i,j}}{k_j} - \frac{Q_{i,j(i)}}{k_{j(i)}} \right) = t\, U_i$$

where $j \in N_i$ and $j \neq j(i)$. If $k_i \geq 3$, then

$$\partial p_i^L / \partial \epsilon_i \leq t \left(Q_{i,i} \frac{2 - k_i}{k_i} + \sum_{j \in N_i} Q_{i,j} \frac{1}{k_j} \right)$$

By construction, $Q_{i,i} > 1/2$ and $\sum_{j \in N_i} Q_{i,j} < 1/2$. Hence, if $k_v \geq 3$, for every $v \in N_i$, then

$$\partial p_i^L / \partial \epsilon_i < t \left(\frac{-1}{6} + \frac{1}{6} \right) = 0.$$

Furthermore, if $k_i \geq 4$ and $k_v \geq 2$, for every $v \in N_i$, then

$$\partial p_i^L / \partial \epsilon_i < t \left(\frac{-1}{4} + \frac{1}{4} \right) = 0.$$

□

2.1.4 Space bounded information

Given $m + 1$ vertices $x_0, \ldots, x_m$ with the property that there are roads $R_{x_0,x_1}, \ldots, R_{x_{m-1},x_m}$ the *(ordered) m path R* is

$$R = (R_{x_0,x_1}, \ldots, R_{x_{m-1},x_m}).$$

Let $\mathcal{R}(i, j; m)$ be the set of all m (ordered) paths $R = (R_{x_0,x_1}, \ldots, R_{x_{m-1},x_m})$ starting at $i = x_0$ and ending at $j = x_m$. Given a m order path $R = (R_{x_0,x_1}, \ldots, R_{x_{m-1},x_m})$, the corresponding *weight* is

$$k(R) = \prod_{q=0}^{m-1} k_{x_q,x_{q+1}}.$$

The matrix $\mathbf{K}^0$ is the identity matrix and, for $n \geq 1$, the elements of the matrix $\mathbf{K}^m$ are

$$k_{i,j}^m = \sum_{R \in \mathcal{R}(i,j;m)} k(R).$$

Definition 2.1.5. *A Hotelling town has* n space bounded information (n-I) *if for every* $1 \leq m \leq n$*, for every firm* F_i *and for every non-empty set* $\mathcal{R}(i,j;m)$*: (i) firm* F_i *knows the cost* c_j *and the average length road* L_j *and the firm deviation* Y_j *of firm* F_j*; (ii) for every* m *path* $R \in \mathcal{R}(i,j;m)$*, firm* F_i *knows the corresponding weight* $k(R)$*.*

The n *local optimal price vector* is

$$\mathbf{P}(n) = \sum_{m=0}^{n} 2^{-(m+1)}\, \mathbf{K}^m \left(\mathbf{C} + t\left(\mathbf{L} + \mathbf{Y}\right)\right).$$

We observe that in a n-I Hotelling town, the firms might not be able to compute $\mathbf{K}$, $\mathbf{C}$, $\mathbf{L}$ or $\mathbf{Y}$. However, every firm F_i is able to compute his n local optimal price $p_i(n)$

$$p_i(n) = \sum_{m=0}^{n} 2^{-(m+1)} \sum_{v \in V} k_{i,v}^m \left(c_v + t\left(L_v + Y_v\right)\right).$$

By (2.5), the best response $\mathbf{P}'$ to $\mathbf{P}(n)$ is given by

$$\begin{aligned}
\mathbf{P}' &= \frac{1}{2}\left(\mathbf{C} + t\left(\mathbf{L} + \mathbf{Y}\right)\right) + \frac{1}{2}\mathbf{K}\,\mathbf{P}(n) \\
&= \frac{1}{2}\left(\mathbf{C} + t\left(\mathbf{L} + \mathbf{Y}\right)\right) + \sum_{m=0}^{n} 2^{-(m+2)}\mathbf{K}^{m+1}\left(\mathbf{C} + t\left(\mathbf{L} + \mathbf{Y}\right)\right) \\
&= \sum_{m=0}^{n+1} 2^{-(m+1)}\mathbf{K}^m\left(\mathbf{C} + t\left(\mathbf{L} + \mathbf{Y}\right)\right) = \mathbf{P}(n+1).
\end{aligned}$$

Hence, $\mathbf{P}(n+1)$ is the best response to $\mathbf{P}(n)$ for n sufficiently large.

Let G denote the number of nodes in the network and let $e = \Delta(c) +$

$3\,t\,(l_M + 2\,\epsilon)$.

Theorem 2.1.4. *A Hotelling town satisfying the WB condition has a local optimum price strategy $\boldsymbol{P}^L$ that is well approximated by the n local optimal price $\boldsymbol{P}(n)$ with the following 2^{-n} bound*

$$0 \le p_i^L - p_i(n) \le 2^{-(n+1)} G\,(c_M + t\,(l_M + 2\,\epsilon)).$$

The profit $\pi_i(\boldsymbol{P}^L)$ is well approximated by $\pi_i(\boldsymbol{P}(n))$ with the following bound

$$|\pi_i(\boldsymbol{P}^L) - \pi_i(\boldsymbol{P}(n))| \le 2^{-(n+2)}\, G\, t^{-1}\,(c_M + t\,(l_M + 2\,\epsilon))\,\left(k_i\, e + 4\, t\, \epsilon\right).$$

Proof. By Proposition 2.1.1, if a Hotelling town satisfies the WB condition then there is local optimum price strategy $\mathbf{P}^L$ given by

$$\mathbf{P}^L = \sum_{m=0}^{\infty} 2^{-(m+1)} \mathbf{K}^m\,(\mathbf{C} + t\,(\mathbf{L} + \mathbf{Y}))\,.$$

Considering $\mathbf{Q} = \sum_{m=0}^{\infty} 2^{-(m+1)} \mathbf{K}^m$, we can write the equilibrium prices as

$$p_i^L = \sum_{v \in V} Q_{i,v}\,(c_v + t\,(L_v + Y_v)), \qquad \text{where} \qquad Q_{i,v} = \sum_{m=0}^{\infty} 2^{-(m+1)} k_{i,v}^m.$$

For the space bounded information Hotelling town, the n local optimal price $\mathbf{P}(n)$ is given by

$$\mathbf{P}(n) = \sum_{m=0}^{n} 2^{-(m+1)} \mathbf{K}^m\,(\mathbf{C} + t\,(\mathbf{L} + \mathbf{Y}))$$

and

$$p_i(n) = \sum_{v \in V} Q_{i,v}(n)\,(c_v + t\,(L_v + Y_v)), \qquad \text{where} \qquad Q_{i,v}(n) = \sum_{m=0}^{n} 2^{-(m+1)} k_{i,v}^m.$$

The difference $R_i(n)$ between p_i^L and $p_i(n)$ is positive and is given by

$$R_i(n) = \sum_{v \in V} (Q_{i,v} - Q_{i,v}(n))\ (c_v + t\,(L_v + Y_v)).$$

We note that

$$Q_{i,v} - Q_{i,v}(n) = \sum_{m=n+1}^{\infty} 2^{-(m+1)} k_{i,v}^m.$$

Since $0 \le k_{i,v}^m \le 1$, for all $m \in \mathbb{N}$ and all $i, v \in V$ and

$$\sum_{m=n+1}^{\infty} 2^{-(m+1)} = 2^{-(n+1)},$$

we have that

$$Q_{i,v} - Q_{i,v}(n) \le 2^{-(n+1)}.$$

Hence,

$$R_i(n) \le \sum_{v \in V} 2^{-(n+1)}\,(c_v + t\,(L_v + Y_v)).$$

Since $L_v \le l_M$, $Y_v \le 2\,\epsilon$ and $c_v \le c_M$, we have that

$$R_i(n) \le 2^{-(n+1)}\,G\,(c_M + t\,(l_M + 2\,\epsilon)). \tag{2.26}$$

Therefore,

$$0 \le p_i^L - p_i(n) \le 2^{-(n+1)}\,G\,(c_M + t\,(l_M + 2\,\epsilon)).$$

The profit for firm F_i for the local optimal price is given by

$$\pi_i(\mathbf{P}^L) = (2t)^{-1}\,(p_i^L - c_i)\left(\sum_{j \in N_i}(p_j^L - p_i^L + t\,\tilde{l}_{i,j}) - 2\,t\,(k_i - 2)\,\epsilon_i\right) \tag{2.27}$$

and the profit for firm F_i when all firms have n-space bounded information is

$$\pi_i(\mathbf{P}(n)) = (2t)^{-1}\,(p_i(n) - c_i)\left(\sum_{j\in N_i}(p_j(n) - p_i(n) + t\,\tilde{l}_{i,j}) - 2\,t\,(k_i - 2)\,\epsilon_i\right)$$

Let $R_{j,i}(n) = R_j(n) - R_i(n)$ and

$$\begin{aligned} Z_i &= \sum_{j\in N_i}(p_j(n) - p_i(n) + t\,\tilde{l}_{i,j} + R_{j,i}(n)) - 2\,t\,(k_i - 2)\,\epsilon_i \\ &= \sum_{j\in N_i}(p_j^L - p_i^L + t\,\tilde{l}_{i,j}) - 2\,t\,(k_i - 2)\,\epsilon_i. \end{aligned}$$

Since $p_i^L = p_i(n) + R_i(n)$, we can write the local equilibrium profit (2.27) for firm i as

$$\pi_i(\mathbf{P}^L) = (2t)^{-1}\,(p_i(n) - c_i + R_i(n))\,Z_i$$

Hence,

$$\pi_i(\mathbf{P}^L) = \pi_i(\mathbf{P}(n)) + (2t)^{-1}\left((p_i(n) - c_i)\sum_{j\in N_i} R_{j,i}(n) + R_i(n)\,Z_i\right)$$

The difference between the equilibrium profit and the profit where all firms have n-space bounded information is

$$\pi_i(\mathbf{P}^L) - \pi_i(\mathbf{P}(n)) = (2t)^{-1}\left((p_i(n) - c_i)\sum_{j\in N_i} R_{j,i}(n) + R_i(n)\,Z_i\right).$$

Hence,

$$|\pi_i(\mathbf{P}^L) - \pi_i(\mathbf{P}(n))| \le (2t)^{-1}\left((p_i(n) - c_i)\sum_{j\in N_i} |R_{j,i}(n)| + R_i(n)\,Z_i\right).$$

Since $p_j^L - p_i^L + t\,\tilde{l}_{i,j} \le 2\,t\,\tilde{l}_{i,j} \le 2\,t\,(l_M + 2\,\epsilon)$, we have

$$Z_i \le 2\,t\,k_i\,(l_M + 2\,\epsilon) - 2\,t\,(k_i - 2)\,\epsilon < 2\,t\,(k_i\,(l_M + 2\,\epsilon) + 2\,\epsilon_i).$$

Let $Z = \Delta(c) + t\,(l_M + 2\,\epsilon)$. Since $p_i(n) - c_i \le p_i^L - c_i$, from (2.8) we have $p_i(n) - c_i \le \Delta(c) + t\,(l_M + 2\,\epsilon) = Z$. Hence,

$$|\pi_i(\mathbf{P}^L) - \pi_i(\mathbf{P}(n))| < (2t)^{-1} \left(Z \sum_{j \in N_i} |R_{j,i}(n)| + 2\,t\,R_i(n)\,(k_i\,(l_M + 2\,\epsilon) + 2\,\epsilon) \right)$$

Let $Z_M = c_M + t\,(l_M + 2\,\epsilon)$. By (2.26), $R_i(n) \le 2^{-(n+1)}\,G\,Z_M$. Then, also, $|R_{j,i}(n)| \le 2^{-(n+1)}\,G\,Z_M$. Therefore,

$$\sum_{j \in N_i} |R_{j,i}(n)| \le 2^{-(n+1)}\,k_i\,G\,Z_M.$$

Hence,

$$|\pi_i(\mathbf{P}^L) - \pi_i(\mathbf{P}(n))| \le 2^{-(n+2)}\,G\,t^{-1}\,Z_M\ \left(k_i\,(\Delta(c) + 3\,t\,(l_M + 2\,\epsilon)) + 4\,t\,\epsilon\right).$$

□

2.1.5 Static Analysis

For simplicity of notation, in this subsection, we assume that the firms are located at the nodes of the network. Let s be the gross consumer surplus, i.e., the maximum consumer willingness to pay for the commodity. Let us assume that the market is covered, i.e., s is sufficiently large for all consumers to be willing to buy. The utility for each consumer x is given by

$$U_x = s - p - t\,d(x)$$

where p is the price to pay and $d(x)$ is the distance between x and the location of the firm where it buys. Since the consumers with the lowest utility are the indifferent consumers, we may say that the market is covered if the indifferent consumer buys. Hence, if $\mathbf{P}$ is a local price strategy then the market is covered if for every road $R_{i,j}$

$$s - p_i - \frac{p_j - p_i + t\, l_{i,j}}{2} \geq 0.$$

Thus, the market is covered if

$$s \geq \frac{p_i + p_j + t\, l_{i,j}}{2}. \tag{2.28}$$

Let us define $s_{i,j} = s - \frac{1}{2}\,(p_i + p_j + t\, l_{i,j})$. We note that $s_{i,j} \geq 0$.
Recall that the Hotelling town admissible market size $\mathbf{L}$ is the vector whose coordinates are

$$L_i = k_i^{-1} \sum_{j \in N_i} l_{i,j}.$$

Let $a \in V$, $j \in N_i$ and $b \in V \setminus \{i, N_i\}$. Hence, $\partial L_i/\partial c_a = 0$, $\partial L_i/\partial t = 0$ and

$$\partial L_i/\partial l_{i,j} = k_i^{-1}, \quad \partial L_j/\partial l_{i,j} = k_j^{-1} \quad \text{and} \quad \partial L_b/\partial l_{i,j} = 0. \tag{2.29}$$

Similarly, we have

$$\partial l_{i,j}/\partial L_i = k_i, \quad \partial l_{i,j}/\partial L_j = k_j \quad \text{and} \quad \partial l_{i,j}/\partial L_b = 0. \tag{2.30}$$

By Proposition 2.1.1, if a Hotelling town satisfies the SB condition then the unique Hotelling town Nash equilibrium price for firm F_i is given by

$$p_i^* = \sum_{v \in V} Q_{i,v}\,(c_v + t\, L_v), \quad \text{where} \quad Q_{i,v} = \sum_{m=0}^{\infty} 2^{-(m+1)} k_{i,v}^m.$$

Let us define $A_i(r, s) = k_r^{-1}\, Q_{i,r} + k_s^{-1}\, Q_{i,s}$.

Corollary 2.1.3. *If a Hotelling town satisfies the SB condition, equilibrium prices are increasing in production costs, admissible local firm market sizes, transportation cost and road lengths.*

Proof. Let $a \in V$. Hence,

$$\partial p_i^*/\partial c_a = Q_{i,a} = \sum_{m=0}^{\infty} 2^{-(m+1)}\, k_{i,a}^m > 0, \tag{2.31}$$

$$\partial p_i^*/\partial L_a = t\, Q_{i,a} = t \sum_{m=0}^{\infty} 2^{-(m+1)}\, k_{i,a}^m > 0 \tag{2.32}$$

and

$$\partial p_i^*/\partial t = \sum_{v \in V} Q_{i,v} L_v > 0. \tag{2.33}$$

Let $R_{r,s} \in E$. Since $\partial L_v/\partial l_{r,s} = 0$, for $v \neq r$ and $v \neq s$, from (2.29) and (2.32), we have

$$\begin{aligned}
\partial p_i^*/\partial l_{r,s} &= \sum_{v \in V} \partial p_i^*/\partial L_v \cdot \partial L_v/\partial l_{r,s} \\
&= \partial p_i^*/\partial L_r \cdot \partial L_r/\partial l_{r,s} + \partial p_i^*/\partial L_s \cdot \partial L_s/\partial l_{r,s}. \\
&= t\,(k_r^{-1}\, Q_{i,r} + k_s^{-1}\, Q_{i,s}) = t\, A_i(r,s) > 0 \qquad (2.34)
\end{aligned}$$

From (2.31), (2.32), (2.33) and (2.34), prices are increasing in production costs, admissible local firm market sizes, transportation cost and road lengths. □

Corollary 2.1.4. *If a Hotelling town satisfies the SB condition, equilibrium profits are decreasing in his own production cost and increasing in production costs of other firms, admissible local firm market sizes, transportation costs and road lengths.*

Proof. From Proposition 2.1.1, if F_i is a firm located at a node of degree k_i, his profit in equilibrium is given by

$$\pi_i^* = (2t)^{-1}\, k_i\, (p_i^* - c_i)^2.$$

Hence,

$$\partial \pi_i^* / \partial p_i^* = k_i\, t^{-1}\, (p_i^* - c_i). \tag{2.35}$$

Let $a \in V \setminus \{i\}$. From (2.35) and (2.31), we get

$$\partial \pi_i^* / \partial c_a = \partial \pi_i^* / \partial p_i^* \cdot \partial p_i^* / \partial c_a = k_i\, t^{-1}\, (p_i^* - c_i)\, Q_{i,a} > 0.$$

Similarly,

$$\partial \pi_i^* / \partial c_i = k_i\, t^{-1}\, (p_i^* - c_i)\, (\partial p_i^* / \partial c_i - 1) = k_i\, t^{-1}\, (p_i^* - c_i)\, (Q_{i,i} - 1)\,.$$

Since $Q_{i,i} < 1$, $\partial \pi_i^* / \partial c_i < 0$. Hence, profits are increasing in production costs of other firms and are decreasing in own production cost.
Let $b \in V$. From (2.35) and (2.32), we get

$$\partial \pi_i^* / \partial L_b = \partial \pi_i^* / \partial p_i^* \cdot \partial p_i^* / \partial L_b = k_i\, (p_i^* - c_i)\, Q_{i,b} > 0$$

and profits are increasing in admissible local firm market sizes.
From (2.35) and (2.33), we get

$$\begin{aligned}
\partial \pi_i^* / \partial t &= \partial \pi_i^* / \partial p_i^* \cdot \partial p_i^* / \partial t - \frac{k_i}{2}\, t^{-2}\, (p_i^* - c_i)^2 \\
&= k_i\, t^{-1}\, (p_i^* - c_i) \sum_{v \in V} Q_{i,v} L_v - \frac{k_i}{2}\, t^{-2}\, (p_i^* - c_i)^2 \\
&= k_i\, t^{-2}\, (p_i^* - c_i) \left(t \sum_{v \in V} Q_{i,v} L_v - \frac{1}{2}\, (p_i^* - c_i) \right).
\end{aligned}$$

Then, $\partial\pi_i^*/\partial t > 0$ if and only if

$$c_i > p_i^* - 2\,t \sum_{v \in V} Q_{i,v}\,L_v = \sum_{v \in V} Q_{i,v}\,(c_v - t\,L_v).$$

Since Q is stochastic, $c_i = \sum_{v \in V} Q_{i,v}\,c_i$, and $\partial\pi_i^*/\partial t > 0$ if and only if

$$\sum_{v \in V} Q_{i,v}\,(c_i - c_v + t\,L_v) > 0. \tag{2.36}$$

Since $L_v \geq l_m$, then $c_i - c_v + t\,L_v \geq c_m - c_M + t\,l_m = t\,l_m - \Delta(c)$. By the WB condition, $c_i - c_v + t\,L_v > 0$. Since $\mathbf{Q}$ is a non-negative matrix, condition (2.36) holds and $\partial\pi_i^*/\partial t > 0$. Hence, profits are increasing in transportation cost.

