THE EINSTEIN RELATION FOR THE KPZ EQUATION

PATRÍCIA GONÇALVES AND MILTON JARA

ABSTRACT. We compute the non-universal constants in the KPZ equation in one dimension, in terms of the thermodynamical quantities associated to the underlying microscopic dynamics. In particular, we derive the second-order Einstein relation $\lambda = \frac{a}{2}\frac{d^2}{d\rho^2}\chi(\rho)D(\rho)$ for the transport coefficient λ of the KPZ equation, in terms of the conserved quantity ρ, the diffusion coefficient D, the strength of the asymmetry a and the static compressibility of the system χ.

1. INTRODUCTION

One of the most challenging problems in statistical mechanics is the study of the evolution of nonequilibrium systems, and in particular the derivation of effective equations in terms of the relevant thermodynamical quantities of those systems. One particular problem which has received a lot of attention recently, is the evolution of random growing interfaces governed by *local* stochastic rules. In the seminal work [9], Kardar, Parisi and Zhang proposed an effective equation, nowadays wide known as the KPZ equation, for the evolution of the fluctuations around the mean of a flat growing interface. They argued that the evolution of a fluctuating interface is governed by three competing factors: *roughening*, represented by the presence of a space-time white noise, *smoothing*, represented by a diffusive term appearing in the form of a Laplacian operator, and a slope-dependent *growth*, represented by a nonlinear transport term. Taking these three ingredients into account, [9] proposed the equation

$$\partial_t h = D\Delta h + \lambda(\nabla h)^2 + \sqrt{2D\chi}\mathcal{W}, \tag{1.1}$$

where $\mathcal{W}$ is a normalized space-time white noise, that is, a Gaussian noise with correlations given by $\langle \mathcal{W}(x,t)\mathcal{W}(x',t')\rangle = \delta(x,x')\delta(t,t')$ and D, λ, χ are constants.

The main result we want to report here is the computation of the constant λ in terms of the constants D and χ. We will be more specific later on, here we just mention that D is the *diffusivity* of the system and χ is the *static compressibility* of the system. The quadratic dependence on the slope ∇h is the simplest nonlinear one that can be coupled with the Ornstein-Uhlenbeck equation, which corresponds to consider $\lambda = 0$ in (1.1).

In [9], starting from the KPZ equation the authors argued that a growing interface has a, statistically, self-similar structure with universal scaling exponents for its width and for its spatial correlations. In particular, they predicted, starting

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from the KPZ equation above, that a one-dimensional interface has fluctuations of order $t^{1/3}$, in contrast with the Edwards-Wilkinson (EW) exponent $t^{1/4}$ of the fluctuations of interfaces in equilibrium. In the EW class the fluctuations evolve according to the Ornstein-Uhlenbeck equation. In principle, the aforementioned feature should be shared by any discrete or continuous, one-dimensional growth models with local stochastic dynamics. Starting from [21], this question has been investigated through Monte Carlo simulations for various simplified models, like the Eden model, the random deposition model and the polynuclear growth model. Universal exponents were confirmed by those simulations, giving support to the KPZ conjecture. We say that a model belongs to the KPZ universality class if its corresponding fluctuations follow the exponents predicted by [9]. For an early review and further references, we refer to [10].

A first ground-breaking contribution, and also the first mathematically rigorous result in this direction, was obtained by [4]. In that article, the authors derived the so-called Cole-Hopf solution of the KPZ equation as the scaling limit of the fluctuations of an interface associated to a particular interacting particle system: the *weakly asymmetric* simple exclusion process. The simple exclusion process is a system of interacting particles evolving on $\mathbb{Z}$ with the following stochastic dynamics. Let $p, q \geq 0$ be such that $p + q = 1$. Independently, the position of a particle is updated with rate 1. The update rules are the following: the particle tries to jump to the right neighboring site with probability p and to the left neighboring site with probability q. The jump is successful if there is no particle at the target position. The weakly asymmetric scaling corresponds to the choice $p = \frac{1}{2} + a\sqrt{\varepsilon}$, where $\varepsilon > 0$ is a scaling parameter which goes to 0 and a can be understood as the strength of the asymmetry.