Let $R_{r,s} \in E$. From (2.35) and (2.34)

$$\begin{aligned} \partial\pi_i^*/\partial l_{r,s} &= \partial\pi_i^*/\partial p_i^* \cdot \partial p_i^*/\partial l_{r,s} = k_i\,t^{-1}\,(p_i^* - c_i)\,t\,A_i(r,s) \\ &= k_i\,(p_i^* - c_i)\,A_i(r,s) > 0. \end{aligned}$$

Hence, profits are increasing in road lengths. □

The *road consumer surplus* $CS_{i,j}(\mathbf{P})$ for the road $R_{i,j}$ is the integral of the difference $s - E(x;\mathbf{P})$ between the valuation s of the consumers for the commodity and the expenditure $E(x;\mathbf{P})$ for all the consumers living in the road $R_{i,j}$. Then,

$$\begin{aligned} CS_{i,j}^* &= CS_{i,j}(\mathbf{P}^*) = \int_0^{x_{i,j}^*} s - p_i^* - t\,x\,\mathrm{d}x + \int_{x_{i,j}^*}^{l_{i,j}} s - p_j^* - t\,(l_{i,j} - x)\,\mathrm{d}x \\ &= s\,l_{i,j} + t\,(x_{i,j}^*)^2 - \frac{t}{2} l_{i,j}^2 - p_j^*\,l_{i,j} \\ &= s\,l_{i,j} + (4\,t)^{-1}(p_j^* - p_i^* + t\,l_{i,j})^2 - \frac{t}{2} l_{i,j}^2 - p_j^*\,l_{i,j}. \end{aligned} \tag{2.37}$$

Corollary 2.1.5. *If a Hotelling town satisfies the SB condition, the road consumer surplus on road $R_{i,j}$, $CS_{i,j}(\boldsymbol{P}^*)$, is decreasing in production costs,* *decreasing in other road lengths, increasing in own length, decreasing in transportation costs, increasing in admissible local firm market sizes L_i and L_j and decreasing in other admissible local firm market sizes.*

Proof. From (2.37), we have

$$\partial CS^*_{i,j}/\partial p^*_i = -(2t)^{-1}(p^*_j - p^*_i + t\, l_{i,j}) = -x^*_{i,j} \tag{2.38}$$

and

$$\partial CS^*_{i,j}/\partial p^*_j = (2\, t)^{-1}(p^*_j - p^*_i + t\, l_{i,j}) - l_{i,j} = x^*_{i,j} - l_{i,j}. \tag{2.39}$$

Since $0 < x^*_{i,j} < l_{i,j}$, $\partial CS^*_{i,j}/\partial p^*_i < 0$ and $\partial CS^*_{i,j}/\partial p^*_j < 0$.
Let $a \in V$. Hence, from (2.38), (2.39) and (2.31) and

$$\begin{aligned} \partial CS^*_{i,j}/\partial c_a &= \partial CS^*_{i,j}/\partial p^*_i \cdot \partial p^*_i/\partial c_a + \partial CS^*_{i,j}/\partial p^*_j \cdot \partial p^*_j/\partial c_a \\ &= -x^*_{i,j}\, Q_{i,a} + (x^*_{i,j} - l_{i,j})\, Q_{j,a}. \end{aligned}$$

Since $\mathbf{Q}$ is a non-negative matrix, $\partial CS^*_{i,j}/\partial c_a < 0$ and road consumer surplus on road $R_{i,j}$ is decreasing in production costs.
Let $b \in V \setminus \{i, j\}$. Hence, from (2.38), (2.39), (2.32), (2.30),

$$\begin{aligned} \partial CS^*_{i,j}/\partial L_b &= \partial CS^*_{i,j}/\partial p^*_i \cdot \partial p^*_i/\partial L_b + \partial CS^*_{i,j}/\partial p^*_j \cdot \partial p^*_j/\partial L_b \\ &= -x^*_{i,j}\, t\, Q_{i,b} + (x^*_{i,j} - l_{i,j})\, t\, Q_{j,b}. \end{aligned}$$

Since $\mathbf{Q}$ is a non-negative matrix, $\partial CS^*_{i,j}/\partial L_b < 0$ and consumer surplus on road $R_{i,j}$ is decreasing in other admissible local firm market sizes.
Similarly, from (2.38), (2.39), (2.32), (2.30)

$$\begin{aligned} \partial CS^*_{i,j}/\partial L_i &= \partial CS^*_{i,j}/\partial p^*_i \cdot \partial p^*_i/\partial L_i + \partial CS^*_{i,j}/\partial p^*_j \cdot \partial p^*_j/\partial L_i + s_{i,j}\, k_i \\ &= -x^*_{i,j}\, t\, Q_{i,i} + (x^*_{i,j} - l_{i,j})\, t\, Q_{j,i} + s_{i,j}\, k_i \\ &= t\, x^*_{i,j}\, (Q_{j,i} - Q_{i,i}) - t\, l_{i,j}\, Q_{j,i} + s_{i,j}\, k_i. \end{aligned}$$

If

$$s_{i,j} > \frac{t}{k_i} \left(l_{i,j}\, Q_{j,i} + x^*_{i,j}\, (Q_{i,i} - Q_{j,i})\right)$$

then $\partial CS^*_{i,j}/\partial L_i > 0$. Otherwise, $\partial CS^*_{i,j}/\partial L_i < 0$.
Similarly,

$$\partial CS^*_{i,j}/\partial L_j = t\, x^*_{i,j}\, (Q_{j,j} - Q_{i,j}) - t\, l_{i,j}\, Q_{j,j} + s_{i,j}\, k_j.$$

If

$$s_{i,j} > \frac{t}{k_j} \left(l_{i,j}\, Q_{j,j} + x^*_{i,j}\, (Q_{i,j} - Q_{j,j})\right)$$

then $\partial CS^*_{i,j}/\partial L_j > 0$. Otherwise, $\partial CS^*_{i,j}/\partial L_j < 0$.
Since we consider the valuation s sufficiently large, we have $CS^*_{i,j}/\partial L_i > 0$ and $CS^*_{i,j}/\partial L_j > 0$. Hence, road consumer surplus on road $R_{i,j}$ is increasing in admissible local firm market sizes L_i and L_j.
From (2.37), (2.38), (2.39) and (2.33)

$$\begin{aligned}
&\partial CS^*_{i,j}/\partial t = \\
&= \partial CS^*_{i,j}/\partial p^*_i \cdot \partial p^*_i/\partial t + \partial CS^*_{i,j}/\partial p^*_j \cdot \partial p^*_j/\partial t + x^*_{i,j}\left(l_{i,j} - x^*_{i,j}\right) - \frac{l^2_{i,j}}{2} \\
&= -x^*_{i,j} \sum_{v\in V} Q_{i,v}\, L_v + (x^*_{i,j} - l_{i,j}) \sum_{v\in V} Q_{j,v}\, L_v + x^*_{i,j}\left(l_{i,j} - x^*_{i,j}\right) - \frac{l^2_{i,j}}{2} \\
&= x^*_{i,j} \left(\sum_{v\in V} (Q_{j,v} - Q_{i,v})\, L_v + l_{i,j} - x^*_{i,j}\right) - l_{i,j} \left(\sum_{v\in V} Q_{j,v}\, L_v + \frac{l_{i,j}}{2}\right).
\end{aligned}$$

Since $0 < x^*_{i,j} < l_{i,j}$ and $\sum_{v\in V} Q_{j,v}\, L_v + \dfrac{l_{i,j}}{2} > 0$, if

$$\sum_{v\in V} (Q_{j,v} - Q_{i,v})\, L_v + l_{i,j} - x^*_{i,j} < \sum_{v\in V} Q_{j,v}\, L_v + \frac{l_{i,j}}{2} \tag{2.40}$$

then $\partial CS^*_{i,j}/\partial t < 0$. But condition (2.40) is equivalent to

$$\sum_{v\in V} Q_{i,v} L_v + x^*_{i,j} - \frac{1}{2}\, l_{i,j} > 0. \tag{2.41}$$

Since $L_v \geq l_m$, then $\sum_{v\in V} Q_{i,v}\, L_v \geq \sum_{v\in V} Q_{i,v}\, l_m = l_m$. Hence,

$$\sum_{v\in V} Q_{i,v} L_v + x^*_{i,j} - \frac{1}{2}\, l_{i,j} \geq l_m + x^*_{i,j} - \frac{1}{2}\, l_{i,j} > l_m - \frac{1}{2}\, l_{i,j}.$$

From the WB condition we know that $l_m > l_M/2$. Hence, $l_m > l_{i,j}/2$, condition (2.41) holds, and $\partial CS^*_{i,j}/\partial t < 0$. Therefore, consumer surplus on road $R_{i,j}$ is decreasing in transportation cost.
Let $R_{r,s} \in E \setminus \{R_{i,j}\}$. Hence, from (2.38), (2.39) and (2.34)

$$\begin{aligned} \partial CS^*_{i,j}/\partial l_{r,s} &= \partial CS^*_{i,j}/\partial p^*_i \cdot \partial p^*_i/\partial l_{r,s} + \partial CS^*_{i,j}/\partial p^*_j \cdot \partial p^*_j/\partial l_{r,s} \\ &= -x^*_{i,j}\, t\, A_i(r,s) + (x^*_{i,j} - l_{i,j})\, t\, A_j(r,s). \end{aligned}$$

Hence, $\partial CS^*_{i,j}/\partial l_{r,s} < 0$ and road consumer surplus on road $R_{i,j}$ is decreasing in other road lengths.
From (2.37), (2.38), (2.39) and (2.34)

$$\begin{aligned} \partial CS^*_{i,j}/\partial l_{i,j} &= \partial CS^*_{i,j}/\partial p^*_i \cdot \partial p^*_i/\partial l_{i,j} + \partial CS^*_{i,j}/\partial p^*_j \cdot \partial p^*_j/\partial l_{i,j} + s_{i,j} \\ &= -x^*_{i,j}\, t\, A_i(i,j) + (x^*_{i,j} - l_{i,j})\, t\, A_j(i,j) + s_{i,j} \\ &= x^*_{i,j}\, t\, (A_j(i,j) - A_i(i,j)) - l_{i,j}\, t\, A_j(i,j) + s_{i,j}. \end{aligned}$$

If $s_{i,j} > l_{i,j}\, t\, A_j(i,j) + x^*_{i,j}\, t\, (A_i(i,j) - A_j(i,j))$ then $CS^*_{i,j}/\partial l_{i,j} > 0$. Otherwise, $CS^*_{i,j}/\partial l_{i,j} < 0$. Since we consider the valuation s sufficiently large, we have $CS^*_{i,j}/\partial l_{i,j} > 0$ and road consumer surplus on road $R_{i,j}$ is increasing in local road length. □

The (total) *consumer surplus* $CS(\mathbf{P})$ is

$$CS(\mathbf{P}) = \sum_{R_{i,j} \in E} CS_{i,j}(\mathbf{P}).$$

Hence, $CS^* = CS(\mathbf{P}^*)$ is given by

$$CS^* = (4\,t)^{-1} \sum_{R_{i,j} \in E} \left(4\,t\,s\,l_{i,j} + (p_j^* - p_i^*)^2 - 2\,t\,l_{i,j}\,(p_j^* + p_i^*) - t^2\,l_{i,j}^2\right). \quad (2.42)$$

Corollary 2.1.6. *If a Hotelling town satisfies the SB condition, the consumer surplus is decreasing in production costs, increasing in road lengths, decreasing in transportation costs and increasing in admissible local firm market sizes.*

Proof. Let $D(u,w) = p_u^* - p_w^* - t\,l_{u,w}$. From (2.42), we obtain that, for any road $R_{u,w}$

$$\partial CS^*/\partial p_u^* = (2\,t)^{-1}\,(p_u^* - p_w^* - t\,l_{u,w}) = \frac{D(u,w)}{2\,t} < 0 \quad (2.43)$$

and

$$\partial CS^*/\partial p_w^* = (2\,t)^{-1}\,(p_w^* - p_u^* - t\,l_{u,w}) = \frac{D(w,u)}{2\,t} < 0. \quad (2.44)$$

Let $a \in V$. From (2.42), (2.43), (2.44) and (2.31)

$$\begin{aligned} \partial CS^*/\partial c_a &= (2\,t)^{-1} \sum_{R_{i,j} \in E} \left(D(i,j) \cdot \partial p_i^*/\partial c_a + D(j,i) \cdot \partial p_j^*/\partial c_a\right) \\ &= (2\,t)^{-1} \sum_{R_{i,j} \in E} D(i,j)\,Q_{i,a} + D(j,i)\,Q_{j,a}. \end{aligned}$$

Since $D(i,j) < 0$, $D(j,i) < 0$ and $\mathbf{Q}$ is non-negative, $\partial CS^*/\partial c_a < 0$ and consumer surplus is decreasing in production costs.

From (2.42), (2.43), (2.44), (2.32) and (2.30)

$$
\begin{aligned}
\partial CS^*/\partial L_a &= (2\,t)^{-1} \sum_{R_{i,j}\in E} D(i,j)\,\cdot \partial p_i^*/\partial L_a + D(j,i)\,\cdot \partial p_j^*/\partial L_a + \\
&+ (4\,t)^{-1} \sum_{R_{i,j}\in E} \left(4\,t\,s - 2\,t\,(p_j^* + p_i^*) - 2\,t^2\,l_{i,j}\right)\,\partial l_{i,j}/\partial L_a \\
&= (2\,t)^{-1} \sum_{R_{i,j}\in E} D(i,j)\,t\,Q_{i,a} + D(j,i)\,t\,Q_{j,a} + k_a \sum_{b\in N_a} s_{a,b} \\
&+ \frac{1}{2} \sum_{b\in N_a} (2\,s - p_b^* - p_a^* - t\,l_{a,b})\;k_a \\
&= \frac{1}{2}\left(\sum_{R_{i,j}\in E} D(i,j)\,Q_{i,a} + D(j,i)\,Q_{j,a} + k_a \sum_{b\in N_a} s_{a,b}\right).
\end{aligned}
$$

Since we consider s sufficiently high, $\partial CS^*/\partial L_a > 0$ and consumer surplus is increasing in admissible local firm market sizes.
From (2.42), (2.43), (2.44) and (2.33)

$$
\begin{aligned}
\partial CS^*/\partial t &= (2\,t)^{-1} \sum_{R_{i,j}\in E} \left(D(i,j)\,\cdot \partial p_i^*/\partial t + D(j,i)\,\cdot \partial p_j^*/\partial t\right) - \\
&- (2\,t)^{-2} \sum_{R_{i,j}\in E} (p_j^* - p_i^*)^2 + t^2\,l_{i,j}^2 \\
&= (2\,t)^{-1} \sum_{R_{i,j}\in E} D(i,j) \sum_{v\in V} Q_{i,v}\,L_v + D(j,i) \sum_{v\in V} Q_{j,v}\,L_v - \\
&- (2\,t)^{-2} \sum_{R_{i,j}\in E} (p_j^* - p_i^*)^2 + t^2\,l_{i,j}^2.
\end{aligned}
$$

Since $D(i,j) < 0$, $D(j,i) < 0$ and $(p_j^* - p_i^*)^2 + t^2\,l_{i,j}^2 > 0$, we have $\partial CS^*/\partial t < 0$. Hence, consumer surplus is decreasing in transportation cost.

Let $R_{r,s} \in E$. From (2.42), (2.43), (2.44) and (2.34)

$$\begin{aligned}\partial CS^*/\partial l_{r,s} &= (2\,t)^{-1} \sum_{R_{i,j}\in E} D(i,j)\ \cdot \partial p_i^*/\partial l_{r,s} + D(j,i)\ \cdot \partial p_j^*/\partial l_{r,s} + s_{r,s} \\ &= (2\,t)^{-1} \sum_{R_{i,j}\in E} D(i,j)\ t\,A_i(r,s) + D(i,j)\ t\,A_j(r,s) + s_{r,s} \\ &= \frac{1}{2}\left(2\,s_{r,s} + \sum_{R_{i,j}\in E} D(i,j)\,A_i(r,s) + D(j,i)\,A_j(r,s)\right).\end{aligned}$$

Since we consider s sufficiently high, $\partial CS^*/\partial l_{r,s} > 0$ and consumer surplus is increasing road lengths. □

The (total) *welfare* $W(\mathbf{P})$ is

$$W(\mathbf{P}) = \sum_{i\in V} \pi_i(\mathbf{P}) + CS(\mathbf{P}).$$

Hence, $W^* = W(\mathbf{P}^*)$ is given by

$$W^* = (4\,t)^{-1} \sum_{R_{i,j}\in E} 2\,t\,l_{i,j}\,(2\,s - c_i - c_j) + 2\,(p_j^* - p_i^*)\ (c_j - c_i) - (p_j^* - p_i^*)^2 - t^2\,l_{i,j}^2 \tag{2.45}$$

Corollary 2.1.7. *If a Hotelling town satisfies the SB condition, the welfare is increasing in road lengths and in local firm market sizes. The marginal rates on production and transportation costs are inconclusive.*

Proof. Let $G(u,w) = p_w - c_w - p_u + c_u$. From (2.45), we obtain that for any road $R_{u,w}$

$$\partial W^*/\partial p_u^* = (2\,t)^{-1}\,(p_w - c_w - p_u + c_u) = \frac{G(u,w)}{2\,t}; \tag{2.46}$$

and

$$\partial W^*/\partial p_w^* = (2\,t)^{-1}\,(p_u - c_u - p_w + c_w) = -\frac{G(u,w)}{2\,t}. \tag{2.47}$$

Let $a \in V$. From (2.46), (2.47) and (2.31)

$$\begin{aligned}\partial W^*/\partial c_a &= (2\,t)^{-1} \sum_{R_{i,j}\in E} G(i,j)\cdot \partial p_i^*/\partial c_a - G(i,j)\cdot \partial p_j^*/\partial c_a \\ &= (2\,t)^{-1} \sum_{R_{i,j}\in E} G(i,j)\ (Q_{i,a} - Q_{j,a})\,.\end{aligned}$$

Let $H(i,j) = 2\,s - c_i - c_j - l_{i,j}$. From (2.45), (2.46), (2.47) (2.32) and (2.30)

$$\begin{aligned}\partial W^*/\partial L_a &= (2\,t)^{-1} \sum_{R_{i,j}\in E} G(i,j)\cdot \partial p_i^*/\partial L_a - G(i,j)\cdot \partial p_j^*/\partial L_a + \\ &+ \frac{1}{2} \sum_{R_{i,j}\in E} (2\,s - c_i - c_j - t\,l_{i,j})\cdot \partial l_{i,j}/\partial L_a \\ &= (2\,t)^{-1} \sum_{R_{i,j}\in E} t\,Q_{i,a}\,G(i,j) - t\,Q_{j,a}\,G(i,j) + \frac{1}{2}\sum_{b\in N_a} H(a,b)\,k_a \\ &= \frac{1}{2}\left(\sum_{R_{i,j}\in E} G(i,j)\ (Q_{i,a} - Q_{j,a}) + k_a \sum_{b\in N_a} H(a,b)\right).\end{aligned}$$

Since we consider s sufficiently high, $\partial W^*/\partial L_a > 0$ and welfare is increasing in admissible local firm market sizes.

From (2.45), (2.46), (2.47) and (2.33)

$$\begin{aligned}
\partial W^*/\partial t &= (2\,t)^{-1} \sum_{R_{i,j}\in E} G(i,j)\cdot \partial p_i^*/\partial t - G(i,j)\cdot \partial p_j^*/\partial t - \\
&- (2\,t)^{-2} \sum_{R_{i,j}\in E} 2\,(p_j^* - p_i^*)\,(c_j - c_i) - (p_j^* - p_i^*)^2 + t^2\, l_{i,j}^2 \\
&= (2\,t)^{-1} \sum_{R_{i,j}\in E} G(i,j) \sum_{v\in V} L_v\,(Q_{i,v} - Q_{j,v}) + \\
&+ (2\,t)^{-2} \sum_{R_{i,j}\in E} 2\,(p_j^* - p_i^*)\,(c_i - c_j) + (p_j^* - p_i^*)^2 - t^2\, l_{i,j}^2 \\
&= (2\,t)^{-2} \left(\sum_{R_{i,j}\in E} 2\,G(i,j) \sum_{v\in V} L_v\,(Q_{i,v} - Q_{j,v}) \right) + \\
&+ (2\,t)^{-2} \left(2\,(p_j^* - p_i^*)\,(c_i - c_j) + (p_j^* - p_i^*)^2 - t^2\, l_{i,j}^2 \right).
\end{aligned}$$

Let $R_{r,s} \in E$. From (2.45), (2.46), (2.47) and (2.34)

$$\begin{aligned}
\partial W^*/\partial l_{r,s} &= (2\,t)^{-1} \sum_{R_{i,j}\in E} G(i,j)\cdot \partial p_i^*/\partial l_{r,s} - G(i,j)\cdot \partial p_j^*/\partial l_{r,s} + \\
&+ \frac{1}{2}\,(2\,s - c_r - c_s - t\, l_{r,s}) \\
&= (2\,t)^{-1} \sum_{R_{i,j}\in E} \left(G(i,j)\,(t\,A_i(r,s) - t\,A_j(r,s)) + \frac{1}{2}\,H(r,s) \right) \\
&= \frac{1}{2} \left(\sum_{R_{i,j}\in E} G(i,j)\,(A_i(r,s) - A_j(r,s)) + H(r,s) \right).
\end{aligned}$$

Since we consider s sufficiently high, $\partial W^*/\partial l_{r,s} > 0$ and welfare is increasing road lengths. □

2.2 Quadratic transportation costs

This section extends the Hotelling model with quadratic transportation costs to networks.

A consumer located at a point x of the network who decides to buy at firm F_i *spends*

$$E(x; i, \mathbf{P}) = p_i + t\, d^2(x, y_i)$$

the price p_i charged by the firm F_i plus the *transportation cost* that is proportional t to the square of the minimal distance measured in the network between the position y_i of the firm F_i and the position x of the consumer.

2.2.1 Local optimal equilibrium price strategy

For every $v \in V$, let $\epsilon_v = d(v, y_v)$ and $j(v)$ be the node with the property that y_v is at the road $R_{v,j(v)}$. The *shift location matrix* $\mathbf{S}(v)$ associated to node v is defined by

$$s_{i,j}(v) = \begin{cases} \epsilon_v & \text{if } i = v \text{ and } j \in N_v \setminus \{j(v)\} \ ; \\ -\epsilon_v & \text{if } i = v \text{ and } j = j(v) \ ; \\ \epsilon_v & \text{if } j = v \text{ and } i \in N_v \setminus \{j(v)\} \ ; \\ -\epsilon_v & \text{if } j = v \text{ and } i = j(v) \ ; \\ 0 & \text{otherwise.} \end{cases}$$

The distance $\tilde{l}_{i,j} = d(y_i, y_j)$ between the location of firms F_i and F_j is given by

$$\tilde{l}_{i,j} = l_{i,j} + \sum_{v \in \{i,j\}} s_{i,j}(v). \tag{2.48}$$

Let $\epsilon = \max_{v \in V} \epsilon_v$. Hence, for every $i, j \in V$ we have

$$l_{i,j} - 2\,\epsilon \leq \tilde{l}_{i,j} \leq l_{i,j} + 2\,\epsilon.$$

We observe that, for every road $R_{i,j}$ there is an *indifferent consumer* located at a distance

$$0 < x_{i,j} = \frac{p_j - p_i + t\,\tilde{l}_{i,j}^2}{2\,t\,\tilde{l}_{i,j}} < \tilde{l}_{i,j} \tag{2.49}$$

of firm F_i if and only if $|p_i - p_j| < t\,\tilde{l}_{i,j}^2$. Thus, a price strategy $\mathbf{P}$ determines a local market structure if and only if $|p_i - p_j| < t\,\tilde{l}_{i,j}^2$ for every road $R_{i,j}$. Hence, if

$$|p_i - p_j| < t\,(l_{i,j} - 2\,\epsilon)^2 = t\,l_{i,j}^2 - 4\,t\,l_{i,j}\,\epsilon + 4\,t\,\epsilon^2 \tag{2.50}$$

then condition (2.49) is satisfied. Therefore, if condition (2.50) holds then the price strategy $\mathbf{P}$ determines a local market structure.