The work of [4] does not say anything about the scaling exponents of the fluctuations of the system, but it clarifies the role of the KPZ equation as an effective model for the evolution of fluctuations in one-dimensional growth models. In particular, the behavior of asymmetric growth models like the TASEP (which corresponds to the choice $p = 1$ above), the polynuclear growth model, etc, should be related with the *long-time behavior* of the KPZ equation.

A second ground-breaking contribution is contained in the paper [15], where it is computed the probability distribution of the height function on a discrete model, known as the corner growth model, with a particular initial configuration, namely, the wedge profile (see also [2, 17]). There it is shown that the fluctuations of the height function of that model are given by the Tracy-Widom (TW) distribution [20], and that the corresponding scaling exponent is effectively $\frac{1}{3}$. We point out that the TW distribution is observed in a strongly asymmetric regime, which in the case of exclusion processes corresponds to the choice $p \neq \frac{1}{2}$ independently of the scaling. In [1, 18], the authors showed that the KPZ equation serves as a crossover equation connecting the KPZ universality class ($p \neq \frac{1}{2}$ in the case of the exclusion process) and the EW universality class ($p = \frac{1}{2}$ in the case of the exclusion process). More precisely, they showed that, as $t \to \infty$, $t^{-1/3}\bar{h}(t, 0)$ converges to a TW distribution, and as $t \to 0$, $t^{-1/4}\bar{h}(t, 0)$ converges to a normal distribution. Here $\bar{h}(t, 0)$ is the centered solution of the KPZ equation with wedge initial profile.

In this sense, the KPZ equation is the universal object that serves as separation between the EW ($t^{1/4}$-scaling) and KPZ ($t^{1/3}$-scaling) universality classes.

More recently, there has been a new way to solve the KPZ equation. The analysis is performed model-by-model and the idea is to obtain some determinantal formulas related to some functionals of the model in question. These formulas can be solved by using the machinery of random matrix theory, which provides very detailed information on the limiting distribution of those functionals, for more details see [5], [7] and references therein.

The main drawback of all the approaches described above is the lack of generality. All of them are based on intricate combinatoric properties of the model; in [19] the term *stochastic integrability* is coined to point out this fact. In particular, the non-universal constants in the KPZ equation do not depend on the thermodynamical properties of the models. This is not true in general, as we will see below. From the point of view of the thermodynamical properties of microscopic growth models, it is more convenient to adopt a different approach. For that purpose, define $\mathcal{Y}_t = -\nabla h_t$. Since h_t solves the KPZ equation (1.1), then the field $\mathcal{Y}_t$ solves the *stochastic Burgers equation* (SBE) given by

$$\partial_t \mathcal{Y}_t = D\Delta \mathcal{Y}_t + \lambda \nabla \mathcal{Y}_t^2 + \sqrt{2\chi D}\nabla \mathcal{W}. \tag{SBE}$$

If we start with a discrete growth model $\{\zeta_t(x); x \in \mathbb{Z}\}$ with local interactions, the discrete gradients defined as $\eta_t(x) = \zeta_t(x) - \zeta_t(x+1)$ can be interpreted as a *conservative* interacting particle system, and vice-versa, for a given conservative interacting particle system, the cumulative currents of the system can be interpreted as an interface growth model. Therefore, from now on we stick to the interacting particle systems interpretation.

An important feature which is not captured by the results of [4][1], is the dependence of λ in the thermodynamical properties of the system. At principle there is no obvious relation between λ and the underlying particle system, but in fact, there is a relation between them and the purpose of this paper is to describe precisely that relation. Notice that, the steady states of the exclusion process are parametrized by its density ρ. Taking the jump rates $p = \frac{1}{2} + a\sqrt{\varepsilon}$, we get to the SBE equation above with $\lambda = a$, which does not depend on ρ. This is very particular of the chosen jump rates and in general is not true. We will show here a *second-order Einstein relation*, namely the relation

$$\lambda = \frac{a}{2}\frac{d^2}{d\rho^2}\chi(\rho)D(\rho).$$

The outline of this paper is as follows. In Section 2 we present the model that we have chosen, namely, the gradient Kawasaki dynamics in order to present our main result. In Section 3 we describe its equilibrium fluctuations in the setting of the KPZ scaling. In Section 4 we present some conclusions and future directions.

[1]We point out that in [9], the authors use generic constants in front of the three terms of this equation, and they do not discuss their meanings in terms of thermodynamical quantities of the underlying systems.

2. The gradient Kawasaki dynamics

Consider the space $\Omega = \{0,1\}^{\mathbb{Z}}$ of binary sequences. We call $\eta = \{\eta(x) \,:\, x \in \mathbb{Z}\}$ the elements of Ω and we interpret $\eta(x) = 1$ as having a particle at the site x and $\eta(x) = 0$ as having a hole at the site x. Given a Gibbs measure μ in Ω, the Kawasaki dynamics is a local, particle-conservative, interacting particle system for which the measure μ is invariant and reversible. Given a Hamiltonian $\mathcal{H}$, in [16] it is explained how to choose a Kawasaki dynamics satisfying an additional property, the so-called *gradient condition*. Let $J = \{J_A \,:\, A \subseteq \mathbb{Z}\}$ be a finite-range, translation-invariant potential, that is, there exists a constant R such that $J_A = 0$ whenever $\mathrm{diam}(A) > R$ and $J_A = J_{A+x}$ for any $x \in \mathbb{Z}$. For $\Lambda \subset \mathbb{Z}$, define the Hamiltonian $\mathcal{H}_\Lambda$ as

$$\mathcal{H}_\Lambda(\eta) = \sum_{A \cap \Lambda \neq \emptyset} J_A \eta(A),$$

where $\eta(A) = \prod_{x \in A} \eta(x)$. Consider an inverse temperature β such that the Gibbs measure

$$\mu_\beta = \lim_{\Lambda} \frac{1}{Z_{\Lambda,\beta}} e^{-\beta \mathcal{H}_\Lambda}$$

is well-defined. Above, $Z_{\Lambda,\beta}$ is a normalizing constant. For $x \in \mathbb{Z}$ and $\eta \in \Omega$, let us denote by $\eta^{x,x+1}$ the configuration obtained from η by exchanging the occupation variables at the sites x and $x+1$, namely,

$$\eta^{x,x+1}(y) = \eta(x+1)\mathbf{1}_{y=x} + \eta(x)\mathbf{1}_{y=x+1} + \eta(x)\mathbf{1}_{y \neq x,x+1}.$$

Let θ be the standard spatial shift in $\mathbb{Z}$, that is, for $x \in \mathbb{Z}$ and $\eta \in \Omega$,

$$\theta_x(\eta)(y) = \eta(x+y).$$

In [16], it is shown the existence of a function $c : \Omega \to \mathbb{R}$ such that:

i) *Local dynamics.*
The value of $c(\eta)$ depends only on $\{\eta(x) \,:\, |x| \leq r\}$ for some $r > 0$.

ii) *Ergodicity and exclusion rule.*
$c(\eta) > 0$ if $\eta(0) \neq \eta(1)$ and $c(\eta) = 0$ if $\eta(0) = \eta(1)$.

iii) *Detailed balance.*
If $\eta(0) \neq \eta(1)$,

$$\frac{c(\eta)}{c(\eta^{0,1})} = e^{-\beta(\mathcal{H}(\eta^{0,1}) - \mathcal{H}(\eta))}.$$

iv) *Gradient condition.*
There exists a local function $\omega : \Omega \to \mathbb{R}$ such that

$$c(\eta)(\eta(0) - \eta(1)) = \omega(\eta) - \omega(\theta_1 \eta).$$

Since c is local, it is bounded. Therefore, without loss of generality, we can assume that c is bounded by 1. Now we define the Kawasaki dynamics associated to c.