Let k_i denote is the cardinality of the set N_i that is equal to the degree of the vertex i. If the price strategy determines a local market structure then

$$S(i, \mathbf{P}) = (2 - k_i)\,\epsilon_i + \sum_{j \in N_i} x_{i,j}$$

and

$$\begin{aligned} \pi_i(\mathbf{P}, \mathbf{C}) &= (p_i - c_i)\,S(i, \mathbf{P}) \\ &= \frac{p_i - c_i}{2\,t}\left(2\,t\,(2 - k_i)\,\epsilon_i + \sum_{j \in N_i} \frac{p_j - p_i + t\,\tilde{l}_{i,j}^2}{\tilde{l}_{i,j}}\right). \end{aligned} \tag{2.51}$$

Given a pair of price strategies $\mathbf{P}$ and $\mathbf{P}^*$ and a firm F_i, we define the price vector $\tilde{\mathbf{P}}(i, \mathbf{P}, \mathbf{P}^*)$ whose coordinates are $\tilde{p}_i = p_i^*$ and $\tilde{p}_j = p_j$, for every $j \in V \setminus \{i\}$. Let $\mathbf{P}$ and $\mathbf{P}^*$ be price strategies that determine local market structures. The price strategy $\mathbf{P}^*$ is a *local best response* to the price strategy $\mathbf{P}$, if for every $i \in V$ the price strategy $\tilde{\mathbf{P}}(i, \mathbf{P}, \mathbf{P}^*)$ determines a local market structure and

$$\frac{\partial \pi_i(\tilde{\mathbf{P}}(i, \mathbf{P}, \mathbf{P}^*), \mathbf{C})}{\partial \tilde{p}_i} = 0 \quad \text{and} \quad \frac{\partial^2 \pi_i(\tilde{\mathbf{P}}(i, \mathbf{P}, \mathbf{P}^*), \mathbf{C})}{\partial \tilde{p}_i^2} < 0.$$

Let $\tilde{l}_i = \sum_{j \in N_i} \dfrac{1}{\tilde{l}_{i,j}}$. The Hotelling town *admissible market size* $\mathbf{L}$ is the vector whose coordinates are the *admissible local firm market sizes*

$$L_i = \frac{1}{\tilde{l}_i} \sum_{j \in N_i} l_{i,j},$$

The Hotelling town *neighboring market structure* $\mathbf{K}$ is the matrix whose elements are (i) $k_{i,j} = \tilde{l}_i^{-1} \, \tilde{l}_{i,j}^{-1}$, if there is a road $R_{i,j}$ between the firms F_i and F_j; and (ii) $k_{i,j} = 0$, if there is not a road $R_{i,j}$ between the firms F_i and F_j. The Hotelling town *firm deviation* is the vector $\mathbf{Y}$ whose coordinates are

$$Y_i = \tilde{l}_i^{-1} \left((2 - k_i) \, \epsilon_i + \sum_{j \in N_i} s_{i,j}(j) \right).$$

Let $\mathbf{1}$ denote the identity matrix.

Lemma 2.2.1. *Let* $\boldsymbol{P}$ *and* $\boldsymbol{P}^*$ *be price strategies that determine local market structures. The price strategy* $\boldsymbol{P}^*$ *is the local best response to price strategy* $\boldsymbol{P}$ *if and only if*

$$\boldsymbol{P}^* = \frac{1}{2} \, (\boldsymbol{C} + t \, (\boldsymbol{L} + \boldsymbol{Y})) + \frac{1}{2} \, \boldsymbol{K} \, \boldsymbol{P} \tag{2.52}$$

and the price strategies $\tilde{\boldsymbol{P}}(i, \boldsymbol{P}, \boldsymbol{P}^*)$ *determine local market structures for all* $i \in V$ *.*

Proof.
By (2.51), the *profit function* $\pi_i(\mathbf{P}, \mathbf{C})$ of firm F_i, in a local market structure, is given by

$$\pi_i(\mathbf{P}, \mathbf{C}) = (2 \, t)^{-1} (p_i - c_i) \left(2 \, t \, (2 - k_i) \, \epsilon_i + \sum_{j \in N_i} \frac{p_j - p_i + t \, \tilde{l}_{i,j}^2}{\tilde{l}_{i,j}} \right)$$

Let $\tilde{\mathbf{P}}(i, \mathbf{P}, \mathbf{P}^*)$ be the price vector whose coordinates are $\tilde{p}_i = p_i^*$ and $\tilde{p}_j = p_j$, for every $j \in V \setminus \{i\}$. Since $\mathbf{P}$ and $\mathbf{P}^*$ are local price strategies, the local best response of firm F_i to the price strategy $\mathbf{P}$, is given by computing $\partial\pi_i(\tilde{\mathbf{P}}(i,\mathbf{P},\mathbf{P}^*),\mathbf{C})/\partial\tilde{p}_i = 0$. Hence,

$$p_i^* = \frac{1}{2}\left(c_i + \frac{2\,t\,(2-k_i)}{\tilde{l}_i}\,\epsilon_i + \frac{1}{\tilde{l}_i}\sum_{j\in N_i} t\,\tilde{l}_{i,j} + \frac{p_j}{\tilde{l}_{i,j}}\right). \tag{2.53}$$

By (2.48), we obtain

$$p_i^* = \frac{1}{2}\left(c_i + \frac{2\,t\,(2-k_i)}{\tilde{l}_i}\,\epsilon_i + \frac{t}{\tilde{l}_i}\sum_{j\in N_i}\sum_{v\in\{i,j\}} s_{i,j}(v) + \frac{1}{\tilde{l}_i}\sum_{j\in N_i} t\,l_{i,j} + \frac{p_j}{\tilde{l}_{i,j}}\right).$$

We note that

$$\sum_{j\in N_i}\sum_{v\in\{i,j\}} s_{i,j}(v) = \sum_{j\in N_i} s_{i,j}(i) + \sum_{j\in N_i} s_{i,j}(j) = (k_i-2)\,\epsilon_i + \sum_{j\in N_i} s_{i,j}(j).$$

Hence,

$$p_i^* = \frac{1}{2}\left(c_i + \frac{t}{\tilde{l}_i}\left((2-k_i)\,\epsilon_i + \sum_{j\in N_i} s_{i,j}(j)\right) + \frac{1}{\tilde{l}_i}\sum_{j\in N_i} t\,l_{i,j} + \frac{p_j}{\tilde{l}_{i,j}}\right).$$

Therefore, since $\partial^2\pi_i(\tilde{\mathbf{P}}(i,\mathbf{P},\mathbf{P}^*),\mathbf{C})/\partial\tilde{p}_i^2 = -\tilde{l}_i/t < 0$, the local best response strategy prices $\mathbf{P}^*$ is given by

$$\mathbf{P}^* = \frac{1}{2}\,(\mathbf{C} + t\,(\mathbf{Y}+\mathbf{L}) + \mathbf{K}\,\mathbf{P})\,.$$

□

Definition 2.2.1. *A Hotelling town satisfies the* weak bounded length and costs (WB) *condition, if*

$$\Delta(c) + t\,\Delta_2(l) < t\,(l_m - 2\,\epsilon)^2 - 4\,t\,\epsilon\,(l_M + l_m).$$

Let $\mathbf{P}$ and $\mathbf{P}^*$ be price strategies that determine local market structures.

A price strategy $\mathbf{P}^*$ is a *local optimum price strategy* if $\mathbf{P}^*$ is the local best response to $\mathbf{P}^*$.

Proposition 2.2.1. *If the Hotelling town satisfies the WB condition, then there is unique local optimum price strategy given by*

$$\begin{aligned}\boldsymbol{P}^L &= \frac{1}{2}\left(\boldsymbol{1}-\frac{1}{2}\boldsymbol{K}\right)^{-1}(\boldsymbol{C}+t\,(\boldsymbol{L}+\boldsymbol{Y})) \\ &= \sum_{m=0}^{\infty} 2^{-(m+1)}\boldsymbol{K}^m\,(\boldsymbol{C}+t\,(\boldsymbol{L}+\boldsymbol{Y}))\,. \end{aligned} \tag{2.54}$$

The local optimum price strategy $\boldsymbol{P}^L$ determines a local market structure. Furthermore, the local optimal equilibrium prices p_i^L are bounded by

$$t\,(l_m-2\,\epsilon)^2+\frac{1}{2}\,(c_i+c_m)\leq p_i^L\leq t\,(l_M+2\,\epsilon)^2+\frac{1}{2}\,(c_i+c_M). \tag{2.55}$$

The local optimal profit $\pi_i^L=\pi_i^L(\boldsymbol{P},\boldsymbol{C})$ of firm F_i is given by

$$\pi_i^L(\boldsymbol{P},\boldsymbol{C})=(2t)^{-1}\,\tilde{l}_i\,(p_i^L-c_i)^2$$

and it is bounded by

$$\frac{k_i\,(2\,t\,(l_m-2\,\epsilon)^2-\Delta(c))^2}{8\,t\,(l_M+2\,\epsilon)}\leq\pi_i^L(\boldsymbol{P},\boldsymbol{C})\leq\frac{k_i\,(2\,t\,(l_M+2\,\epsilon)^2+\Delta(c))^2}{8\,t\,(l_m-2\,\epsilon)}$$

Corollary 2.2.1. *Consider a Hotelling town where all firms are located at the nodes. If $\Delta(c)+t\,\Delta_2(l)<t\,l_m^2$, then there is unique local optimum price strategy given by*

$$\boldsymbol{P}^L=\sum_{m=0}^{\infty}2^{-(m+1)}\boldsymbol{K}^m\,(\boldsymbol{C}+t\,\boldsymbol{L})\,.$$

The local optimum price strategy $\boldsymbol{P}^L$ determines a local market structure.

Furthermore, the local optimal equilibrium prices p_i^L are bounded by

$$t\,l_m^2 + \frac{1}{2}\,(c_i + c_m) \le p_i^L \le t\,l_M^2 + \frac{1}{2}\,(c_i + c_M).$$

The local optimal profit $\pi_i^L = \pi_i^L(\boldsymbol{P}, \boldsymbol{C})$ of firm F_i is given by

$$\pi_i^L(\boldsymbol{P}, \boldsymbol{C}) = (2t)^{-1}\ (p_i^L - c_i)^2 \sum_{j \in N_i} \frac{1}{l_{i,j}}$$

and it is bounded by

$$\frac{k_i\,(2\,t\,l_m^2 - \Delta(c))^2}{8\,t\,l_M} \le \pi_i^L(\boldsymbol{P}, \boldsymbol{C}) \le \frac{k_i\,(2\,t\,l_M^2 + \Delta(c))^2}{8\,t\,l_m}$$

Proof of Proposition 2.2.1.
Let, first, prove that $\mathbf{K}$ is a stochastic matrix (i.e., $\sum_{j \in V} k_{i,j} = 1$, for every $i \in V$). Since $k_{i,j} = \tilde{l}_i^{-1}\,\tilde{l}_{i,j}^{-1}$ and $\tilde{l}_i = \sum_{j \in N_i} \dfrac{1}{\tilde{l}_{i,j}}$, we have

$$\sum_{j \in V} k_{i,j} = \sum_{j \in N_i} k_{i,j} = \sum_{j \in N_i} \tilde{l}_i^{-1}\,\tilde{l}_{i,j}^{-1} = \tilde{l}_i^{-1} \sum_{j \in N_i} \tilde{l}_{i,j}^{-1} = \frac{1}{\sum_{j \in N_i} \dfrac{1}{\tilde{l}_{i,j}}} \sum_{j \in N_i} \frac{1}{\tilde{l}_{i,j}} = 1.$$

Then, K is a stochastic matrix, and we have $\|\mathbf{K}\| = 1$. Hence, the matrix $\mathbf{Q}$ is well-defined by

$$\mathbf{Q} = \frac{1}{2}\left(\mathbf{1} - \frac{1}{2}\,\mathbf{K}\right)^{-1} = \sum_{m=0}^{\infty} 2^{-(m+1)}\,\mathbf{K}^m$$

and Q is also a non-negative and stochastic matrix. By Lemma 2.2.1, a local optimum price strategy satisfy equality (2.52). Therefore,

$$\begin{aligned}\mathbf{P}^L &= \frac{1}{2}\left(\mathbf{1}-\frac{1}{2}\mathbf{K}\right)^{-1}(\mathbf{C}+t\,(\mathbf{L}+\mathbf{Y}))\\ &= \sum_{m=0}^{\infty} 2^{-(m+1)}\,\mathbf{K}^m\,(\mathbf{C}+t\,(\mathbf{L}+\mathbf{Y}))\,,\end{aligned}$$

and so $\mathbf{P}^L$ satisfies (2.54). By construction,

$$p_i^L = \sum_{v\in V} Q_{i,v}(c_v + t\,(L_v+Y_v)). \tag{2.56}$$

Let us prove that the price strategy $\mathbf{P}^L$ is local, i.e., the indifferent consumer $x_{i,j}$ satisfies $0 < x_{i,j} < \tilde{l}_{i,j}$ for every $R_{i,j}\in E$. We note that

$$\frac{l_m - 2\,\epsilon}{k_v} = \frac{1}{\dfrac{k_v}{l_m-2\,\epsilon}} \le \tilde{l}_v^{-1} = \frac{1}{\sum_{j\in N_v}\dfrac{1}{\tilde{l}_{v,j}}} \le \frac{1}{\dfrac{k_v}{l_M+2\,\epsilon}} \le \frac{l_M+2\,\epsilon}{k_v}. \tag{2.57}$$

Hence,

$$\frac{l_m-2\,\epsilon}{k_v}\sum_{j\in N_v} l_{v,j} \le L_v = \tilde{l}_v^{-1}\sum_{j\in N_v} l_{v,j} \le \frac{l_M+2\,\epsilon}{k_v}\sum_{j\in N_v} l_{v,j}.$$

Therefore,

$$l_m\,(l_m-2\,\epsilon) \le L_v \le l_M\,(l_M+2\,\epsilon). \tag{2.58}$$

We note that

$$-k_v\,\epsilon \le \sum_{j\in N_v} s_{v,j}(j) \le k_v\,\epsilon$$

If $k_v = 1$ then $Y_v = \tilde{l}_v^{-1}\left(\epsilon_v + \sum_{j\in N_v} s_{v,j}(j)\right)$, and from (2.57)

$$-\epsilon(l_m-2\,\epsilon) \le (l_m-2\,\epsilon)\,(\epsilon_v-\epsilon) \le Y_v \le (l_M+2\,\epsilon)\,(\epsilon_v+\epsilon) \le 2\,\epsilon(l_M+2\,\epsilon); \tag{2.59}$$

if $k_v = 2$ then $Y_v = \tilde{l}_v^{-1} \sum_{j \in N_v} s_{v,j}(j)$, and from (2.57)

$$-\epsilon\,(l_m - 2\,\epsilon) = -\frac{l_m - 2\,\epsilon}{2}\,2\,\epsilon \le Y_v \le \frac{l_M + 2\,\epsilon}{2}\,2\,\epsilon = \epsilon\,(l_M + 2\,\epsilon); \qquad (2.60)$$

and if $k_v \ge 3$ then $Y_v = \tilde{l}_v^{-1}\left((2 - k_v)\,\epsilon_v + \sum_{j \in N_v} s_{v,j}(j)\right)$, and from (2.57)

$$-((k_v - 2)\,\epsilon_v + k_v\,\epsilon)\,\frac{l_m - 2\,\epsilon}{k_v} \le Y_v \le \frac{l_M + 2\,\epsilon}{k_v}\,((2 - k_v)\,\epsilon_v + k_v\,\epsilon)\,.$$

Hence, if $k_v \ge 3$ then

$$-2\,\epsilon\,(l_m - 2\,\epsilon) \le -\frac{l_m - 2\,\epsilon}{k_v}\,(k_v\,(\epsilon + \epsilon_v)) \le Y_v \le \epsilon\,(l_M + 2\,\epsilon). \qquad (2.61)$$

Therefore, from (2.59), (2.60) and (2.61), we have

$$-2\,\epsilon\,(l_m - 2\,\epsilon) \le Y_v = \tilde{l}_v^{-1}\left(\sum_{j \in N_v} s_{v,j}(j) - \epsilon_v(k_v - 2)\right) \le 2\,\epsilon\,(l_M + 2\,\epsilon). \quad (2.62)$$

Since $\mathbf{Q}$ is a nonnegative and stochastic matrix, we obtain

$$\sum_{v \in V} Q_{i,v}(c_m + t\,(l_m - 2\,\epsilon)^2) = c_m + t\,(l_m - 2\,\epsilon)^2$$

and

$$\sum_{v \in V} Q_{i,v}(c_M + t\,(l_M + 2\,\epsilon)^2) = c_M + t\,(l_M + 2\,\epsilon)^2.$$

Hence, putting (2.56), (2.58) and (2.62) together we obtain that

$$c_m + t\,(l_m^2 - 4\,l_m\,\epsilon + 4\,\epsilon^2) \le p_i^L \le c_M + t\,(l_M^2 + 4\,l_M\,\epsilon + 4\,\epsilon^2).$$

Therefore,

$$c_m + t\,(l_m - 2\,\epsilon)^2 \le p_i^L \le c_M + t\,(l_M + 2\,\epsilon)^2.$$

Since the last relation is satisfied for every firm, we obtain

$$p_i^L - p_j^L \geq -\left(c_M - c_m + t(\Delta_2(l) + 4\,\epsilon(l_M + l_m))\right)$$

and

$$p_i^L - p_j^L \leq c_M - c_m + t(\Delta_2(l) + 4\,\epsilon(l_M + l_m)).$$

Therefore,

$$|p_i^L - p_j^L| \leq \Delta(c) + t(\Delta_2(l) + 4\,\epsilon(l_M + l_m)).$$

Hence, by the WB condition, we conclude that

$$|p_i^L - p_j^L| < t\,(l_m - 2\,\epsilon)^2.$$

Thus, by equation (2.50), we obtain that the indifferent consumer is located at $0 < x_{i,j} < \tilde{l}_{i,j}$ for every road $R_{i,j} \in E$. Hence, the price strategy $\mathbf{P}^L$ is local and is the unique local optimum price strategy.

From (2.56), (2.58) and (2.62), we obtain

$$p_i^L \geq \sum_{v\in V} Q_{i,v}\,t(l_m - 2\,\epsilon)^2 + \sum_{v\in V\setminus\{i\}} Q_{i,v}\,c_m + Q_{i,i}\,c_i.$$

By construction of matrix $\mathbf{Q}$, we have $Q_{i,i} > 1/2$. Furthermore, since $\mathbf{Q}$ is stochastic,

$$\sum_{v\in V\setminus\{i\}} Q_{i,v} < 1/2,$$

$\sum_{v\in V} Q_{i,v}\,t(l_m - 2\,\epsilon)^2 = t(l_m - 2\,\epsilon)^2$. Hence,

$$p_i^L \geq t\,(l_m - 2\,\epsilon)^2 + \frac{1}{2}\,(c_i + c_m).$$

Similarly, we obtain

$$p_i^L \leq t\,(l_M + 2\,\epsilon)^2 + \frac{1}{2}\,(c_i + c_M),$$

and so the local optimal equilibrium prices p_i^L are bounded and satisfy (2.55).

We can write the the profit function (2.51) of firm F_i for the price strategy P^L as

$$\pi_i^L = \pi_i(\mathbf{P}^L, \mathbf{C}) = (2t)^{-1}(p_i^L - c_i) \left(2\,t\,(2-k_i)\,\epsilon_i - \tilde{l}_i\, p_i^L + \sum_{j \in N_i} \frac{p_j^L + t\,\tilde{l}_{i,j}^2}{\tilde{l}_{i,j}} \right) \tag{2.63}$$

Since $\mathbf{P}^L$ satisfies the best response function (2.53), we have

$$2\,p_i^L = c_i + \frac{2\,t\,(2-k_i)}{\tilde{l}_i}\,\epsilon_i + \frac{1}{\tilde{l}_i} \sum_{j \in N_i} \frac{p_j^L + t\,\tilde{l}_{i,j}^2}{\tilde{l}_{i,j}}.$$

Therefore, $\sum_{j \in N_i} \frac{p_j^L + t\,\tilde{l}_{i,j}^2}{\tilde{l}_{i,j}} = 2\,\tilde{l}_i\,p_i^L - \tilde{l}_i\,c_i - 2\,t\,(2-k_i)\,\epsilon_i$, and replacing this sum in the profit function (2.63), we obtain

$$\pi_i^L = (2t)^{-1}(p_i^L - c_i)\,\tilde{l}_i\,(p_i^L - c_i) = (2t)^{-1}\,\tilde{l}_i\,(p_i^L - c_i)^2.$$

Hence, since

$$\frac{k_i}{l_M + 2\,\epsilon} \leq \tilde{l}_i \leq \frac{k_i}{l_m - 2\,\epsilon},$$

using the price bounds (2.55), we conclude

$$\frac{k_i\,(2\,t\,(l_m - 2\,\epsilon)^2 - \Delta(c))^2}{8\,t\,(l_M + 2\,\epsilon)} \leq \pi_i^L \leq \frac{k_i\,(2\,t\,(l_M + 2\,\epsilon)^2 + \Delta(c))^2}{8\,t\,(l_m - 2\,\epsilon)}.$$

□

Let $a \in V$, $R_{b,c} \in E$ and $d \in V \setminus \{i\}$. The marginal rates of the local optimal equilibrium prices p_i^L are positive with respect to the production costs c_a, admissible local firm market sizes L_a, transportation costs t and road lengths $l_{b,c}$. The marginal rates of the local optimal equilibrium profit π_i^L are negative with respect to the production costs c_i and positive with respect to the production costs c_d, admissible local firm market sizes L_a, transportation costs t and road lengths $l_{b,c}$.