Independently, for each $x \in \mathbb{Z}$, we exchange the occupation variables $\eta_t(x)$ and $\eta_t(x+1)$ at rate 1. The exchange is accepted with probability $c(\theta_{-x}\eta_t)$ and otherwise rejected. We call the stochastic process $\{\eta_t \,:\, t \geq 0\}$ defined in this way the *gradient Kawasaki dynamics* associated to c. The gradient Kawasaki dynamics is particle-conservative in the sense that particles are nor destroyed neither created by the dynamics. This conservation law implies that a family of invariant measures can be obtained introducing a *fugacity*, denoted by ϕ, which controls the average number of particles in the system. For that purpose, define the Hamiltonian $\mathcal{H}_{\Lambda,\beta,\phi}$ by

$$\mathcal{H}_{\Lambda,\beta,\phi}(\eta) = \beta\mathcal{H}_{\Lambda}(\eta) + \phi \sum_{x\in\Lambda} \eta(x).$$

Then, the measure

$$\mu_{\beta,\phi} = \lim_{\Lambda} \frac{1}{Z_{\Lambda,\beta,\phi}} e^{-\mathcal{H}_{\Lambda,\beta,\phi}}$$

is invariant under the Kawasaki dynamics described above. Here $Z_{\Lambda,\beta,\phi}$ is a normalizing constant.

We can introduce an asymmetry in this dynamics by redefining

$$c_\gamma(\eta) = c(\eta)(1 - \gamma\eta(1)(1 - \eta(0)))$$

for $\gamma \in (0,1]$. It turns out that the measures $\mu_{\beta,\phi}$ are also invariant with respect to the perturbed dynamics, that is, the dynamics associated to c_γ. In fact, in [16] it is shown that this property is equivalent to the gradient condition stated in iv) above.

Let us define the *density of particles* as

$$\rho(\phi) = \int \eta(0) d\mu_{\beta,\phi}.$$

Since particles are conserved by the dynamics, it is natural to consider the fugacity as a function of the density of particles, that is, $\phi = \phi(\rho)$ and to reparametrize the invariant measures by the density of particles: we write $\nu_{\beta,\rho} = \mu_{\beta,\phi(\rho)}$. In order to keep notation simple, from now on we use the notation ν_ρ to denote $\nu_{\beta,\rho}$. The two basic thermodynamical quantities associated to the symmetric dynamics are the *diffusivity*, defined as

$$D(\rho) = \frac{d}{d\rho} \int \omega d\nu_\rho,$$

and the *static compressibility*, defined as

$$\chi(\rho) = \lim_{n\to\infty} \int \Big(\frac{1}{\sqrt{n}} \sum_{x=1}^{n} (\eta(x) - \rho) \Big)^2 d\nu_\rho.$$

In the asymmetric case, another meaningful quantity is given by the *flux function* defined as

$$H(\rho) = \int j d\nu_\rho,$$

where $j(\eta) = c(\eta)\eta(0)(1-\eta(1))$. One of the versions of the fluctuation-dissipation relation states that

$$\int c(\eta)(\eta(1) - \eta(0))^2 d\nu_\rho = 2\chi(\rho)D(\rho).$$

Due to the gradient condition, we have the relation

$$c(\eta)\eta(1)(1 - \eta(0)) = \tfrac{1}{2}c(\eta)(\eta(1) - \eta(0))^2 + \tfrac{1}{2}(\omega(\theta_1\eta) - \omega(\eta)),$$

and in particular,

$$H(\rho) = \chi(\rho)D(\rho). \tag{2.1}$$

3. THE KPZ SCALING

Let $\varepsilon \in (0,1)$ be a scaling parameter which will go to 0. As pointed out in [4], the SBE appears as the scaling limit for the density of particles on a *weakly asymmetric*, particle-conservative system, under a *diffusive scaling* of time. This corresponds, in our case, to the choice

$$\gamma = a\sqrt{\varepsilon}.$$

Fix $\rho \in (0,1)$. Let $\{\eta_{t\varepsilon^{-2}} : t \geq 0\}$ be the gradient Kawasaki dynamics with initial distribution ν_ρ, associated to c_γ for $\gamma = a\sqrt{\varepsilon}$. Notice that the system is being speeded up in the diffusive time scaling. Since the solutions of the SBE are distribution-valued, it is convenient to define the density fluctuation field $\{\mathcal{Y}_t^\varepsilon : t \geq 0\}$ through its action over test functions $F : \mathbb{R} \to \mathbb{R}$ belonging to the Schwartz space $\mathcal{S}(\mathbb{R})$ as:

$$\langle \mathcal{Y}_t^\varepsilon, F \rangle = \sqrt{\varepsilon} \sum_{x \in \mathbb{Z}} (\eta_{t\varepsilon^{-2}}(x) - \rho) F(\varepsilon x).$$