2.2.2 Nash equilibrium price strategy

The price strategy $\mathbf{P}^*$ is a *best response* to the price strategy $\mathbf{P}$, if

$$(\tilde{p}_i - c_i)\, S(i, \tilde{\mathbf{P}}(i, \mathbf{P}, \mathbf{P}^*)) \geq (p'_i - c_i)\, S(i, \mathbf{P}'_i),$$

for all $i \in V$ and for all price strategies $\mathbf{P}'_i$ whose coordinates satisfy $p'_i \geq c_i$ and $p'_j = p_j$ for all $j \in V \setminus \{i\}$. A price strategy $\mathbf{P}^*$ is a Hotelling town *Nash equilibrium* if $\mathbf{P}^*$ is the best response to $\mathbf{P}^*$.

Lemma 2.2.2. *In a Hotelling town satisfying the WB condition, if there is a Nash price $\boldsymbol{P}^*$ then $\boldsymbol{P}^*$ is unique and $\boldsymbol{P}^* = \boldsymbol{P}^L$.*

Hence, the local optimum price strategy $\mathbf{P}^L$ is the only candidate to be a Nash equilibrium price strategy. However, $\mathbf{P}^L$ might not be a Nash equilibrium price strategy because there can be a firm F_i that by decreasing his price is able to absorb markets of other firms in such a way that increases its own profit. Therefore, the best response price strategy $\mathbf{P}^{L,*}$ to the local optimum price strategy $\mathbf{P}^L$ might be different from $\mathbf{P}^L$.

Proof of Lemma 2.2.2.
Suppose that P^* is a Nash price strategy and that $\mathbf{P}^* \neq \mathbf{P}^L$. Hence, $\mathbf{P}^*$ does not determine a local market structure, i.e., there exists $i \in V$ such that

$$M(i, \mathbf{P}^*) \not\subset \cup_{j \in N_i} R_{i,j}.$$

Hence, there exists $j \in N_i$ such that $M(j, \mathbf{P}^*) = 0$ and, therefore, $\pi^*_j = 0$. Moreover, in this case, we have that

$$p^*_j > p^*_i + t\, \tilde{l}^2_{i,j}.$$

Consider, now, that F_j changes his price to $p_j = c_j + t\,\Delta_2(l) + 4\,t\,\epsilon\,(l_M + l_m)$. Since $p_i^* > c_i$ and $c_j - c_i \leq \Delta(c)$ we have that

$$p_j - p_i^* = c_j + t\,\Delta_2(l) + 4\,t\,\epsilon\,(l_M + l_m) - p_i^* < \Delta(c) + t\,\Delta_2(l) + 4\,t\,\epsilon\,(l_M + l_m).$$

Since the Hotelling town satisfies the WB condition, we obtain

$$p_j - p_i^* < t\,(l_m - 2\,\epsilon)^2 \leq t\,(l_{i,j} - 2\,\epsilon)^2 \leq t\,\tilde{l}_{i,j}^2.$$

Hence, $M(j, \tilde{\mathbf{P}}(j, \mathbf{P}^*, \mathbf{P})) > 0$ and

$$\pi_j = (t\,\Delta_2(l) + 4\,t\,\epsilon\,(l_M + l_m))\,S(j, \tilde{\mathbf{P}}(j, \mathbf{P}^*, \mathbf{P})) > 0.$$

Therefore, F_j will change its price and so $\mathbf{P}^*$ is not a Nash equilibrium price strategy. Hence, if there is a Nash price $\mathbf{P}^*$ then $\mathbf{P}^* = \mathbf{P}^L$. □

Let $\cup_{j \in N_i} R_{i,j}$ be the *1-neighbourhood* $\mathcal{N}(i,1)$ of a firm $i \in V$. Let $\cup_{j \in N_i} \cup_{k \in N_j} R_{j,k}$ be the *2-neighbourhood* $\mathcal{N}(i,2)$ of a firm $i \in V$.

Lemma 2.2.3. *In a Hotelling town satisfying the WB condition,*

$$M(i, \tilde{\boldsymbol{P}}(i, \boldsymbol{P}^L, \boldsymbol{P}^{L,*})) \subset \mathcal{N}(i,2)$$

for every $i \in V$.

Hence, a consumer $x \in R_{j,k}$ might not buy in its local firms F_j and F_k. However, the consumer $x \in R_{j,k}$ still has to buy in a firm F_i that is a neighboring firm of its local firms F_j and F_k, i.e. $i \in N_j \cup N_k$.

Proof of Lemma 2.2.3.
By contradiction, let us consider a consumer $z \in M(i, \tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*}))$ and $z \notin \mathcal{N}(i,2)$. The price that consumer z pays to buy in firm F_i is given by

$$e = p_i + t\,\left(\tilde{l}_{i_1,i_2} + \tilde{l}_{i_2,i_3} + d\,(y_{i_3}, z)\right)^2 \geq p_i + t\,\left(l_{i_1,i_2} + l_{i_2,i_3} - 2\,\epsilon + d\,(y_{i_3}, z)\right)^2$$

where $p_i = p_i^{L,*}$ is the coordinate of the vector $\tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*})$ and for the 2-path $(R_{i_1,i_2}, R_{i_2,i_3})$ with $i_1 = i$. If the consumer z buys at firm F_{i_3}, then the price that has to pay is

$$\tilde{e} = p_{i_3}^L + t\, d^2\,(y_{i_3}, z).$$

Since, by hypothesis, $z \in M(i, \tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*}))$, we have $e < \tilde{e}$. Therefore

$$p_i < p_{i_3}^L - t\, \left(l_{i_1,i_2} + l_{i_2,i_3} - 2\,\epsilon\right)^2 - 2\,t\,(l_{i_1,i_2} + l_{i_2,i_3} - 2\,\epsilon)\, d(y_{i_3}, z).$$

Since $l_{i,j} \geq l_m$ for all $R_{i,j} \in E$,

$$p_i < p_{i_3}^L - 4\,t\, \left(l_m - \epsilon\right)^2 - 4\,t\,(l_m - \epsilon)\, d(y_{i_3}, z).$$

By (2.55), $p_i^L \leq t\,(l_M + 2\,\epsilon)^2 + c_M$ for all $i \in V$. Hence,

$$p_i < c_M + t\,(l_M + 2\,\epsilon)^2 - 4\,t\, \left(l_m - \epsilon\right)^2 - 4\,t\,(l_m - \epsilon)\, d(y_{i_3}, z).$$

Furthermore,

$$\begin{aligned} p_i - c_i &< \Delta(c) + t\,(l_M + 2\,\epsilon)^2 - 4\,t\, \left(l_m - \epsilon\right)^2 \\ &= \Delta(c) + t\,\Delta_2(l) - 3\,t\,l_m^2 + 4\,t\,\epsilon\,(l_M + l_m) + 4\,t\,l_m\,\epsilon + 4\,t\,\epsilon^2 - 4\,t\,\epsilon^2 \\ &= \Delta(c) + t\,\Delta_2(l) - t\,(l_m - 2\,\epsilon)^2 + 4\,t\,\epsilon\,(l_M + l_m) + 2\,t\,(2\,\epsilon^2 - l_m^2). \end{aligned}$$

Since $l_m > 2\,\epsilon$, by the WB condition, $p_i - c_i < 0$. Hence, $\pi_i^{L,*} < 0$ which contradicts the fact that p_i is the best response to $\mathbf{P}^L$ (since $\pi_i^L > 0$). Therefore, $z \in \mathcal{N}(i, 2)$ and $M(i, \tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*})) \subset \mathcal{N}(i, 2)$. $\square$

Definition 2.2.2. *A Hotelling town satisfies the* strong bounded length and costs (SB) *condition, if*

$$\Delta(c) + t\,\Delta_2(l) \leq \frac{(2\,t\,(l_m - 2\,\epsilon)^2 - \Delta(c))^2}{8\,t\,k_M\,(l_M + 2\,\epsilon)^2} - 4\,t\,\epsilon\,(l_M + l_m).$$

The SB condition implies the WB condition, and so under the SB condition the only candidate to be a Nash equilibrium price strategy is the local optimum strategy price $\mathbf{P}^L$. On the other hand, the condition

$$\Delta(c) + t\,\Delta_2(l) \leq \frac{t\,l_M^4}{8\,k_M\,(l_M + 2\,\epsilon)^2} - 4\,t\,\epsilon\,(l_M + l_m).$$

together with the WB condition implies the SB condition. Hence, we note that the condition

$$\Delta(c) + t\,\Delta_2(l) \leq \frac{t\,(l_m - 2\,\epsilon)^4}{8\,k_M\,(l_M + 2\,\epsilon)^2} - 4\,t\,\epsilon\,(l_M + l_m).$$

implies the WB and SB conditions.

Theorem 2.2.1. *If a Hotelling town satisfies the SB condition then there is a unique Hotelling town Nash equilibrium price strategy $\boldsymbol{P}^* = \boldsymbol{P}^L$.*

Hence, the Nash equilibrium price strategy for the Hotelling town satisfying the SB condition determines a local market structure, i.e. every consumer located at $x \in R_{i,j}$ spends less by shopping at his local firms F_i or F_j than in any other firm in the town and so the consumer at x will buy either at his local firm F_i or at his local firm F_j.

For ϵ small enough, a *cost and length uniform* Hotelling town, i.e. $c_m = c_M$ and $l_m = l_M$, has a unique pure network Nash price strategy.

Corollary 2.2.2. *Consider a Hotelling town where all firms are located at the nodes. If*

$$\Delta(c) + t\,\Delta_2(l) \leq \frac{(2\,t\,l_m^2 - \Delta(c))^2}{8\,t\,k_M\,l_M^2}$$

then there is a unique Hotelling town Nash equilibrium price strategy $\boldsymbol{P}^* = \boldsymbol{P}^L$.

Proof of Theorem 2.2.1.
By Proposition 2.2.1 and Lemma 2.2.2, if there is a Nash equilibrium price strategy $\mathbf{P}^*$ then $\mathbf{P}^*$ is unique and $\mathbf{P}^* = \mathbf{P}^L$.

We note that if $M(i, \tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*})) \subset \mathcal{N}(i, 1)$ for every $i \in V$ then $\tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*}) = p_i^L$ and so $\mathbf{P}^L$ is a Nash equilibrium.
By Lemma 2.2.3, we have that $M(i, \tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*})) \subset \mathcal{N}(i, 2)$ for every $i \in V$. Now, we will prove that the SB condition implies that firm F_i earns more competing only in the 1-neighborhood than competing in a 2-neighborhood. By Proposition 2.2.1,

$$\pi_i^L \geq \frac{k_i\,(2\,t\,(l_m - 2\,\epsilon)^2 - \Delta(c))^2}{8\,t\,(l_M + 2\,\epsilon)} \tag{2.64}$$

By Lemma 2.2.3,

$$\begin{aligned} \pi_i(\tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*}), \mathbf{C}) &\leq (p_i - c_i) \sum_{j \in N_i} \left(\tilde{l}_{i,j} + \sum_{k \in N_j \setminus \{i\}} \tilde{l}_{j,k} \right) \\ &\leq (p_i - c_i) \sum_{j \in N_i} \sum_{k \in N_j} \tilde{l}_{j,k}, \end{aligned}$$

where $p_i = p_i^{L,*}$ is the coordinate of the vector $\tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*})$. Hence,

$$\pi_i(\tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*}), \mathbf{C}) \leq (p_i - c_i) \sum_{j \in N_i} \sum_{k \in N_j} (l_{j,k} + \epsilon) \leq (p_i - c_i) k_i\, k_M\, (l_M + \epsilon). \tag{2.65}$$

By contradiction, let us consider a consumer $z \in M(i, \tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*}))$ and $z \notin \mathcal{N}(i, 1)$. Let $i_2 \in N_i$ be the vertex such that $z \in \mathcal{N}(i_2, 1)$. The price that consumer z pays to buy in firm F_i is given by

$$e = p_i + t\,(\tilde{l}_{i,i_2} + d\,(y_{i_2}, z))^2 \geq p_i + t\,(l_{i,i_2} - 2\,\epsilon + d\,(y_{i_2}, z))^2.$$

If the consumer y buys at firm F_{i_2}, then the price that has to pay is

$$\tilde{e} = p_{i_2}^L + t\,d^2\,(y_{i_2}, z).$$

Since, by hypothesis, $z \in M(i, \tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*}))$, we have $e < \tilde{e}$. Therefore

$$p_i < p_{i_2}^L - t\,(l_{i,i_2} - 2\,\epsilon)^2 - 2\,t\,(l_{i,i_2} - 2\,\epsilon)\,d\,(y_{i_2}, z).$$

By (2.55), $p_{i_2}^L \leq t\,(l_M + 2\,\epsilon)^2 + \dfrac{1}{2}(c_M + c_{i_2}) \leq c_M + t\,(l_M + 2\,\epsilon)^2$. Since $l_{i,i_2} \geq l_m$, we have

$$p_i < c_M + t\,(l_M + 2\,\epsilon)^2 - t\,(l_m - 2\,\epsilon)^2 - 2\,t\,(l_m - 2\,\epsilon)\,d\,(y_{i_2}, z).$$

Thus,

$$p_i - c_i < \Delta(c) + t\,(l_M + 2\,\epsilon)^2 - t\,(l_m - 2\,\epsilon)^2.$$

Hence, from (2.65) we obtain

$$\pi_i(\tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*}), \mathbf{C}) < k_i\,k_M\,(l_M + \epsilon)\,\left(\Delta(c) + t\,(l_M + 2\,\epsilon)^2 - t\,(l_m - 2\,\epsilon)^2\right).$$

Hence,

$$\pi_i(\tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*}), \mathbf{C}) < k_i\,k_M\,(l_M + 2\,\epsilon)\,\left(\Delta(c) + t\,(l_M + 2\,\epsilon)^2 - t\,(l_m - 2\,\epsilon)^2\right).$$

By the SB condition,

$$\pi_i(\tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*}), \mathbf{C}) < \frac{k_i\,(2\,t\,(l_m + 2\,\epsilon)^2 - \Delta(c))^2}{8\,t\,(l_M + 2\,\epsilon)}. \tag{2.66}$$

By inequalities (2.64) and (2.66), $\pi_i^L > \pi_i(\tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*}), \mathbf{C})$, which contradicts the fact that p_i is the best response to $\mathbf{P}^L$. Therefore, $z \in \mathcal{N}(i, 1)$ and $M(i, \tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*})) \subset \mathcal{N}(i, 1)$. Hence, $\tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*}) = p_i^L$ and so $\mathbf{P}^L$ is a Nash equilibrium. □

We are going to present an example satisfying the WB condition but not the SB condition. Furthermore, we will show that in this example the local optimal prices do not form a Nash price equilibrium. Consider the Hotelling town network presented in figure 2.4. The parameter values are $\epsilon_i = 0$, $c_i = 0$,

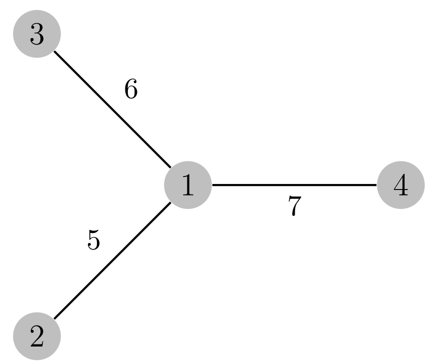


Figure 2.4: Star Network not satisfying the SB condition

$l_m = 5$, $l_M = 7$, $\Delta_2(l) = 24$ and $k_M = 3$. Hence, Network 2.4 satisfies the WB condition. By Proposition 2.2.1, the local optimal equilibrium prices are

$$\mathbf{P}^L = t\left(\frac{3780}{107}, \frac{6455}{214}, \frac{3816}{107}, \frac{9023}{214}\right)$$

and the correspondent profits are

$$\pi^L = t\left(\frac{34020}{107}, \frac{8333405}{91592}, \frac{1213488}{11449}, \frac{11630647}{91592}\right).$$

We will show that the local optimum price strategy is not a Nash equilibrium. The profits of the firms are given by $\pi_i^L = p_i\, S(i, \mathbf{P}^L)$, and the local market sizes $S(i, \mathbf{P}^L)$ are

$$S(i, \mathbf{P}^L) = \frac{\pi_i^L}{p_i^L} = \frac{l_i\, p_i^L}{2\,t}$$

Hence, the local market sizes are

$$S(1, \mathbf{P}^L) = 9; \quad S(2, \mathbf{P}^L) = \frac{6455}{2140}; \quad S(3, \mathbf{P}^L) = \frac{3816}{1284}; \quad S(4, \mathbf{P}^L) = \frac{9023}{2996}.$$

Suppose that firm F_2 decides to lower its price in order to capture the market of firm F_1. The firm F_2 captures the market of F_1, excluding F_1 from the game, if the firm F_2 charges a price p_2 such that $p_2 + 25\,t < p_1^L$ or, equivalently

$p_2 < 1105\,t/107$. Let us consider $p_2 = 1105\,t/107 - \delta$, where δ is sufficiently small. Hence, for this new price, firm F_2 keeps the market $M(2, \mathbf{P}^L)$ and, since the price of F_2 at location of F_1 is less that p_1^L, firm F_2 gains at least the market of firm F_1. Thus, the new market $M(2, \mathbf{P})$ of firm F_2 is such that $S(2, \mathbf{P}) > S(1, \mathbf{P}^L) + S(2, \mathbf{P}^L)$. Therefore, $S(2, \mathbf{P}) > 5143/428$ and so

$$\pi_2 > p_2\, S(2, \mathbf{P}) = \left(\frac{1105}{107}\, t - \delta\right) \frac{5143}{428} = \frac{11366030}{91592}\, t - \frac{5143}{428}\, \delta.$$

Thus $\pi_2 > \dfrac{8333405}{91592}\, t = \pi_2^L$, and so firm F_2 prefers to alter its price p_2^L. Therefore, $\mathbf{P}^L$ is not a Nash equilibrium price.

2.2.3 Space bounded information

The notation in this subsection has already been introduced in subsection 2.2.3. However, we duplicate the information in order to guarantee the independence of the sections.

Given $m+1$ vertices $x_0, \ldots, x_m$ with the property that there are roads $R_{x_0,x_1}, \ldots, R_{x_{m-1},x_m}$ the *(ordered) m path R* is

$$R = (R_{x_0,x_1}, \ldots, R_{x_{m-1},x_m}).$$

Let $\mathcal{R}(i, j; m)$ be the set of all m (ordered) paths $R = (R_{x_0,x_1}, \ldots, R_{x_{m-1},x_m})$ starting at $i = x_0$ and ending at $j = x_m$. Given a m order path $R = (R_{x_0,x_1}, \ldots, R_{x_{m-1},x_m})$, the corresponding *weight* is

$$k(R) = \prod_{q=0}^{m-1} k_{x_q,x_{q+1}}.$$

The matrix $\mathbf{K}^0$ is the identity matrix and, for $n \geq 1$, the elements of the matrix $\mathbf{K}^m$ are

$$k^m_{i,j} = \sum_{R \in \mathcal{R}(i,j;m)} k(R).$$

Definition 2.2.3. *A Hotelling town has* n space bounded information (n-I) *if for every* $1 \leq m \leq n$*, for every firm* F_i *and for every non-empty set* $\mathcal{R}(i,j;m)$*: (i) firm* F_i *knows the cost* c_j *and the average length road* L_j *and the firm deviation* Y_j *of firm* F_j*; (ii) for every* m *path* $R \in \mathcal{R}(i,j;m)$*, firm* F_i *knows the corresponding weight* $k(R)$*.*

The n *local optimal price vector* is

$$\mathbf{P}(n) = \sum_{m=0}^{n} 2^{-(m+1)} \, \mathbf{K}^m \left(\mathbf{C} + t\left(\mathbf{L} + \mathbf{Y}\right)\right).$$

We observe that in a n-I Hotelling town, the firms might not be able to compute $\mathbf{K}$, $\mathbf{C}$, $\mathbf{L}$ or $\mathbf{Y}$. However, every firm F_i is able to compute his n local optimal price $p_i(n)$

$$p_i(n) = \sum_{m=0}^{n} 2^{-(m+1)} \sum_{v \in V} k^m_{i,v} \left(c_v + t\left(L_v + Y_v\right)\right).$$

By (2.52), the best response $\mathbf{P}'$ to $\mathbf{P}(n)$ is given by

$$\begin{aligned}
\mathbf{P}' &= \frac{1}{2}\left(\mathbf{C} + t\left(\mathbf{L} + \mathbf{Y}\right)\right) + \frac{1}{2}\mathbf{K}\,\mathbf{P}(n) \\
&= \frac{1}{2}\left(\mathbf{C} + t\left(\mathbf{L} + \mathbf{Y}\right)\right) + \sum_{m=0}^{n} 2^{-(m+2)}\mathbf{K}^{m+1}\left(\mathbf{C} + t\left(\mathbf{L} + \mathbf{Y}\right)\right) \\
&= \sum_{m=0}^{n+1} 2^{-(m+1)}\mathbf{K}^{m}\left(\mathbf{C} + t\left(\mathbf{L} + \mathbf{Y}\right)\right) = \mathbf{P}(n+1).
\end{aligned}$$

Hence, $\mathbf{P}(n+1)$ is the best response to $\mathbf{P}(n)$ for n sufficiently large.