For $x \in \mathbb{Z}$, $\eta \in \Omega$ and $f : \Omega \to \mathbb{R}$ let us introduce the notation $f_x(\eta) = f(\theta_{-x}\eta)$. Using the Markovian character of the evolution of $\eta_{t\varepsilon^{-2}}$, the total time derivative of $\langle \mathcal{Y}_t^\varepsilon, F \rangle$ is equal to the sum of three terms:

a diffusive term

$$\sqrt{\varepsilon} \sum_{x \in \mathbb{Z}} \omega_x(\eta_{t\varepsilon^{-2}}) \Delta F(\varepsilon x),$$

a transport term

$$a \sum_{x \in \mathbb{Z}} j_x(\eta_{t\varepsilon^{-2}}) \nabla F(\varepsilon x)$$

and a noise term of instantaneous variance given by

$$\varepsilon \sum_{x \in \mathbb{Z}} c_x(\eta_{t\varepsilon^{-2}}) (\eta_{t\varepsilon^{-2}}(x+1) - \eta_{t\varepsilon^{-2}}(x))^2 (\nabla F(\varepsilon x))^2.$$

Above, $\Delta F(\varepsilon x)$ and $\nabla F(\varepsilon x)$ denote the second and the first space derivatives of F evaluated at εx, respectively. In order to identify the stochastic partial differential equation ruling the space-time evolution of the limit of $\mathcal{Y}_t^\varepsilon$, we need to "close" the terms above as functions of $\mathcal{Y}_t$. For that purpose, we need to use what is called in the literature as the Boltzmann-Gibbs principle, which was first introduced by [6] and was proved in this context by [8]. This principle allows to replace the diffusive term above by

$$D(\rho)\sqrt{\varepsilon} \sum_{x \in \mathbb{Z}} (\eta_{t\varepsilon^{-2}}(x) - \rho) \Delta F(\varepsilon x) = \langle \mathcal{Y}_t^\varepsilon, D(\rho) \Delta F \rangle.$$

The ergodic theorem shows that the instantaneous variance of the noise term is well approximated by

$$2\chi(\rho) D(\rho) \int (\nabla F)^2 dx.$$

The main novelty comes from the analysis of the transport term. First notice that there is no $\sqrt{\varepsilon}$ factor in front of the sum. Therefore, at a first glance it is not reasonable to think that this term is bounded. It turns out that the Boltzmann-Gibbs principle shows that this term is well approximated by

$$aH'(\rho) \sum_{x \in \mathbb{Z}} (\eta_{t\varepsilon^{-2}}(x) - \rho) \nabla F(\varepsilon x) = a\varepsilon^{-\frac{1}{2}} H'(\rho) \langle \mathcal{Y}_t^\varepsilon, \nabla F \rangle.$$

Therefore, we need to impose $H'(\rho) = 0$ if we want to see a non-trivial limit of $\mathcal{Y}_t^\varepsilon$. As already noticed by [9], this is a natural assumption, since the fluctuations should be observed around the *characteristic lines* of the system. Moreover, it is not a real restriction, because after a Galilean transformation of the field, the thermodynamical constants $\chi(\rho)$, $D(\rho)$ remain unchanged, while the new flux satisfies

$$\tilde{H}'(\rho) = H'(\rho) - v,$$

where v is the velocity of the Galilean transformation.

Notice that, since we are assuming $H'(\rho) = 0$, the usual Boltzmann-Gibbs principle does not give any useful information about the behavior of the transport term. In order to study the limit of the transport term, in [11, 12, 13, 14] we introduced, what we call, the *second-order Boltzmann-Gibbs principle*, see for example Theorem 7 in [13]. For that purpose, let $\iota : \mathbb{R} \to \mathbb{R}$ be a positive test function with $\int_{\mathbb{R}} \iota(x)dx = 1$, and define the approximation of the identity $\iota_\delta(x) = \frac{1}{\delta}\iota(\frac{x}{\delta})$, $\delta > 0$. Let $f : \Omega \to \mathbb{R}$ be a local function and let

$$\tilde{f}(\rho) = \int f d\nu_\rho.$$

Assume that $\tilde{f}'(\rho) = 0$. Then, the second-order Boltzmann-Gibbs principle asserts that for any $T > 0$ and for any test function $F \in \mathcal{S}(\mathbb{R})$, the time integral

$$\int_0^T \sum_{x\in\mathbb{Z}} f_x(\eta_{t\varepsilon^{-2}})F(\varepsilon x)\, dt$$

is very well approximated by

$$\int_0^T \tfrac{1}{2}\tilde{f}''(\rho)\varepsilon \sum_{x\in\mathbb{Z}} F(\varepsilon x)\Big(\langle \mathcal{Y}_t^\varepsilon, \iota_\delta^{\varepsilon x}\rangle^2 - \frac{\kappa\chi(\rho)}{\delta}\Big)\, dt,$$

see the statement of Theorem 7 of [13] or Theorem 3.2 of [14]. Above, we denote by $\iota_\delta^{\varepsilon x}$ the approximation of the identity centered at εx and $\kappa = \int_{\mathbb{R}} \iota(x)^2 dx$. The approximation holds when $\varepsilon \to 0$ *and then* $\delta \to 0$. Notice the Wick renormalization represented by the diverging term $\frac{\kappa\chi(\rho)}{\delta}$. The justification of this approximation comes from the equivalence of ensembles, which now we explain. Fix a local function $f : \Omega \to \mathbb{R}$ and let $\psi_f^\ell(\sigma)$ be the expectation of f with respect to the canonical ensemble in $\Lambda_\ell = \{1, \dots, \ell\}$ with density of particles $= \frac{1}{\ell}\sum_{x=1}^{\ell} \eta(x) = \sigma$:

$$\psi_f^\ell(\sigma) = E_\rho^\ell\Big[f \,\Big|\, \frac{1}{\ell}\sum_{x=1}^{\ell} \eta(x) = \sigma\Big],$$

where E_ρ^ℓ denotes the expectation with respect to the canonical ensemble on Λ_ℓ. Then, morally, this condition expectation satisfies a "Taylor" expansion, in the sense that

$$\psi_f^\ell(\sigma) = \tilde{f}(\rho) + \tilde{f}'(\rho)(\sigma-\rho) + \tfrac{1}{2}\tilde{f}''(\rho)(\sigma-\rho)^2$$

plus an error term of order $(\sigma - \rho)^3$.

Applying the second-order Boltzmann-Gibbs principle for $f = j$, we see that the transport term is well approximated by

$$\tfrac{a}{2}H''(\rho)\varepsilon \sum_{x\in\mathbb{Z}} \nabla F(\varepsilon x)\Big(\langle \mathcal{Y}_t^\varepsilon, \iota_\delta^{\varepsilon x}\rangle^2 - \frac{\kappa\chi(\rho)}{\delta}\Big).$$

Notice that the Wick renormalisation is not needed in this case, since $\sum_{x\in\mathbb{Z}} \nabla F(\varepsilon x) = 0$. In conclusion, we have just shown that the total time-derivative of $\langle \mathcal{Y}_t^\varepsilon, F\rangle$ satisfies

$$\tfrac{d}{dt}\langle \mathcal{Y}_t^\varepsilon, F\rangle = D(\rho)\langle \mathcal{Y}_t^\varepsilon, \Delta F\rangle + \tfrac{a}{2}H''(\rho)\langle (\mathcal{Y}_t^\varepsilon * \iota_\varepsilon)^2, \nabla F\rangle + \sqrt{2\chi(\rho)D(\rho)}\langle \mathcal{W}_t, \nabla F\rangle$$