Let G denote the number of nodes in the network and let

$$e = \frac{\Delta(c) + t\,(l_M + 2\,\epsilon)^2}{l_m - 2\,\epsilon} + 2\,t\,(l_M + 2\,\epsilon).$$

Theorem 2.2.2. *A Hotelling town satisfying the WB condition has a local optimum price strategy $\boldsymbol{P}^L$ that is well approximated by the n local optimal price $\boldsymbol{P}(n)$ with the following 2^{-n} bound*

$$0 \leq p_i^L - p_i(n) \leq 2^{-(n+1)} G\,(c_M + t\,(l_M + 2\,\epsilon)^2).$$

The profit $\pi_i(\boldsymbol{P}^L)$ is well approximated by $\pi_i(\boldsymbol{P}(n))$ with the following bound

$$|\pi_i(\boldsymbol{P}^L) - \pi_i(\boldsymbol{P}(n))| \leq 2^{-(n+2)}\,G\,t^{-1}\,(c_M + t\,(l_M + 2\,\epsilon)^2)\,\left(k_i\,e + 4\,t\,\epsilon\right).$$

Proof. By Proposition 2.2.1, if a Hotelling town satisfies the WB condition then there is local optimum price strategy $\mathbf{P}^L$ given by

$$\mathbf{P}^L = \sum_{m=0}^{\infty} 2^{-(m+1)} \mathbf{K}^m \left(\mathbf{C} + t\,(\mathbf{L} + \mathbf{Y})\right).$$

Considering $\mathbf{Q} = \sum_{m=0}^{\infty} 2^{-(m+1)} \mathbf{K}^m$, we can write the equilibrium prices as

$$p_i^L = \sum_{v \in V} Q_{i,v}\,(c_v + t\,(L_v + Y_v)), \qquad \text{where} \qquad Q_{i,v} = \sum_{m=0}^{\infty} 2^{-(m+1)} k_{i,v}^m.$$

For the space bounded information Hotelling town, the n local optimal price $\mathbf{P}(n)$ is given by

$$\mathbf{P}(n) = \sum_{m=0}^{n} 2^{-(m+1)} \mathbf{K}^m \left(\mathbf{C} + t\,(\mathbf{L} + \mathbf{Y})\right)$$

and

$$p_i(n) = \sum_{v \in V} Q_{i,v}(n)\,(c_v + t\,(L_v + Y_v)), \quad \text{where} \quad Q_{i,v}(n) = \sum_{m=0}^{n} 2^{-(m+1)} k_{i,v}^m.$$

The difference $R_i(n)$ between p_i^L and $p_i(n)$ is positive and is given by

$$R_i(n) = \sum_{v \in V} (Q_{i,v} - Q_{i,v}(n))\,(c_v + t\,(L_v + Y_v)).$$

We note that

$$Q_{i,v} - Q_{i,v}(n) = \sum_{m=n+1}^{\infty} 2^{-(m+1)} k_{i,v}^m.$$

Since $0 \leq k_{i,v}^m \leq 1$, for all $m \in \mathbb{N}$ and all $i, v \in V$ and

$$\sum_{m=n+1}^{\infty} 2^{-(m+1)} = 2^{-(n+1)},$$

we have that

$$Q_{i,v} - Q_{i,v}(n) \leq 2^{-(n+1)}.$$

Hence,

$$R_i(n) \leq \sum_{v \in V} 2^{-(n+1)}\,(c_v + t\,(L_v + Y_v)).$$

Since $L_v \leq l_M\,(l_M + 2\,\epsilon)$, $Y_v \leq 2\,\epsilon\,(l_M + 2\,\epsilon)$ and $c_v \leq c_M$, we have that

$$R_i(n) \leq 2^{-(n+1)}\,G\,(c_M + t\,(l_M + 2\,\epsilon)^2). \tag{2.67}$$

Therefore,

$$0 \leq p_i^L - p_i(n) \leq 2^{-(n+1)}\,G\,(c_M + t\,(l_M + 2\,\epsilon)^2).$$

The profit for firm F_i for the local optimal price is given by

$$\pi_i(\mathbf{P}^L) = (2t)^{-1}\,(p_i^L - c_i)\left(2\,t\,(2-k_i)\,\epsilon_i + \sum_{j\in N_i}\frac{p_j^L - p_i^L + t\,\tilde{l}_{i,j}^2}{\tilde{l}_{i,j}}\right) \qquad (2.68)$$

and the profit for firm F_i when all firms have n-space bounded information is

$$\pi_i(\mathbf{P}(n)) = (2t)^{-1}\,(p_i(n) - c_i)\left(2\,t\,(2-k_i)\,\epsilon_i + \sum_{j\in N_i}\frac{p_j(n) - p_i(n) + t\,\tilde{l}_{i,j}^2}{\tilde{l}_{i,j}}\right)$$

Let $R_{j,i}(n) = R_j(n) - R_i(n)$ and

$$\begin{aligned} Z_i &= 2\,t\,(2-k_i)\,\epsilon_i + \sum_{j\in N_i}\frac{p_j(n) - p_i(n) + R_{j,i}(n) + t\,\tilde{l}_{i,j}^2}{\tilde{l}_{i,j}} \\ &= 2\,t\,(2-k_i)\,\epsilon_i + \sum_{j\in N_i}\frac{p_j^L - p_i^L + t\,\tilde{l}_{i,j}^2}{\tilde{l}_{i,j}}. \end{aligned}$$

Since $p_i^L = p_i(n) + R_i(n)$, we can write the local equilibrium profit (2.68) for firm i as

$$\pi_i(\mathbf{P}^L) = (2t)^{-1}\,(p_i(n) - c_i + R_i(n))\,Z_i$$

Hence,

$$\pi_i(\mathbf{P}^L) = \pi_i(\mathbf{P}(n)) + (2t)^{-1}\left((p_i(n) - c_i)\sum_{j\in N_i}\frac{R_{j,i}(n)}{\tilde{l}_{i,j}} + R_i(n)\,Z_i\right)$$

The difference between the equilibrium profit and the profit where all firms have n-space bounded information is

$$\pi_i(\mathbf{P}^L) - \pi_i(\mathbf{P}(n)) = (2t)^{-1}\left((p_i(n) - c_i)\sum_{j\in N_i}\frac{R_{j,i}(n)}{\tilde{l}_{i,j}} + R_i(n)\,Z_i\right).$$

Hence,

$$|\pi_i(\mathbf{P}^L) - \pi_i(\mathbf{P}(n))| \leq (2t)^{-1} \left((p_i(n) - c_i) \sum_{j \in N_i} \frac{|R_{j,i}(n)|}{\tilde{l}_{i,j}} + R_i(n)\, Z_i \right).$$

Since

$$\frac{p_j^L - p_i^L + t\, \tilde{l}_{i,j}^2}{\tilde{l}_{i,j}} \leq 2\, t\, \tilde{l}_{i,j} \leq 2\, t\, (l_M + 2\, \epsilon),$$

we have

$$Z_i \leq 2\, t\, (2 - k_i)\, \epsilon_i + 2\, t\, k_i\, (l_M + 2\, \epsilon) < 2\, t\, (k_i\, (l_M + 2\, \epsilon) + 2\, \epsilon_i).$$

Let $Z = \Delta(c) + t\, (l_M + 2\, \epsilon)^2$. Since $p_i(n) - c_i \leq p_i^L - c_i$, from (2.55) we have $p_i(n) - c_i \leq \Delta(c) + t\, (l_M + 2\, \epsilon)^2 = Z$. Hence,

$$|\pi_i(\mathbf{P}^L) - \pi_i(\mathbf{P}(n))| < (2t)^{-1} \left(Z \sum_{j \in N_i} \frac{|R_{j,i}(n)|}{\tilde{l}_{i,j}} + 2\, t\, R_i(n)\, (k_i\, (l_M + 2\, \epsilon) + 2\, \epsilon) \right)$$

Let $Z_M = c_M + t\, (l_M + 2\, \epsilon)^2$. By (2.67), $R_i(n) \leq 2^{-(n+1)}\, G\, Z_M$. Then, also, $|R_{j,i}(n)| \leq 2^{-(n+1)}\, G\, Z_M$. Therefore,

$$\sum_{j \in N_i} \frac{|R_{j,i}(n)|}{\tilde{l}_{i,j}} \leq 2^{-(n+1)}\, \frac{k_i}{l_m - 2\, \epsilon}\, G\, Z_M.$$

We note that

$$\frac{Z}{l_m - 2\, \epsilon} + 2\, t\, (l_M + 2\, \epsilon) = e.$$

Hence,

$$|\pi_i(\mathbf{P}^L) - \pi_i(\mathbf{P}(n))| \leq 2^{-(n+2)}\, G\, t^{-1}\, Z_M\, \left(k_i\, e + 4\, t\, \epsilon\right).$$

□

2.3 Different transportation costs

This section extends the Hotelling model with different linear transportation costs to networks. For simplicity of notation, we assume that the firms are located at the nodes of the network.

A consumer located at a point x of the network who decides to buy at firm F_i *spends*

$$E(x; i, \mathbf{P}) = p_i + t_i\, d(x, i)$$

the price p_i charged by the firm F_i plus the *transportation cost* that is proportional t_i to the minimal distance measured in the network between the position i of the firm F_i and the position x of the consumer.

2.3.1 Local optimal equilibrium price strategy

We observe that, for every road $R_{i,j}$ there is an *indifferent consumer* located at a distance

$$0 < x_{i,j} = \frac{p_j - p_i + t_j\, l_{i,j}}{t_i + t_j} < l_{i,j}$$

of firm F_i if and only if $-t_i\, l_{i,j} < p_i - p_j < t_j\, l_{i,j}$. Thus, a price strategy $\mathbf{P}$ determines a local market structure if and only if

$$-t_i\, l_{i,j} < p_i - p_j < t_j\, l_{i,j} \tag{2.69}$$

for every road $R_{i,j}$.

If the price strategy determines a local market structure then

$$S(i, \mathbf{P}) = \sum_{j \in N_i} x_{i,j}$$

and

$$\begin{aligned} \pi_i(\mathbf{P},\mathbf{C}) &= (p_i - c_i)\, S(i,\mathbf{P}) \\ &= (p_i - c_i) \sum_{j \in N_i} \frac{p_j - p_i + t_j\, l_{i,j}}{t_i + t_j}. \end{aligned} \tag{2.70}$$

Given a pair of price strategies $\mathbf{P}$ and $\mathbf{P}^*$ and a firm F_i, we define the price vector $\tilde{\mathbf{P}}(i,\mathbf{P},\mathbf{P}^*)$ whose coordinates are $\tilde{p}_i = p_i^*$ and $\tilde{p}_j = p_j$, for every $j \in V \setminus \{i\}$. Let $\mathbf{P}$ and $\mathbf{P}^*$ be price strategies that determine local market structures. The price strategy $\mathbf{P}^*$ is a *local best response* to the price strategy $\mathbf{P}$, if for every $i \in V$ the price strategy $\tilde{\mathbf{P}}(i,\mathbf{P},\mathbf{P}^*)$ determines a local market structure and

$$\frac{\partial \pi_i(\tilde{\mathbf{P}}(i,\mathbf{P},\mathbf{P}^*),\mathbf{C})}{\partial \tilde{p}_i} = 0 \quad \text{and} \quad \frac{\partial^2 \pi_i(\tilde{\mathbf{P}}(i,\mathbf{P},\mathbf{P}^*),\mathbf{C})}{\partial \tilde{p}_i^2} < 0.$$

Let $T_i = \sum_{j \in N_i} \dfrac{1}{t_i + t_j}$. The Hotelling town *admissible market size* $\mathbf{L}$ is the vector whose coordinates are the *admissible local firm market sizes*

$$L_i = T_i^{-1} \sum_{j \in N_i} \frac{t_j\, l_{i,j}}{t_i + t_j}.$$

The Hotelling town *neighboring market structure* $\mathbf{K}$ is the matrix whose elements are (i) $k_{i,j} = \dfrac{1}{T_i\,(t_i + t_j)}$, if there is a road $R_{i,j}$ between the firms F_i and F_j; and (ii) $k_{i,j} = 0$, if there is not a road $R_{i,j}$ between the firms F_i and F_j. Let $\mathbf{1}$ denote the identity matrix.

Lemma 2.3.1. *Let $\boldsymbol{P}$ and $\boldsymbol{P}^*$ be price strategies that determine local market structures. The price strategy $\boldsymbol{P}^*$ is the local best response to price strategy $\boldsymbol{P}$ if and only if*

$$\boldsymbol{P}^* = \frac{1}{2}\,(\boldsymbol{C} + \boldsymbol{L}) + \frac{1}{2}\,\boldsymbol{K}\,\boldsymbol{P} \tag{2.71}$$

and the price strategies $\tilde{\boldsymbol{P}}(i,\boldsymbol{P},\boldsymbol{P}^*)$ determine local market structures for all

$i \in V$.

Proof. By (2.70), the *profit function* $\pi_i(\mathbf{P}, \mathbf{C})$ of firm F_i, in a local market structure, is given by

$$\pi_i(\mathbf{P}, \mathbf{C}) = (p_i - c_i) \sum_{j \in N_i} \frac{p_j - p_i + t_j\, l_{i,j}}{t_i + t_j}.$$

Let $\tilde{\mathbf{P}}(i, \mathbf{P}, \mathbf{P}^*)$ be the price vector whose coordinates are $\tilde{p}_i = p_i^*$ and $\tilde{p}_j = p_j$, for every $j \in V \setminus \{i\}$. Since $\mathbf{P}$ and $\mathbf{P}^*$ are local price strategies, the local best response of firm F_i to the price strategy $\mathbf{P}$, is given by computing $\partial \pi_i(\tilde{\mathbf{P}}(i, \mathbf{P}, \mathbf{P}^*), \mathbf{C})/\partial \tilde{p}_i = 0$. Hence,

$$p_i^* = \frac{1}{2} \left(c_i + \frac{1}{T_i} \sum_{j \in N_i} \frac{p_j + t_j\, l_{i,j}}{t_i + t_j} \right). \tag{2.72}$$

Therefore, since $\partial^2 \pi_i(\tilde{\mathbf{P}}(i, \mathbf{P}, \mathbf{P}^*), \mathbf{C})/\partial \tilde{p}_i^2 = -2\, T_i < 0$, the local best response strategy prices $\mathbf{P}^*$ is given by

$$\mathbf{P}^* = \frac{1}{2} \left(\mathbf{C} + \mathbf{L} + \mathbf{K}\, \mathbf{P} \right).$$

□

We denote by t_M (resp. t_m) the maximum (resp. minimum) transportation cost of the Hotelling town

$$t_M = \max\{t_i : i \in V\} \quad \text{and} \quad t_m = \min\{t_i : i \in V\}.$$

Let $\Delta(t) = t_M - t_m$.

Definition 2.3.1. *A Hotelling town satisfies the* weak bounded length and costs (WB) *condition, if*

$$\Delta(c) + \frac{l_M\, t_M^3 - l_m\, t_m^3}{t_m\, t_M} < t_m\, l_m.$$

Let $\mathbf{P}$ and $\mathbf{P}^*$ be price strategies that determine local market structures. A price strategy $\mathbf{P}^*$ is a *local optimum equilibrium* if $\mathbf{P}^*$ is the local best response to $\mathbf{P}^*$.

Proposition 2.3.1. *If the Hotelling town satisfies the WB condition, then there is unique local optimum price strategy given by*

$$\boldsymbol{P}^L = \frac{1}{2}\left(\boldsymbol{1} - \frac{1}{2}\boldsymbol{K}\right)^{-1}(\boldsymbol{C} + \boldsymbol{L}) = \sum_{m=0}^{\infty} 2^{-(m+1)} \boldsymbol{K}^m (\boldsymbol{C} + \boldsymbol{L}) . \tag{2.73}$$

The local optimum price strategy $\boldsymbol{P}^L$ determines a local market structure. Furthermore, the local optimal equilibrium prices p_i^L are bounded by

$$\frac{l_m\, t_m^2}{t_M} + \frac{1}{2}(c_i + c_m) \le p_i^L \le \frac{l_M\, t_M^2}{t_m} + \frac{1}{2}(c_i + c_M) \tag{2.74}$$

The local optimal profit $\pi_i^L = \pi_i^L(\boldsymbol{P}, \boldsymbol{C})$ of firm F_i is given by

$$\pi_i^L(\boldsymbol{P}, \boldsymbol{C}) = T_i\,(p_i^L - c_i)^2$$

and it is bounded by

$$\frac{k_i}{t_i + t_M}\left(\frac{l_m\, t_m^2}{t_M} - \frac{\Delta(c)}{2}\right)^2 \le \pi_i^L \le \frac{k_i}{t_i + t_m}\left(\frac{l_M\, t_M^2}{t_m} + \frac{\Delta(c)}{2}\right)^2 .$$

Proof. Let, first prove that $\mathbf{K}$ is a stochastic matrix (i.e., $\sum_{j \in V} k_{i,j} = 1$, for every $i \in V$). Since

$$T_i = \sum_{j \in N_i} \frac{1}{t_i + t_j} \qquad \text{and} \qquad k_{i,j} = \frac{1}{T_i\,(t_i + t_j)}$$

we have

$$\begin{aligned}\sum_{j\in V} k_{i,j} &= \sum_{j\in N_i} k_{i,j} = \sum_{j\in N_i} T_i^{-1}\frac{1}{t_i+t_j} = T_i^{-1}\sum_{j\in N_i}\frac{1}{t_i+t_j}\\ &= \frac{1}{\sum_{j\in N_i}\frac{1}{t_i+t_j}}\sum_{j\in N_i}\frac{1}{t_i+t_j} = 1.\end{aligned}$$

Then $\mathbf{K}$ is a stochastic matrix, and we have $\|\mathbf{K}\| = 1$. Hence, the matrix Q is well-defined by

$$\mathbf{Q} = \frac{1}{2}\left(\mathbf{1} - \frac{1}{2}\mathbf{K}\right)^{-1} = \sum_{m=0}^{\infty} 2^{-(m+1)}\,\mathbf{K}^m$$

and Q is also a non-negative and stochastic matrix. By Lemma 2.3.1, a local optimum price strategy satisfy equality (2.71). Therefore,

$$\mathbf{P}^L = \frac{1}{2}\left(\mathbf{1} - \frac{1}{2}\mathbf{K}\right)^{-1}(\mathbf{C}+\mathbf{L}) = \sum_{m=0}^{\infty} 2^{-(m+1)}\,\mathbf{K}^m\,(\mathbf{C}+\mathbf{L}),$$

and so $\mathbf{P}^L$ satisfies (2.73). By construction,

$$p_i^L = \sum_{v\in V} Q_{i,v}(c_v + L_v). \tag{2.75}$$

Let us prove that the price strategy $\mathbf{P}^L$ is local, i.e., the indifferent consumer $x_{i,j}$ satisfies $0 < x_{i,j} < l_{i,j}$ for every $R_{i,j} \in E$.
We note that

$$\frac{k_v}{t_v+t_M} \le T_v = \sum_{j\in N_v}\frac{1}{t_v+t_j} \le \frac{k_v}{t_v+t_m} \tag{2.76}$$

Hence,

$$\frac{t_v+t_m}{k_v} \le T_v^{-1} \le \frac{t_v+t_M}{k_v}$$

and, therefore,

$$\frac{t_v + t_m}{k_v}\, k_v\, \frac{t_m\, l_m}{t_v + t_M} \leq L_v = T_v^{-1} \sum_{j \in N_v} \frac{t_j\, l_{v,j}}{t_v + t_j} \leq \frac{t_v + t_M}{k_v}\, k_v\, \frac{t_M\, l_M}{t_v + t_m}$$

Hence,

$$\frac{l_m\, t_m^2}{t_M} \leq \frac{t_m\, l_m\, (t_v + t_m)}{t_v + t_M} L_v \leq \frac{t_M\, l_M\, (t_v + t_M)}{t_v + t_m} \leq \frac{l_M\, t_M^2}{t_m}. \tag{2.77}$$

Since $\mathbf{Q}$ is a nonnegative and stochastic matrix, we obtain

$$\sum_{v \in V} Q_{i,v} \left(c_m + \frac{l_m\, t_m^2}{t_M} \right) = c_m + \frac{l_m\, t_m^2}{t_M}$$

and

$$\sum_{v \in V} Q_{i,v} \left(c_M + \frac{l_M\, t_M^2}{t_m} \right) = c_M + \frac{l_M\, t_M^2}{t_m}.$$

Hence, putting (2.75) and (2.77) together we obtain that

$$c_m + \frac{l_m\, t_m^2}{t_M} \leq p_i^L \leq c_M + \frac{l_M\, t_M^2}{t_m}.$$

Since the last relation is satisfied for every firm, we obtain

$$-\left(c_M - c_m + \frac{l_M\, t_M^2}{t_m} - \frac{l_m\, t_m^2}{t_M} \right) \leq p_i^L - p_j^L \leq c_M - c_m + \frac{l_M\, t_M^2}{t_m} - \frac{l_m\, t_m^2}{t_M}.$$

Therefore,

$$|p_i^L - p_j^L| \leq \Delta(c) + \frac{l_M\, t_M^3 - l_m\, t_m^3}{t_m\, t_M}.$$

Hence, by the WB condition, we conclude that

$$|p_i^L - p_j^L| < t_m\, l_m.$$

Thus, by equation (2.69), we obtain that the indifferent consumer is located at $0 < x_{i,j} < l_{i,j}$ for every road $R_{i,j} \in E$. Hence, the price strategy $\mathbf{P}^L$ is local and is the unique local optimum price strategy.

From (2.75) and (2.77), we obtain

$$p_i^L \geq \sum_{v \in V} Q_{i,v} \frac{l_m\, t_m^2}{t_M} + \sum_{v \in V \setminus \{i\}} Q_{i,v}\, c_m + Q_{i,i}\, c_i.$$

By construction of matrix $\mathbf{Q}$, we have $Q_{i,i} > 1/2$. Furthermore, since $\mathbf{Q}$ is stochastic,

$$\sum_{v \in V \setminus \{i\}} Q_{i,v} < 1/2, \quad \text{and} \quad \sum_{v \in V} Q_{i,v} \frac{l_m\, t_m^2}{t_M} = \frac{l_m\, t_m^2}{t_M}.$$

Hence,

$$p_i^L \geq \frac{l_m\, t_m^2}{t_M} + \frac{1}{2}\,(c_i + c_m).$$

Similarly, we obtain

$$p_i^L \leq \frac{l_M\, t_M^2}{t_m} + \frac{1}{2}\,(c_i + c_M),$$

and so the local optimal equilibrium prices p_i^L are bounded and satisfy (2.74).

We can write the the profit function (2.70) of firm F_i for the price strategy P^L as

$$\pi_i^L = \pi_i(\mathbf{P}^L, \mathbf{C}) = (p_i^L - c_i) \left(-p_i^L\, T_i + \sum_{j \in N_i} \frac{p_j^L + t_j\, l_{i,j}}{t_i + t_j} \right). \tag{2.78}$$

Since $\mathbf{P}^L$ satisfies the best response function (2.72), we have

$$2\, p_i^L = c_i + \frac{1}{T_i} \sum_{j \in N_i} \frac{p_j^L + t_j\, l_{i,j}}{t_i + t_j}.$$

Therefore,

$$\sum_{j \in N_i} \frac{p_j^L + t_j\, l_{i,j}}{t_i + t_j} = 2\, p_i^L\, T_i - c_i\, T_i,$$

and replacing this sum in the profit function (2.78), we obtain

$$\pi_i^L = (p_i^L - c_i)\left(-p_i^L\, T_i + 2\, p_i^L\, T_i - c_i\, T_i\right) = T_i\,(p_i^L - c_i)^2.$$

Hence, from (2.76), and using the price bounds (2.74), we conclude

$$\frac{k_i}{t_i + t_M}\left(\frac{l_m\, t_m^2}{t_M} - \Delta(c)/2\right)^2 \le \pi_i^L \le \frac{k_i}{t_i + t_m}\left(\frac{l_M\, t_M^2}{t_m} + \Delta(c)/2\right)^2.$$

□

Let $a \in V$, $R_{b,c} \in E$ and $d \in V \setminus \{i\}$. The marginal rates of the local optimal equilibrium prices p_i^L are positive with respect to the production costs c_a, admissible local firm market sizes L_a, transportation costs t and road lengths $l_{b,c}$. The marginal rates of the local optimal equilibrium profit π_i^L are negative with respect to the production costs c_i and positive with respect to the production costs c_d, admissible local firm market sizes L_a, transportation costs t and road lengths $l_{b,c}$.

2.3.2 Nash equilibrium price strategy

The price strategy $\mathbf{P}^*$ is a *best response* to the price strategy $\mathbf{P}$, if

$$(\tilde{p}_i - c_i)\, S(i, \tilde{\mathbf{P}}(i, \mathbf{P}, \mathbf{P}^*)) \ge (p_i' - c_i)\, S(i, \mathbf{P}_i'),$$

for all $i \in V$ and for all price strategies $\mathbf{P}_i'$ whose coordinates satisfy $p_i' \ge c_i$ and $p_j' = p_j$ for all $j \in V \setminus \{i\}$. A price strategy $\mathbf{P}^*$ is a Hotelling town *Nash equilibrium* if $\mathbf{P}^*$ is the best response to $\mathbf{P}^*$.

Lemma 2.3.2. *In a Hotelling town satisfying the WB condition, if there is a Nash price $\boldsymbol{P}^*$ then $\boldsymbol{P}^*$ is unique and $\boldsymbol{P}^* = \boldsymbol{P}^L$.*

Hence, the local optimum price strategy $\mathbf{P}^L$ is the only candidate to be a Nash equilibrium price strategy. However, $\mathbf{P}^L$ might not be a Nash equilibrium price strategy because there can be a firm F_i that by decreasing his price is able to absorb markets of other firms in such a way that increases its own profit. Therefore, the best response price strategy $\mathbf{P}^{L,*}$ to the local optimum price strategy $\mathbf{P}^L$ might be different from $\mathbf{P}^L$.

Proof of Lemma 2.3.2.
Suppose that P^* is a Nash price strategy and that $\mathbf{P}^* \neq \mathbf{P}^L$. Hence, $\mathbf{P}^*$ does not determine a local market structure, i.e., there exists $i \in V$ such that

$$M(i, \mathbf{P}^*) \not\subset \cup_{j \in N_i} R_{i,j}.$$

Hence, there exists $j \in N_i$ such that $M(j, \mathbf{P}^*) = 0$ and, therefore, $\pi_j^* = 0$. Moreover, in this case, we have that

$$p_j^* > p_i^* + t_i\, l_{i,j}.$$

Consider, now, that F_j changes his price to

$$p_j = c_j + \frac{l_M\, t_M^3 - l_m\, t_m^3}{t_m\, t_M}.$$

Since $p_i^* > c_i$ and $c_j - c_i \leq \Delta(c)$ we have that

$$p_j - p_i^* < p_j - c_i = c_j + \frac{l_M\, t_M^3 - l_m\, t_m^3}{t_m\, t_M} - c_i \leq \Delta(c) + \frac{l_M\, t_M^3 - l_m\, t_m^3}{t_m\, t_M}.$$

Since the Hotelling town satisfies the WB condition, we obtain

$$p_j - p_i^* < t_m\, l_m \leq t_i\, l_{i,j}.$$

Hence, $M(j, \tilde{\mathbf{P}}(j, \mathbf{P}^*, \mathbf{P})) > 0$ and

$$\pi_j = \left(\frac{l_M\, t_M^3 - l_m\, t_m^3}{t_m\, t_M} \right) S(j, \tilde{\mathbf{P}}(j, \mathbf{P}^*, \mathbf{P})) > 0.$$

Therefore, F_j will change its price and so $\mathbf{P}^*$ is not a Nash equilibrium price strategy. Hence, if there is a Nash price $\mathbf{P}^*$ then $\mathbf{P}^* = \mathbf{P}^L$. □

Definition 2.3.2. *A Hotelling town satisfies the $WB1$ condition, if*

$$\Delta(c) + \frac{l_M\, t_M^2}{t_m} - l_m\, t_m + \Delta(t)\, l_M \leq l_m\, t_m.$$

Let $\cup_{j \in N_i} R_{i,j}$ be the *1-neighbourhood* $\mathcal{N}(i, 1)$ of a firm $i \in V$. Let $\cup_{j \in N_i} \cup_{k \in N_j} R_{j,k}$ be the *2-neighbourhood* $\mathcal{N}(i, 2)$ of a firm $i \in V$.

Lemma 2.3.3. *In a Hotelling town satisfying the $WB1$ condition,*

$$M(i, \tilde{\boldsymbol{P}}(i, \boldsymbol{P}^L, \boldsymbol{P}^{L,*})) \subset \mathcal{N}(i, 2)$$

for every $i \in V$.

Hence, a consumer $x \in R_{j,k}$ might not buy in its local firms F_j and F_k. However, the consumer $x \in R_{j,k}$ still has to buy in a firm F_i that is a neighboring firm of its local firms F_j and F_k, i.e. $i \in N_j \cup N_k$.

Proof of Lemma 2.3.3.
By contradiction, let us consider a consumer $z \in M(i, \tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*}))$ and $z \notin \mathcal{N}(i, n)$, with $n \geq 2$. The price that consumer z pays to buy in firm F_i is given by

$$e = p_i + t_i \left(\sum_{j=1}^{n} l_{i_j, i_{j+1}} + d\,(i_{n+1}, z) \right)$$

where $p_i = p_i^{L,*}$ is the coordinate of the vector $\tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*})$ and for the n-path $(R_{i_1,i_2}, R_{i_2,i_3}, \ldots, R_{i_n,i_{n+1}})$ with $i_1 = i$. If the consumer z buys at firm

$F_{i_{n+1}}$, then the price that has to pay is

$$\tilde{e} = p^L_{i_{n+1}} + t_{i_{n+1}}\, d\,(i_{n+1}, z).$$

Since, by hypothesis, $z \in M(i, \tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*}))$, we have $e < \tilde{e}$. Therefore

$$p_i < p^L_{i_{n+1}} - t_i \left(\sum_{j=1}^{n} l_{i_j, i_{j+1}} \right) + (t_{i_{n+1}} - t_i)\, d\,(i_{n+1}, z).$$

By inequality (2.74),

$$p^L_i \leq \frac{l_M\, t^2_M}{t_m} + \frac{1}{2}(c_M + c_i)$$

for all $i \in V$. Since $l_{i,j} \geq l_m$ for all $R_{i,j} \in E$,

$$p_i < \frac{l_M\, t^2_M}{t_m} + \frac{1}{2}(c_M + c_{i_{n+1}}) - n\, t_i\, l_m + (t_{i_{n+1}} - t_i)\, d\,(i_{n+1}, z).$$

Since $n \geq 2$, $d\,(i_{n+1}, z) < l_M$ and $t_{i_{n+1}} - t_i \leq \Delta(t)$ we obtain that

$$p_i < \frac{l_M\, t^2_M}{t_m} + \frac{1}{2}(c_M + c_{i_{n+1}}) - 2\, t_i\, l_m + \Delta(t)\, l_M.$$

Since $t_i \geq t_m$ and $c_m \leq c_i \leq c_M$ for all $i \in V$ we conclude that

$$p_i - c_i < \Delta(c) + \frac{l_M\, t^2_M}{t_m} - 2\, t_m\, l_m + \Delta(t)\, l_M.$$

By the $WB1$ condition, $p_i - c_i < 0$. Hence, $\pi^{L,*}_i < 0$ which contradicts the fact that p_i is the best response to $\mathbf{P}^L$ (since $\pi^L_i > 0$). Therefore, $z \in \mathcal{N}(i, 2)$ and $M(i, \tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*})) \subset \mathcal{N}(i, 2)$. □

Definition 2.3.3. *A Hotelling town satisfies the* strong bounded length and costs (SB) *condition, if*

$$\Delta(c) + \frac{l_M\, t^2_M}{t_m} - l_m\, t_m + \Delta(t)\, l_M \leq \frac{\left(2\, l_m\, t^2_m - \Delta(c)\, t_M\right)^2}{4\, t^2_M\, l_M\, k_M\, (t_m + t_M)}.$$

The SB condition implies the WB condition, and so under the SB condition the only candidate to be a Nash equilibrium price strategy is the local optimum strategy price $\mathbf{P}^L$.

Theorem 2.3.1. *If a Hotelling town satisfies the SB condition then there is a unique Hotelling town Nash equilibrium price strategy $\boldsymbol{P}^* = \boldsymbol{P}^L$.*

Hence, the Nash equilibrium price strategy for the Hotelling town satisfying the SB condition determines a local market structure, i.e. every consumer located at $x \in R_{i,j}$ spends less by shopping at his local firms F_i or F_j than in any other firm in the town and so the consumer at x will buy either at his local firm F_i or at his local firm F_j.

Proof of Theorem 2.3.1.
By Proposition 2.3.1 and Lemma 2.3.2, if there is a Nash equilibrium price strategy $\mathbf{P}^*$ then $\mathbf{P}^*$ is unique and $\mathbf{P}^* = \mathbf{P}^L$.

We note that if $M(i, \tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*})) \subset \mathcal{N}(i, 1)$ for every $i \in V$ then $\tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*}) = p_i^L$ and so $\mathbf{P}^L$ is a Nash equilibrium.
We note that the SB condition implies the $WB1$ condition. Hence, by Lemma 2.3.3, we have that $M(i, \tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*})) \subset \mathcal{N}(i, 2)$ for every $i \in V$. Now, we will prove that the SB condition implies that firm F_i earns more competing only in the 1-neighborhood than competing in a 2-neighborhood. By Proposition 2.3.1

$$\pi_i^L \geq \frac{k_i}{t_i + t_M} \left(\frac{l_m \, t_m^2}{t_M} - \frac{\Delta(c)}{2} \right)^2 \tag{2.79}$$

By Lemma 2.3.3,

$$\pi_i(\tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*}), \mathbf{C}) \leq (p_i - c_i) \sum_{j \in N_i} \sum_{k \in N_j} l_{j,k} \leq (p_i - c_i) k_i \, k_M \, l_M \tag{2.80}$$

where $p_i = p_i^{L,*}$ is the coordinate of the vector $\tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*})$.

By contradiction, let us consider a consumer $z \in M(i, \tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*}))$ and $z \notin \mathcal{N}(i, 1)$. Let $i_2 \in N_i$ be the vertex such that $z \in \mathcal{N}(i_2, 1)$. The price that consumer z pays to buy in firm F_i is given by

$$e = p_i + t_i \, (l_{i,i_2} + d\,(i_2, z)).$$

If the consumer z buys at firm F_{i_2}, then the price that has to pay is

$$\tilde{e} = p^L_{i_2} + t_{i_2} \, d\,(i_2, z).$$

Since, by hypothesis, $z \in M(i, \tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*}))$, we have $e < \tilde{e}$. Therefore

$$p_i < p^L_{i_2} - t_i \, l_{i,i_2} + (t_{i_2} - t_i) \, d\,(i_2, z).$$

By inequality (2.74),

$$p^L_i \leq \frac{l_M \, t^2_M}{t_m} + \frac{1}{2}(c_M + c_i)$$

for all $i \in V$. Since $l_{i,j} \geq l_m$ for all $R_{i,j} \in E$,

$$p_i < \frac{l_M \, t^2_M}{t_m} + \frac{1}{2}(c_M + c_{i_2}) - t_i \, l_m + (t_{i_2} - t_i) \, d\,(i_2, z).$$

Since $d\,(i_2, z) < l_M$ and $t_{i_2} - t_i \leq \Delta(t)$ we obtain that

$$p_i < \frac{l_M \, t^2_M}{t_m} + \frac{1}{2}(c_M + c_{i_2}) - t_i \, l_m + \Delta(t) \, l_M.$$

Since $t_i \geq t_m$ and $c_m \leq c_i \leq c_M$ for all $i \in V$ we conclude that

$$p_i - c_i < \Delta(c) + \frac{l_M \, t^2_M}{t_m} - t_m \, l_m + \Delta(t) \, l_M.$$

Hence, from (2.80) we obtain

$$\pi_i(\tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*}), \mathbf{C}) < k_i\, k_M\, l_M\, \left(\Delta(c) + \frac{l_M\, t_M^2}{t_m} - t_m\, l_m + \Delta(t)\, l_M\right).$$

By the SB condition,

$$\pi_i(\tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*}), \mathbf{C}) < \frac{k_i\, \left(2\, l_m\, t_m^2 - \Delta(c)\, t_M\right)^2}{4\, t_M^2\, (t_m + t_M)}. \tag{2.81}$$

Hence, by inequalities (2.79) and (2.81), $\pi_i^L > \pi_i(\tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*}), \mathbf{C})$, which contradicts the fact that p_i is the best response to $\mathbf{P}^L$. Therefore, $z \in \mathcal{N}(i, 1)$ and $M(i, \tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*})) \subset \mathcal{N}(i, 1)$. Hence, $\tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*}) = p_i^L$ and so $\mathbf{P}^L$ is a Nash equilibrium. □

2.4 Uncertainty on the Hotelling Network

In this section, we introduce incomplete information, considering uncertainty on the production costs of the firms, in the Hotelling network with linear transportation costs, and we find the Bayesian Nash equilibrium in prices.

For simplicity of notation, we consider a Hotelling town model where the firms are located at the nodes and where each firm has a specific space of price strategies associated with their production costs.

For every $v \in V$, let the triples (I_v, Ω_v, q_v) represent (finite, countable or uncountable) sets of types I_v with σ-algebras Ω_v and probability measures q_v over I_v. Hence $\mathrm{d}q_v(z_v)$ denotes the probability of the *common believes* of the other firms on the production costs of the firm F_v to be $c_v^{z_v}$.

The Hotelling town *production cost* $\mathbf{C}$ is the vector $(c_1, \ldots, c_{N_v})$ whose coordinates $c_v : I_v \to [c_v^m, c_v^M] \subseteq [c_m, c_M] \subseteq \mathbb{R}_0^+$ are measurable functions. The Hotelling town *average production cost* $E(\mathbf{C})$ is the vector $(E(c_1), \ldots, E(c_{N_v}))$

whose coordinates are the *expected production costs*

$$E(c_v) = \int_{I_v} c_v^{z_v} \mathrm{d}q_v(z_v) < \infty.$$

A price strategy $\mathbf{P}$ is the vector $(p_1, \ldots, p_{N_v})$ whose coordinates $p_v : I_v \to \mathbb{R}_0^+$ are measurable functions. The average $E(\mathbf{P})$ of the price strategy $\mathbf{P}$ is the vector $(E(p_1), \ldots, E(p_{N_v}))$ whose coordinates are the expected prices

$$E(p_v) = \int_{I_v} p_v^{z_v} \mathrm{d}q_v(z_v).$$

For each road $R_{i,j}$, the indifferent consumer $x_{i,j} : I_i \times I_j \to (0, l_{i,j})$ is given by

$$x_{i,j}^{z_i,z_j} = \frac{p_j^{z_j} - p_i^{z_i} + t\, l_{i,j}}{2\, t}. \tag{2.82}$$

Let the type of the neighbours of a firm F_i of degree k_i be denoted by $\mathbf{Z}_{N_i} = (z_{i,1}, z_{i,2}, \ldots, z_{i,k_i})$ which is a vector of dimension k_i. Consider that $I_{N_i} = I_{i,1} \times I_{i,2} \ldots \times I_{i,k_i}$. The ex-post market size of firm F_i, $S_i^{EP} : I_i \times I_{N_i} \to \mathbb{R}_0^+$, is given by

$$S_i^{EP}(i, \mathbf{P}) = \sum_{j \in N_i} x_{i,j}^{z_i,z_j}. \tag{2.83}$$

The ex-post profit of firm F_i, $\pi_i^{EP} : I_i \times I_{N_i} \to \mathbb{R}_0^+$, is given by

$$\begin{aligned} \pi_i^{EP}(z_i, \mathbf{Z}_{N_i}) &= \pi_i^{EP}(\mathbf{P}, \mathbf{C}, z_i, \mathbf{Z}_{N_i}) \\ &= (p_i^{z_i} - c_i^{z_i})\, S_i^{EP}(i, \mathbf{P}) = (p_i^{z_i} - c_i^{z_i}) \sum_{j \in N_i} x_{i,j}^{z_i,z_j}. \end{aligned} \tag{2.84}$$

We assume that $\mathrm{d}q_{N_i}(\mathbf{Z}_{N_i})$ denotes the probability of the *belief* of the firm F_i on the production costs of its neighbours to be $\mathbf{C}_{N_i}^{z_{N_i}} = (c_{i,1}^{z_{i,1}}, c_{i,2}^{z_{i,2}}, \ldots, c_{i,k_i}^{z_{i,k_i}})$. We note that

$$\mathrm{d}q_{N_i}(\mathbf{Z}_{N_i}) = \mathrm{d}q_{i,1}(z_{i,1})\, \mathrm{d}q_{i,2}(z_{i,2}) \ldots \mathrm{d}q_{i,k_i}(z_{i,k_i}).$$

The ex-ante market size of firm F_i, $S_i^{EA} : I_i \to \mathbb{R}_0^+$, is given by

$$S_i^{EA}(i, \mathbf{P}) = \int_{I_{N_i}} S_i^{EP}(i, \mathbf{P})\, \mathrm{d}q_{N_i}(\mathbf{Z}_{N_i}) = \sum_{j \in N_i} \frac{E(p_j) - p_i^{z_i} + t\, l_{i,j}}{2\,t}. \qquad (2.85)$$

The ex-ante profit of firm F_i, $\pi_i^{EA} : I_i \to \mathbb{R}_0^+$, is given by

$$\begin{aligned} \pi_i^{EA}(z_i) &= \pi_i^{EA}(\mathbf{P}, \mathbf{C}, z_i) \\ &= \int_{I_{N_i}} \pi_i^{EP}(z_i, \mathbf{Z}_{N_i})\, \mathrm{d}q_{N_i}(\mathbf{Z}_{N_i}) = (p_i^{z_i} - c_i^{z_i})\, S_i^{EA}(i, \mathbf{P}) \\ &= (p_i^{z_i} - c_i^{z_i}) \sum_{j \in N_i} \frac{E(p_j) - p_i^{z_i} + t\, l_{i,j}}{2\,t}. \qquad (2.86) \end{aligned}$$

The expected profit of firm F_i, $E(\pi_i)$, is given by

$$E(\pi_i) = \int_{I_i} \pi_i^{EA}(z_i)\, \mathrm{d}q_i(z_i) = \int_{I_i} \int_{I_{N_i}} \pi_A^{EP}(z_i, \mathbf{Z}_{N_i})\, \mathrm{d}q_{N_i}(\mathbf{Z}_{N_i})\, \mathrm{d}q_i(z_i)$$

2.4.1 Local optimal equilibrium price strategy

Given a pair of price strategies $\mathbf{P}$ and $\mathbf{P}^*$ and a firm F_i, we define the price vector $\tilde{\mathbf{P}}(i, \mathbf{P}, \mathbf{P}^*)$ whose coordinates are $\tilde{p}_i = p_i^*$ and $\tilde{p}_j = p_j$, for every $j \in V \setminus \{i\}$. Let $\mathbf{P}$ and $\mathbf{P}^*$ be price strategies that determine local market structures. The price strategy $\mathbf{P}^*$ is a *local best response* to the price strategy $\mathbf{P}$, if for every $i \in V$ the price strategy $\tilde{\mathbf{P}}(i, \mathbf{P}, \mathbf{P}^*)$ determines a local market structure and

$$\frac{\partial \pi_i^{EA}(\tilde{\mathbf{P}}(i, \mathbf{P}, \mathbf{P}^*), \mathbf{C}, z_i)}{\partial \tilde{p}_i} = 0 \quad \text{and} \quad \frac{\partial^2 \pi_i^{EA}(\tilde{\mathbf{P}}(i, \mathbf{P}, \mathbf{P}^*), \mathbf{C}, z_i)}{\partial \tilde{p}_i^2} < 0.$$

Consider that $\mathbf{L}$ and $\mathbf{K}$ represent, respectively, the *admissible market size* vector and the *neighboring market structure* matrix defined in section 2.1.1.