plus an error term which vanishes as $\varepsilon \to 0$ and then $\delta \to 0$. We see that when $\varepsilon \to 0$, the process $\mathcal{Y}_t^\varepsilon$ converges to the process $\mathcal{Y}_t$, solution of the (SBE) with $\lambda = \frac{a}{2}H''(\rho)$. Recall (2.1) and that a represents the strength of the asymmetry introduced in the system. In particular, we identify the constant λ of the KPZ/SBE equations with

$$\lambda = \frac{a}{2}H''(\rho) = \frac{a}{2}\frac{d^2}{d\rho^2}\chi(\rho)D(\rho),$$

which proves the second-order Einstein relation for the KPZ/SBE equations. From all these conclusion, the constants in the KPZ/SBE equation are given in terms of the thermodynamical quantities of the system and the SBE equation for $\mathcal{Y}_t$ now reads as

$$\partial_t \mathcal{Y}_t = D(\rho)\Delta \mathcal{Y}_t + \lambda \nabla \mathcal{Y}_t^2 + \sqrt{2\chi(\rho)D(\rho)}\nabla \mathcal{W}. \tag{3.1}$$

4. Conclusions and comments

We have shown that the non-universal coefficients of the KPZ/SBE equations in $d = 1$ can be obtained as functions of the thermodynamical quantities associated to the underlying interacting particle system. More precisely, denote by ρ the average value of the conserved quantity in a one-dimensional, conservative, weakly asymmetry stochastic dynamics and by ν_ρ the stationary state associated to ρ. Let $\chi = \chi(\rho)$ be the static compressibility of the dynamics, which is simply the self-correlation of the conserved quantity with respect to ν_ρ. Let $D = D(\rho)$ be the diffusion coefficient associated to the symmetrized dynamics, computed using the Green-Kubo formula or any other convenient formula. Let a be a parameter regulating the strength of the asymmetry. Then, the space-time fluctuations of the conserved quantity, with respect to the stationary state ν_ρ, are given by the KPZ/SBE equation

$$\partial \mathcal{Y}_t = D\Delta \mathcal{Y}_t + \lambda \nabla \mathcal{Y}_t^2 + \sqrt{2\chi(\rho)D(\rho)}\nabla \mathcal{W},$$

where $\mathcal{W}$ is a space-time white noise of covariance matrix $\delta(x-x')\delta(t-t')$ and $\lambda = \lambda(a,\rho)$ satisfies

$$\lambda = \frac{a}{2}\frac{d^2}{d\rho^2}\chi(\rho)D(\rho).$$

This relation can be understood as an Einstein relation for the KPZ/SBE, since the transport term turns out to be proportional to the strength of the asymmetry, and the constant of proportionality is given in terms of the thermodynamical quantities associated to the model.

Notice that λ is linear in a. This is due to the weak asymmetry of the model, which can be interpreted as a vanishing perturbation of the symmetric dynamics. Therefore, our result can be in interpreted in terms of linear response theory.

Another remark is that our derivation can be carried out only for gradient systems. General Kawasaki dynamics are non-gradient, as well as, many other models. The main obstruction in our derivation is that for non-gradient models, the invariant measures of the perturbed dynamics are not known and even their existence has not been established yet. However, at the level of large deviations, it has been recently shown that weakly asymmetric non-gradient systems behave like gradient systems, in the sense that the relation between the rate function and the thermodynamical variables is the same as in the gradient scenario. Therefore, we expect the same to be true in our case, see [3] for a more detailed discussion.

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DEPARTAMENTO DE MATEMÁTICA, PUC-RIO, RUA MARQUÊS DE SÃO VICENTE, NO. 225, 22453-900, RIO DE JANEIRO, RJ-BRAZIL AND CMAT, CENTRO DE MATEMÁTICA DA UNIVERSIDADE DO MINHO, CAMPUS DE GUALTAR, 4710-057 BRAGA, PORTUGAL.
E-mail address: patricia@mat.puc-rio.br

IMPA, ESTRADA DONA CASTORINA 110, JARDIM BOTÂNICO, CEP 22460-340, RIO DE JANEIRO, BRAZIL
E-mail address: mjara@impa.br