Lemma 2.4.1. *Let $\boldsymbol{P}$ and $\boldsymbol{P}^*$ be price strategies that determine local market structures. The price strategy $\boldsymbol{P}^*$ is the local best response to price strategy*

P *if and only if*

$$\boldsymbol{P}^{*} = \frac{1}{2}\,(\boldsymbol{C} + \boldsymbol{K}\,E(\boldsymbol{P}) + t\,\boldsymbol{L})\,. \tag{2.87}$$

and the price strategies $\tilde{\boldsymbol{P}}(i, \boldsymbol{P}, \boldsymbol{P}^{*})$ *determine local market structures for all* $i \in V$ *. Furthermore,*

$$E(\boldsymbol{P}^{*}) = \frac{1}{2}\,(E(\boldsymbol{C}) + t\,\boldsymbol{L}) + \frac{1}{2}\,\boldsymbol{K}\,E(\boldsymbol{P}). \tag{2.88}$$

Proof. From (2.86), the ex-ante profit for firm F_i in a local market structure is given by

$$\pi_i^{EA}(z_i) = \frac{p_i^{z_i} - c_i^{z_i}}{2\,t}\left(\sum_{j \in N_i} E(p_j) - p_i^{z_i} + t\,l_{i,j}\right) \tag{2.89}$$

Let $\tilde{\mathbf{P}}(i, \mathbf{P}, \mathbf{P}^{*})$ be the price vector whose coordinates are $\tilde{p}_i = p_i^{*}$ and $\tilde{p}_j = p_j$, for every $j \in V \setminus \{i\}$. Since $\mathbf{P}$ and $\mathbf{P}^{*}$ are local price strategies, the local best response of firm F_i to the price strategy $\mathbf{P}$, is given by computing $\partial \pi_i^{EA}(\tilde{\mathbf{P}}(i, \mathbf{P}, \mathbf{P}^{*}), E(\mathbf{C}), z_i)/\partial \tilde{p}_i = 0$. Hence,

$$p_i^{z_i,*} = \frac{1}{2}\left(c_i^{z_i} + \frac{1}{k_i}\sum_{j \in N_i} E(p_j) + t\,l_{i,j}\right). \tag{2.90}$$

and equation (2.87) is satisfied.

Then,

$$E(p_i^{*}) = \int_{I_i} p_i^{z_i,*}\,\mathrm{d}q_i(z_i) = \frac{1}{2}\left(E(c_i) + \frac{1}{k_i}\sum_{j \in N_i} E(p_j) + t\,l_{i,j}\right)$$

Therefore, since $\partial^2 \pi_i^{EA}(\tilde{\mathbf{P}}(i, \mathbf{P}, \mathbf{P}^{*}), \mathbf{C}, z_i)/\partial \tilde{p}_i^2 = -k_i/t < 0$, the local best response strategy prices $\mathbf{P}^*$ satisfy

$$E(\mathbf{P}^*) = \frac{1}{2}\left(E(\mathbf{C}) + t\,\mathbf{L} + \mathbf{K}\,E(\mathbf{P})\right).$$

□

Let $\mathbf{P}$ and $\mathbf{P}^*$ be price strategies that determine local market structures. A price strategy $\mathbf{P}^*$ is a *local optimum price strategy* if $\mathbf{P}^*$ is the local best response to $\mathbf{P}^*$.

Let

$$Q_{i,j} = \sum_{m=0}^{\infty} 2^{-(m+1)}\, k_{i,j}^{m}.$$

Proposition 2.4.1. *If the Hotelling town satisfies the WB condition, then there is unique Bayesian local optimal equilibrium price strategy given by*

$$\boldsymbol{P}^{E} = \frac{1}{2}\left(\boldsymbol{C} + \boldsymbol{K}\,E(\boldsymbol{P}^{E}) + t\,\boldsymbol{L}\right) \tag{2.91}$$

where

$$E(\boldsymbol{P}^{E}) = \frac{1}{2}\left(\boldsymbol{1} - \frac{1}{2}\boldsymbol{K}\right)^{-1}\left(E(\boldsymbol{C}) + t\,\boldsymbol{L}\right).$$

Furthermore, the Bayesian local optimal equilibrium price $\boldsymbol{P}^{E}$ determines a local market structure and the local optimal equilibrium prices p_i^E are bounded by

$$t\,l_m + \frac{1}{2}c_i^{z_i} + \frac{E(c_i) + c_m}{4} \leq p_i^{z_i,E} \leq t\,l_M + \frac{1}{2}c_i^{z_i} + \frac{E(c_i) + c_M}{4}. \tag{2.92}$$

Proof. The matrix $\mathbf{K}$ is a stochastic matrix (i.e., $\sum_{j\in V} k_{i,j} = 1$, for every $i \in V$). Thus, we have $\|\mathbf{K}\| = 1$. Hence, the matrix Q is well-defined by

$$\mathbf{Q} = \frac{1}{2}\left(\mathbf{1} - \frac{1}{2}\mathbf{K}\right)^{-1} = \sum_{m=0}^{\infty} 2^{-(m+1)}\,\mathbf{K}^{m}$$

and Q is also a non-negative and stochastic matrix. By Lemma 2.4.1, a local optimum price strategy satisfy equality (2.88). Therefore,

$$E(\mathbf{P}^E) = \frac{1}{2}\left(\mathbf{1} - \frac{1}{2}\mathbf{K}\right)^{-1}(E(\mathbf{C}) + t\,\mathbf{L}) = \sum_{m=0}^{\infty} 2^{-(m+1)}\,\mathbf{K}^m\,\left(E(\mathbf{C}) + t\,\mathbf{L}\right). \tag{2.93}$$

By construction,

$$E(p_i^E) = \sum_{v\in V} Q_{i,v}(E(c_v) + t\,L_v). \tag{2.94}$$

From equality (2.87), we obtain that the Bayesian local optimal equilibrium price $\mathbf{P}^E$ has coordinates

$$\begin{aligned} p_i^{z_i,E} &= \frac{1}{2}\left(c_i + \frac{1}{k_i}\sum_{j\in N_i} E(p_j^E) + t\,l_{i,j}\right) \\ &= \frac{1}{2}\left(c_i + \frac{1}{k_i}\sum_{j\in N_i}\left(\sum_{v\in V} Q_{j,v}(E(c_v) + t\,L_v) + t\,l_{i,j}\right)\right). \end{aligned} \tag{2.95}$$

Let us prove that the price strategy $\mathbf{P}^E$ is local, i.e., the indifferent consumer $x_{i,j}^{z_i,z_j}$ satisfies $0 < x_{i,j}^{z_i,z_j} < l_{i,j}$ for every $R_{i,j} \in E$ which, from (2.82), is equivalent to

$$\left|p_i^{z_i,E} - p_j^{z_j,E}\right| < t\,l_{i,j}. \tag{2.96}$$

Since $c_m \leq E(c_v) \leq c_M$ for every $v \in V$, from (2.94) we obtain that for every $i \in V$

$$\sum_{v\in V} Q_{i,v}(c_m + t\,L_v) \leq E(p_i^E) \leq \sum_{v\in V} Q_{i,v}(c_M + t\,L_v). \tag{2.97}$$

We note that

$$l_m \leq L_v = k_v^{-1}\sum_{j\in N_v} l_{v,j} \leq l_M. \tag{2.98}$$

Since $\mathbf{Q}$ is a nonnegative and stochastic matrix we obtain

$$\sum_{v \in V} Q_{i,v}(c_m + t\, l_m) = c_m + t\, l_m$$

and

$$\sum_{v \in V} Q_{i,v}(c_M + t\, l_M) = c_M + t\, l_M.$$

Hence, putting (2.97) and (2.98) together, we obtain that

$$c_m + t\, l_m \leq E(p_i^E) \leq c_M + t\, l_M.$$

Then,

$$p_i^{z_i,E} \leq \frac{1}{2}\left(c_i^{z_i} + \frac{1}{k_i}\sum_{j \in N_i} c_M + t\, l_M + t\, l_{i,j}\right) \leq \frac{1}{2}\left(c_i^{z_i} + c_M + 2\, t\, l_M\right)$$

and

$$p_i^{z_i,E} \geq \frac{1}{2}\left(c_i^{z_i} + \frac{1}{k_i}\sum_{j \in N_i} c_m + t\, l_m + t\, l_{i,j}\right) \geq \frac{1}{2}\left(c_i^{z_i} + c_m + 2\, t\, l_m\right).$$

Therefore,

$$c_m + t\, l_m \leq p_i^{z_i,E} \leq c_M + t\, l_M.$$

Since the last relation is satisfied for every firm, we obtain

$$-\left(c_M - c_m + t(l_M - l_m)\right) \leq p_i^{z_i,E} - p_j^{z_j,E} \leq c_M - c_m + t(l_M - l_m).$$

Therefore,

$$\left|p_i^{z_i,E} - p_j^{z_j,E}\right| \leq \Delta(c) + t\,\Delta(l).$$

Hence, by the WB condition, we conclude that

$$\left| p_i^{z_i,E} - p_j^{z_i,E} \right| < t\, l_m.$$

Thus, by equation (2.96), we obtain that the indifferent consumer is located at $0 < x_{i,j}^{z_i,z_j} < l_{i,j}$ for every road $R_{i,j} \in E$. Hence, the price strategy $\mathbf{P}^E$ is local and is the unique local optimal equilibrium price strategy.

From (2.94) and (2.98), we obtain

$$E(p_i^E) \geq \sum_{v \in V} Q_{i,v}\, t\, l_m + \sum_{v \in V \setminus \{i\}} Q_{i,v}\, c_m + Q_{i,i}\, E(c_i).$$

By construction of matrix $\mathbf{Q}$, we have $Q_{i,i} > 1/2$. Furthermore, since $\mathbf{Q}$ is stochastic,

$$\sum_{v \in V \setminus \{i\}} Q_{i,v} < 1/2,$$

and $\sum_{v \in V} Q_{i,v} t\, l_m = t\, l_m$. Hence,

$$E(p_i^E) \geq t\, l_m + \frac{1}{2}\, (E(c_i) + c_m).$$

Similarly, we obtain

$$E(p_i^E) \leq t\, l_M + \frac{1}{2}\, (E(c_i) + c_M).$$

Hence

$$\begin{aligned} p_i^{z_i,E} &\geq \frac{1}{2} \left(c_i + \frac{1}{k_i} \sum_{j \in N_i} t\, l_m + \frac{1}{2}\, (E(c_i) + c_m) + t\, l_{i,j} \right) \\ &\geq t\, l_m + \frac{1}{2}\, c_i + \frac{1}{4}\, (E(c_i) + c_m). \end{aligned}$$

Similarly,

$$p_i^{z_i,E} \leq t\, l_M + \frac{1}{2}\, c_i + \frac{1}{4}\, (E(c_i) + c_M).$$

and so the Bayesian local optimal equilibrium prices p_i^E are bounded and satisfy (2.92). □

Proposition 2.4.2. *If the Hotelling town satisfies the WB condition, the ex-ante local optimal profit $\pi_i^{EA,E}(z_i)$ of firm F_i is given by*

$$\pi_i^{EA,E}(z_i) = \pi_i^{EA}(\boldsymbol{P}^E, E(\boldsymbol{C}), z_i) = \frac{k_i\,(p_i^{z_i,E} - c_i^{z_i})^2}{2\,t}$$

and is bounded by

$$\frac{k_i\,(4\,t\,l_m + E(c_i) + c_m - 2\,c_i^{z_i})^2}{32\,t} \leq \pi_i^{EA,E}(z_i) \leq \frac{k_i\,(4\,t\,l_M + E(c_i) + c_M - 2\,c_i^{z_i})^2}{32\,t}.$$

Proof. We can write the ex-ante profit function (2.89) of firm F_i with respect to the local optimum price strategy P^E by

$$\pi_i^{EA,E}(z_i) = (2t)^{-1}(p_i^{z_i,E} - c_i^{z_i})\left(-k_i\,p_i^{z_i,E} + \sum_{j\in N_i}(E(p_j^E) + t\,l_{i,j})\right) \qquad (2.99)$$

Since $\mathbf{P}^E$ satisfies the best response function (2.87), we have

$$2\,p_i^{z_i,E} = c_i^{z_i} + \frac{1}{k_i}\sum_{j\in N_i}\left(E(p_j^E) + t\,l_{i,j}\right).$$

Therefore,

$$\sum_{j\in N_i}\left(E(p_j^E) + t\,l_{i,j}\right) = 2\,k_i\,p_i^{z_i,E} - k_i\,c_i^{z_i},$$

and replacing this sum in the profit function (2.99), we obtain

$$\pi_i^{EA,E}(z_i) = (2t)^{-1}\,k_i\,(p_i^{z_i,E} - c_i^{z_i})^2.$$

Using the price bounds (2.92), we conclude

$$\frac{k_i\,(4\,t\,l_m + E(c_i) + c_m - 2\,c_i^{z_i})^2}{32\,t} \le \pi_i^{EA,E}(z_i) \le \frac{k_i\,(4\,t\,l_M + E(c_i) + c_M - 2\,c_i^{z_i})^2}{32\,t}.$$

□

2.4.2 Bayesian Nash equilibrium price strategy

The price strategy $\mathbf{P}^*$ is a *best response* to the price strategy $\mathbf{P}$, if

$$(\tilde{p}_i - c_i)\,S^{EA}(i, \tilde{\mathbf{P}}(i, \mathbf{P}, \mathbf{P}^*)) \ge (p_i' - c_i)\,S^{EA}(i, \mathbf{P}_i'),$$

for all $i \in V$ and for all price strategies $\mathbf{P}_i'$ whose coordinates satisfy $p_i' \ge c_i$ and $p_j' = p_j$ for all $j \in V \setminus \{i\}$. A price strategy $\mathbf{P}^*$ is a Hotelling town *Nash equilibrium* if $\mathbf{P}^*$ is the best response to $\mathbf{P}^*$.

Lemma 2.4.2. *In a Hotelling town satisfying the WB condition, if there is a Bayesian Nash price $\boldsymbol{P}^*$ then $\boldsymbol{P}^*$ is unique and $\boldsymbol{P}^* = \boldsymbol{P}^E$.*

Hence, the Bayesian local optimum price strategy $\mathbf{P}^E$ is the only candidate to be a Nash equilibrium price strategy. However, $\mathbf{P}^E$ might not be a Bayesian Nash equilibrium price strategy because there can be a firm F_i that by decreasing his price is able to absorb markets of other firms in such a way that increases its own profit. Therefore, the best response price strategy $\mathbf{P}^{E,*}$ to the optimal local price strategy $\mathbf{P}^E$ might be different from $\mathbf{P}^E$.

Proof of Lemma 2.4.2.
Suppose that P^* is a Nash price strategy and that $\mathbf{P}^* \neq \mathbf{P}^E$. Hence, $\mathbf{P}^*$ does not determine a local market structure, i.e., there exists $i \in V$ such that

$$M(i, \mathbf{P}^*) \not\subset \cup_{j \in N_i} R_{i,j}.$$

Hence, there exists $j \in N_i$ such that $M(j, \mathbf{P}^*) = 0$ and, therefore, $\pi_j^{EA,*} = 0$.

Moreover, in this case, we have that

$$p_j^{z_j,*} > E(p_i^*) + t\, l_{i,j}.$$

Consider, now, that F_j changes his price to $p_j = c_j^{z_j} + t\,\Delta(l)$. Since $E(p_i^*) > c_m$ and $c_j^{z_j} - c_m \leq \Delta(c)$ we have that

$$p_j - E(p_i^*) = c_j^{z_j} + t\,\Delta(l) - E(p_i^*) < c_j^{z_j} + t\,\Delta(l) - c_m \leq \Delta(c) + t\,\Delta(l).$$

Since the Hotelling town satisfies the WB condition, $\Delta(c) + t\,\Delta(l) < t\, l_m$, we have

$$p_j - E(p_i^*) < t\, l_m \leq t\, l_{i,j}.$$

Hence, $M(j, \tilde{\mathbf{P}}(j, \mathbf{P}^*, \mathbf{P})) > 0$ and $\pi_j^{EA} = (c_j + t\,\Delta(l))\, S^{EA}(j, \tilde{\mathbf{P}}(j, \mathbf{P}^*, \mathbf{P})) > 0$. Therefore, F_j will change its price and so $\mathbf{P}^*$ is not a Nash equilibrium price strategy. Hence, if there is a Nash price $\mathbf{P}^*$ then $\mathbf{P}^* = \mathbf{P}^E$. □

Lemma 2.4.3. *In a Hotelling town satisfying the WB condition,*

$$M(i, \tilde{\boldsymbol{P}}(i, \boldsymbol{P}^E, \boldsymbol{P}^{E,*})) \subset \mathcal{N}(i, 2)$$

for every $i \in V$.

Hence, a consumer $x \in R_{j,k}$ might not buy in its local firms F_j and F_k. However, the consumer $x \in R_{j,k}$ still has to buy in a firm F_i that is a neighboring firm of its local firms F_j and F_k, i.e. $i \in N_j \cup N_k$.

Proof of Lemma 2.4.3.
By contradiction, let us consider a consumer $z \in M(i, \tilde{\mathbf{P}}(i, \mathbf{P}^E, \mathbf{P}^{E,*}))$ and $z \notin \mathcal{N}(i, 2)$. For every type $z_i \in I_i$, the price that consumer z pays to buy in firm F_i is given by

$$e = p_i^{z_i} + t\,\left(l_{i_1,i_2} + l_{i_2,i_3} + d\,(y_{i_3}, z)\right)$$

where $p_i = p_i^{E,*}$ is the coordinate of the vector $\tilde{\mathbf{P}}(i, \mathbf{P}^E, \mathbf{P}^{E,*})$ and for the 2-path $(R_{i_1,i_2}, R_{i_2,i_3})$ with $i_1 = i$. If the consumer z buys at firm F_{i_3}, then the price that has to pay for every type $z_{i_3} \in I_{i_3}$ is

$$\tilde{e} = p_{i_3}^{z_{i_3},E} + t\, d\,(y_{i_3}, z).$$

Since, by hypothesis, $z \in M(i, \tilde{\mathbf{P}}(i, \mathbf{P}^E, \mathbf{P}^{E,*}))$, we have $e < \tilde{e}$. Therefore, for every types $z_i \in I_i$ and $z_{i_3} \in I_{i_3}$, we have

$$p_i^{z_i} < p_{i_3}^{z_{i_3},E} - t\ (l_{i_1,i_2} + l_{i_2,i_3})\,.$$

By (2.92), $p_i^{z_i,E} \le t\, l_M + \dfrac{1}{2}\left(c_i^{z_i} + \dfrac{E(c_i) + c_M}{2}\right)$ for all $i \in V$. Since $l_{i,j} \ge l_m$ for all $R_{i,j} \in E$,

$$p_i^{z_i} < t\, l_M + \frac{1}{2}\left(c_{i_3}^{z_{i_3}} + \frac{E(c_{i_3}) + c_M}{2}\right) - 2\, t\, l_m \le c_M + t\, \Delta(l) - t\, l_m.$$

Furthermore,

$$p_i^{z_i} - c_i^{z_i} < \Delta(c) + t\, \Delta(l) - t\, l_m.$$

By the WB condition, $p_i^{z_i} - c_i^{z_i} < 0$. Hence, $\pi_i^{E,*} < 0$ which contradicts the fact that p_i is the best response to $\mathbf{P}^E$ (since $\pi_i^E > 0$). Therefore, $z \in \mathcal{N}(i, 2)$ and $M(i, \tilde{\mathbf{P}}(i, \mathbf{P}^E, \mathbf{P}^{E,*})) \subset \mathcal{N}(i, 2)$. □

Definition 2.4.1. *A Hotelling town satisfies the* strong bounded length and costs (SB) *condition, if*

$$\Delta(c) + t\Delta(l) \le \frac{(2\, t\, l_m - \Delta(c))^2}{8\, t\, k_M\, l_M}. \tag{2.100}$$

Theorem 2.4.1. *If a Hotelling town satisfies the SB condition then there is a unique Hotelling town Bayesian Nash equilibrium price strategy $\boldsymbol{P}^* = \boldsymbol{P}^E$.*

Hence, the Nash equilibrium price strategy for the Hotelling town satisfying the SB condition determines a local market structure, i.e. every consumer located at $x \in R_{i,j}$ spends less by shopping at his local firms F_i or F_j than in any other firm in the town and so the consumer at x will buy either at his local firm F_i or at his local firm F_j.

Proof of Theorem 2.4.1.
By Proposition 2.4.1 and Lemma 2.4.2, if there is a Bayesian Nash equilibrium price strategy $\mathbf{P}^*$ then $\mathbf{P}^*$ is unique and $\mathbf{P}^* = \mathbf{P}^E$.

We note that if $M(i, \tilde{\mathbf{P}}(i, \mathbf{P}^E, \mathbf{P}^{E,*})) \subset \mathcal{N}(i,1)$ for every $i \in V$ then $\tilde{\mathbf{P}}(i, \mathbf{P}^E, \mathbf{P}^{E,*}) = p_i^E$ and so $\mathbf{P}^E$ is a Nash equilibrium.
By Lemma 2.4.3, we have that $M(i, \tilde{\mathbf{P}}(i, \mathbf{P}^E, \mathbf{P}^{E,*})) \subset \mathcal{N}(i,2)$ for every $i \in V$. Now, we will prove that condition (2.100) implies that firm F_i earns more competing only in the 1-neighborhood than competing in a 2-neighborhood. By Lemma 2.4.3,

$$\begin{aligned} \pi_i^{EA}(\tilde{\mathbf{P}}(i, \mathbf{P}^E, \mathbf{P}^{E,*}), \mathbf{C}, z_i) &\leq (p_i^{z_i} - c_i^{z_i}) \sum_{j \in N_i} \left(l_{i,j} + \sum_{k \in N_j \setminus \{i\}} l_{j,k} \right) \\ &\leq (p_i^{z_i} - c_i^{z_i}) \sum_{j \in N_i} \sum_{k \in N_j} l_{j,k}, \end{aligned}$$

where $p_i = p_i^{E,*}$ is the coordinate of the vector $\tilde{\mathbf{P}}(i, \mathbf{P}^E, \mathbf{P}^{E,*})$. Hence,

$$\pi_i^{EA}(\tilde{\mathbf{P}}(i, \mathbf{P}^E, \mathbf{P}^{E,*}), \mathbf{C}, z_i) \leq (p_i^{z_i} - c_i^{z_i}) \sum_{j \in N_i} \sum_{k \in N_j} l_{j,k} \leq (p_i^{z_i} - c_i^{z_i}) k_i \, k_M \, l_M. \tag{2.101}$$

By contradiction, let us consider a consumer $z \in M(i, \tilde{\mathbf{P}}(i, \mathbf{P}^E, \mathbf{P}^{E,*}))$ and $z \notin \mathcal{N}(i,1)$. Let $i_2 \in N_i$ be the vertex such that $z \in \mathcal{N}(i_2, i)$. The price that consumer z pays to buy in firm F_i is given by

$$e = p_i + t \, l_{i,i_2} + t \, d\,(y_{i_2}, z).$$

If the consumer y buys at firm F_{i_2}, then the price that has to pay is

$$\tilde{e} = p^E_{i_2} + t\, d\,(y_{i_2}, z).$$

Since, by hypothesis, $z \in M(i, \tilde{\mathbf{P}}(i, \mathbf{P}^E, \mathbf{P}^{E,*}))$, we have $e < \tilde{e}$. Therefore

$$p_i < p^E_{i_2} - t\, l_{i,i_2}.$$

By (2.92), $p^E_i \leq t\, l_M + \frac{1}{2}\left(c_i + \frac{E(c_i) + c_M}{2}\right)$. Since $l_{i,i_2} \geq l_m$, we have

$$p_i < t\, l_M + \frac{1}{2}\left(c_{i_2} + \frac{E(c_{i_2}) + c_M}{2}\right) - t\, l_m \leq c_M + t\, \Delta(l).$$

Thus,

$$p_i - c_i < \Delta(c) + t\, \Delta(l).$$

Hence, from (2.101) we obtain

$$\pi^{EA}_i(\tilde{\mathbf{P}}(i, \mathbf{P}^E, \mathbf{P}^{E,*}), \mathbf{C}, z_i) < k_i\, k_M\, l_M\, \left(\Delta(c) + t\, \Delta(l)\right).$$

By the SB condition,

$$\pi^{EA}_i(\tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*}), \mathbf{C}, z_i) < (2\, t)^{-1}\, k_i\, (t\, l_m - \Delta(c)/2)^2. \qquad (2.102)$$

By Proposition 2.4.2 and (2.102),

$$\pi^{EA,E}_i(z_i) \geq (2\, t)^{-1}\, k_i\, \left(t\, l_m - \Delta(c)/2\right)^2 > \pi^{EA}_i(\tilde{\mathbf{P}}(i, \mathbf{P}^E, \mathbf{P}^{E,*}), \mathbf{C}, z_i),$$

which contradicts the fact that p_i is the best response to $\mathbf{P}^E$. Therefore, $z \in \mathcal{N}(i, 1)$ and $M(i, \tilde{\mathbf{P}}(i, \mathbf{P}^E, \mathbf{P}^{E,*})) \subset \mathcal{N}(i, 1)$. Hence, $\tilde{\mathbf{P}}(i, \mathbf{P}^E, \mathbf{P}^{E,*}) = p^E_i$ and so $\mathbf{P}^E$ is a Bayesian Nash equilibrium. □

2.5 Future Work: General model

This section presents the initial ideas of the general model for the Hotelling model, allowing that firms can have entire markets and compete with other that its neighbours.

Let $S_{i,j} \subseteq E$ denote the set of edges where F_i and F_j divide consumers

$$S_{i,j} = \{(k,l),(k',l'),\ldots\}$$

Let $l_{k,l}$ denote the length of the roads $R_{k,l}$ and let $L_{i,k}$ and $L_{j,l}$ denote the length between node i and node k and between node j and node l, respectively.

For every edge shared by firm F_i and F_j, $R_{k,l}$ there is an indifferent consumer located at distance

$$x_i(k,l) = \frac{p_j - p_i}{2\,t} + \frac{L_{j,l} - L_{i,k} + l_{k,l}}{2}$$

from firm F_i. Hence,

$$\tilde{x}_i(k,l) = \frac{p_j - p_i}{2\,t} + \frac{L_{j,l} - 3\,L_{i,k} + l_{k,l}}{2}$$

is the distance of the indifferent consumer to the node k.

Let MC_i denote the market of the network that belongs exclusively to firm F_i, i.e., the set of edges where all the consumers buy at F_i. Hence, the total market of firm F_i, M_i is given by

$$M_i = MC_i + \sum_{j\in N}\sum_{(k,l)\in S_{i,j}} \frac{p_j - p_i}{2\,t} + \frac{L_{j,l} - 3\,L_{i,k} + l_{k,l}}{2}$$

and the profit of F_i is given by

$$\pi_i = (p_i - c_i)\,MC_i + (p_i - c_i)\sum_{j\in N}\sum_{(k,l)\in S_{i,j}} \frac{p_j - p_i}{2\,t} + \frac{L_{j,l} - 3\,L_{i,k} + l_{k,l}}{2}$$

Hence

$$\frac{\partial \pi_i}{\partial p_i} = MC_i + \sum_{j\in N} \sum_{(k,l)\in S_{i,j}} \frac{p_j - 2\,p_i + c_i}{2\,t} + \frac{L_{j,l} - 3\,L_{i,k} + l_{k,l}}{2}$$

From the FOC, we obtain

$$2 \sum_{j\in N} \sum_{(k,l)\in S_{i,j}} p_i = 2\,t\,MC_i + \sum_{j\in N} \sum_{(k,l)\in S_{i,j}} p_j + \sum_{j\in N} \sum_{(k,l)\in S_{i,j}} c_i + \sum_{j\in N} \sum_{(k,l)\in S_{i,j}} t\,(L_{j,l} - 3\,L_{i,k} + l_{k,l})$$

Let $k_i = \sum_{j\in N} \sum_{(k,l)\in S_{i,j}}$ denote the number of markets shared by firm F_i and let N_i denote the set of firms that share a market with F_i. Hence

$$2\,k_i\,p_i = 2\,t\,MC_i + \sum_{j\in N_i} p_j\,\sharp(S_{i,j}) + k_i\,c_i + t \sum_{j\in N_i} \sum_{(k,l)\in S_{i,j}} (L_{j,l} - 3\,L_{i,k} + l_{k,l})$$

Let $B_i = \sum_{j\in N_i} \sum_{(k,l)\in S_{i,j}} (L_{j,l} - 3\,L_{i,k} + l_{k,l})$. Then

$$2\,k_i\,p_i = \sum_{j\in N_i} p_j\,\sharp(S_{i,j}) + k_i\,c_i + t\,(2\,MC_i + B_i)$$

and

$$p_i = \frac{1}{2\,k_i} \sum_{j\in N_i} p_j\,\sharp(S_{i,j}) + \frac{c_i}{2} + \frac{t}{2\,k_i}\,(2\,MC_i + B_i)$$

Let $\mathbf{K}$ be the matrix defined by

$$k_{i,j} = \frac{\sharp(S_{i,j})}{k_i}$$

and $\tilde{\mathbf{M}}$ and $\tilde{\mathbf{B}}$ the vectors whose coordinates are

$$\tilde{M}_i = \frac{MC_i}{k_i}$$

and

$$\tilde{B}_i = \frac{B_i}{k_i}.$$

Hence,

$$\left(\mathbf{1} - \frac{1}{2}\,\mathbf{K}\right)\,\mathbf{P} = \frac{1}{2}\,\left(\mathbf{C} + t\,(2\,\tilde{\mathbf{M}} + \tilde{\mathbf{B}})\right).$$

Since K is a stochastic matrix, $\left(\mathbf{1} - \frac{1}{2}\,\mathbf{K}\right)^{-1}$ exists, and

$$\mathbf{P} = \frac{1}{2}\,\left(\mathbf{1} - \frac{1}{2}\,\mathbf{K}\right)^{-1}\,\left(\mathbf{C} + t\,(2\,\tilde{\mathbf{M}} + \tilde{\mathbf{B}})\right).$$

Conclusions

In the first part of this work, we studied the linear and quadratic Hotelling model with uncertainty on the production costs. We introduced a new condition on the exogenous variables that we called the $BUC1$ ($BUCL1$, in the quadratic transportation cost case) condition. We proved that there is a local optimum price strategy if and only if the $BUC1$ ($BUCL1$) condition is satisfied. We gave the explicit formula for the local optimum price strategy and we observed that the formula does not depend on the distributions of the production costs of the firms, except on their first moments. Furthermore, the local optimum price strategy determines prices for both firms that are affine with respect to the expected costs of both firms and to its own costs. The corresponding expected profits are quadratic in the expected cost of both firms, in its own cost and in the transportation cost. We did the ex-ante versus ex-post analysis of the profits. We proved that, under the $A - BUC$ and $B - BUC$ conditions, the ex-post profit of a firm is smaller than its ex-ante profit if and only if the production cost of the competitor firm is greater than its expected cost. Then, we proved that the $A - BUC$ and $B - BUC$ conditions are implied by the $BUC1$ ($BUCL1$) condition, if the distribution of the production costs of both firms coincide (symmetric Hotelling). We introduced a new condition on the exogenous variables that we called the $BUC2$ ($BUCL2$) condition and we proved that under the $BUC1$ ($BUCL1$) and $BUC2$ ($BUCL2$) conditions, the local optimum price strategy is a Bayesian-Nash price strategy.

With quadratic transportation costs, assuming that the firms choose the Bayesian-Nash price strategy, we showed in which conditions the maximal differentiation is a local optimum for the localization strategy of both firms.

In the second part of this work, we presented a model of price competition in a network, extending the linear city presented by Hotelling with linear and quadratic transportation costs to a network where firms are located at the neighbourhood of the nodes and consumers distributed along the edges. Under a condition on lengths and costs (WB condition), we found the local optimum price strategy $\mathbf{P}^L$ for which the Hotelling town has a local market structure, i.e. the consumers prefer to buy at the local firms. Under a condition on lengths and costs and maximum node degree (SB condition), we proved that under the SB condition, the Nash equilibrium price strategy $\mathbf{P}^*$ exists and that $\mathbf{P}^* = \mathbf{P}^L$. We gave an explicit series expansion formula for the Nash price equilibrium that shows explicitly how the Nash price equilibrium of a firm depends on the production costs, road market sizes and firm locations. Furthermore, the influence of a firm in the Nash price equilibrium of other firm decreases exponentially with the distance between the firms. We introduced the notion of space bounded information in the Hotelling town and we showed that firms that only have local knowledge of network are still able to compute good approximations of local optimum prices. All this results were obtained for linear and quadratic transportation costs.

With linear transportation costs, we presented additional results: (a) we proved that, if the firms are located at the neighbourhood of the nodes of degree greater than 2, the local optimal localization of the firms is at the vertices of the network; (b) we determined the Nash equilibrium price strategy for a Hotelling network where each firm has associated a different transportation cost; (c) we determined the Bayesian-Nash equilibrium price strategy with uncertainty on the production costs in the hotelling model; and (d) under a condition on lengths and costs, we showed that the local optimum profits of the firms increases with the degree of the nodes in which they are located.

Further, research work can consist (i) on finding sufficient and necessary conditions for the local optimum price strategy to be a Nash equilibrium; (ii) to solve the localization problem by studying the cases where the firms are not located at the ends of the segment line; (iii) extend the Hotelling town model to general case, without a local market structure.

Bibliography

[1] S. Anderson, A. de Palma and J-F. Thisse, *Discrete Choice Theory of Product Differentiation*, MIT Press, Cambridge, Massachusetts, 1992.

[2] C. D'Aspremont, J. Gabszewicz, and J.-F. Thisse, *On Hotelling's "Stability in Competition"*, Econometrica 47 (5), (1979), pp. 1145–1150.

[3] Biscaia, R. and Mota, I., *Models of spatial competition: A critical review*, Papers in Regional Science 92 (4) (2013) 851-871.

[4] R. Biscaia, P. Sarmento, *Spatial Competition and Firms' Location Decisions under Cost Uncertainty*, FEP Working Papers n445, (2012).

[5] M. Boyer, J. Laffont, P. Mahenc and M. Moreaux, *Location Distortions under Incomplete information*, Regional Science and Urban Economics 24 (4), (1994), pp. 409–440.

[6] M. Boyer, P. Mahenc and M. Moreaux, *Asymmetric Information and Product Differentiation*, Regional Science and Urban Economics 33 (1), (2003a), pp. 93–113.

[7] M. Boyer, P. Mahenc and M. Moreaux, *Entry preventing locations under incomplete information*, International Journal of Industrial Organization 21 (6), (2003b), pp. 809–829.

[8] Y. Bramoull, R. Kranton. M. D'Amours, *Strategic Interaction and Networks*, mimeo., Duke University, (2012).

[9] Y. Chen and M. H. Riordan, *Price and Variety in the Spokes Model*, Economic Journal, Royal Economic Society 117 (522), (2007) pp. 897-921.

[10] P. Dasgupta and E. Maskin, *The Existence of Equilibrium in Discontinuous Economic Games, II: Applications*, Review of Economic Studies, 53 (1986), pp. 27–41.

[11] F. Ferreira, F. A. Ferreira, M. Ferreira and A. A. Pinto, *Flexibility in a Stackelberg leadership with differentiated goods*, Optimization (to appear).

[12] F. Ferreira, F. A. Ferreira, and A. A. Pinto, *Price-setting dynamical duopoly with incomplete information*, in *Nonlinear Science and Complexity*, J. A. Machado, M. F. Silva, R. S. Barbosa, and L. B. Figueiredo, editors, Springer, 2010, pp. 397–404.

[13] F. Ferreira, F. A. Ferreira, and A. A. Pinto, *Flexibility in stackelberg leadership*, in *Intelligent Engineering Systems and Computational Cybernetics*, J. A. Machado, B. Patkai, and I. J. Rudas, editors, Springer Netherlands, 2008, pp. 399–405.

[14] F. Ferreira, F. A. Ferreira, and A. A. Pinto, *Bayesian price leadership*, in *Mathematical Methods in Engineering*, K. Tas et al., editors, Springer, 2007, pp. 359–369.

[15] F. A. Ferreira, F. Ferreira, and A. A. Pinto, *Unknown costs in a duopoly with differentiated products*, in *Mathematical Methods in Engineering*, K. Tas et al., editors, Springer, 2007, pp. 371–379.

[16] M. Ferreira, I.P. Figueiredo, B.M.P.M. Oliveira and A. A. Pinto. *Strategic optimization in R&D Investment*. Optimization: A Journal of Mathematical Programming and Operations Research, 61 (8), (2012) pp. 1013-1023.

[17] F. A. Ferreira, and A. A. Pinto, *Uncertainty on a Bertrand duopoly with product differentiation*, in *Nonlinear Science and Complexity*, J. A. Machado, M. F. Silva, R. S. Barbosa, and L. B. Figueiredo, editors, Springer, 2010, pp. 389–396.

[18] D. Fudenberg and J. Tirole, *Game Theory*, MIT Press, 1993.

[19] A. Galeotti, S. Goyal, M. Jackson, F. Vega-Redondo and L. Yariv, *Network Games*, The Review of Economic Studies 77 (2010), pp. 218–244.

[20] A. Galeotti and F. Vega-Redondo, *Complex networks and local externalities: a strategic approach*, International Journal of Economic Theory, 7 (1) (2011), pp. 77-92.

[21] R. Gibbons, *A Primer in Game Theory*, Financial Times Prentice Hall, 1992.

[22] C. Godsil and G. Royle, *Algebraic Graph Theory*. Springer-Verlag, 2001.

[23] S. Goyal, *Connections: An introduction to the Economics of Networks*, Princeton University Press, 2007.

[24] D. Graitson, *Spatial competition la Hotelling: a selective survey*, The Journal of Industrial Economics 31 (1982), pp. 11–25.

[25] H. Hotelling, *Stability in Competition*, The Economic Journal 39 (1929), pp. 41–57.

[26] P. Lederer and A. Hurter, *Competition of Firms: Discriminatory Pricing and Location*, Econometrica 54 (3), (1986), pp. 623–640.

[27] M. J. Osborne and C. Pitchick, *Equilibrium in Hotelling's Model of Spatial Competition*. Econometrica, 55 (4), (1987) pp. 911–922.

[28] A. A. Pinto, F.A. Ferreira, M. Ferreira, and B.M.P.M. Oliveira, *Cournot duopoly with competition in the R&D expenditures*, Proceedings of Symposia in Pure Mathematics Vol. 7, Wiley-VCH Verlag: Weinheim, 2007.

[29] A. A. Pinto, B. M. P. M. Oliveira, F. A. Ferreira, and F. Ferreira, *Stochasticity favoring the effects of the R&D strategies of the firms*, in *Intelligent Engineering Systems and Computational Cybernetics*, J. A. Machado, B. Patkai, and I. J. Rudas, editors, Springer Netherlands, 2008, pp. 415–423.

[30] A. A. Pinto, and T. Parreira, *A hotelling-type network*, in *Dynamics, Games and Science I*, M. Peixoto, A. A. Pinto, and D. Rand, editors, Springer Proceedings in Mathematics series 1, 2011, pp. 709–720.

[31] A. A. Pinto, and T. Parreira, *Optimal localization of firms in Hotelling networks*, in *Modeling, Dynamics, Optimization and Bioeconomy*, A. A. Pinto, and D. Zilberman, Springer Proceedings in Mathematics and Statistics series, 2014.

[32] A. A. Pinto, and T. Parreira, *Complete versus incomplete information in the Hotelling model*, in *Modeling, Dynamics, Optimization and Bioeconomy*, A. A. Pinto, and D. Zilberman, Springer Proceedings in Mathematics and Statistics series, 2014.

[33] A. A. Pinto, and T. Parreira, *Maximal differentiation in the Hotelling model with uncertainty*, in *Modeling, Dynamics, Optimization and Bioeconomy*, A. A. Pinto, and D. Zilberman, Springer Proceedings in Mathematics and Statistics series, 2014.

[34] A. A. Pinto, and T. Parreira, *Price competition in the Hotelling model with uncertainty on costs*, Optimization: A Journal of Mathematical Programming and Operations Research (accepted).

[35] A. A. Pinto, and T. Parreira, *Bayesian-Nash prices in linear Hotelling model*, Submited.

[36] A. A. Pinto, and T. Parreira, *Localization and prices in the quadratic Hotelling model with uncertainty*, Submited.

[37] S. Salop, *Monopolistic Competition with Outside Goods*, Bell Journal of Economics 10 (1979), pp. 141-156.

[38] T. Tabuchi and J. F. Thisse, *Asymmetric equilibria in spatial competition*, International Journal of Economic Theory 13 (2), (1995), pp. 213–227.

[39] J. Tirole, *The Theory of Industrial Organization* , MIT Press, Cambridge, Massachusetts, 1988.

[40] X. Vives, *Oligopoly Pricing: old ideas and new tools* , MIT Press, Cambridge, Massachusetts, 1999.

[41] S. Ziss, *Entry Deterrence, Cost Advantage and Horizontal Product Differentiation*, Regional Science and Urban Economics 23, (1993), pp. 523–543.