Review on exact and perturbative deformations of the Einstein-Straus model: uniqueness and rigidity results

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Abstract

The Einstein-Straus model consists of a Schwarzschild spherical vacuole in a Friedman-Lemaître-Robertson-Walker (FLRW) dust spacetime (with or without Λ). It constitutes the most widely accepted model to answer the question of the influence of large scale (cosmological) dynamics on local systems. The conclusion drawn by the model is that there is no influence from the cosmic background, since the spherical vacuole is static. Spherical generalizations to other interior matter models are commonly used in the construction of lumpy inhomogeneous cosmological models. On the other hand, the model has proven to be reluctant to admit non-spherical generalizations. In this review, we summarize the known uniqueness results for this model. These seem to indicate that the only reasonable and realistic non-spherical deformations of the Einstein-Straus model require perturbing the FLRW background. We review results about linear perturbations of the Einstein-Straus model, where the perturbations in the vacuole are assumed to be stationary and axially symmetric so as to describe regions (voids in particular) in which the matter has reached an equilibrium regime.

1 Introduction

During a meal in the 19th Jena Meeting on Relativity, in September 1996, Bill Bonnor provocatively asked José Senovilla if the table could be expanding with the Universe. Not surprisingly, Bonnor later took the question seriously and wrote a paper about how the hydrogen atom is affected by the cosmic expansion [11], which is well worth reading.

About five decades before, Einstein and Straus asked a similar question, on a bigger scale, which led them to investigate *the influence of the expansion of space on the gravitational fields surrounding individual stars* [19]. They took the Schwarzschild solution representing the vacuole surrounding a star located in the centre and the Friedman-Lemaitre-Robertson-Walker (FLRW) solution as the cosmological model. At the core of their model was the matching of the two solutions across a spherical surface with constant cosmological radius. Since the expansion kept the vacuole symmetry and time independence, their conclusion was that it does not affect the gravitational fields surrounding stars and, in particular, it does not affect the solar system dynamics. A previous attempt to address the issue of whether the planetary orbits expand with the Universe was made by McVittie [44] who found a smooth model describing a spherically symmetric mass embedded in a flat FLRW.

Since then the research about this problem was scarce, although some alternatives to the McVittie model were suggested, e.g. in [22], and difficulties of the global meaning of the model were also pointed out [48, 49, 50, 51] (see also [15]). Concerning the Einstein-Straus model itself, it was revisited in [2, 8] and stability issues were raised in [57], [32] cf. also the discussion in [30]. However, the Einstein-Straus model has never stopped being considered as the correct answer to the lack of influence of the cosmological expansion on local systems. Moreover, since the vacuole can be inserted anywhere due to the homogeneity of FLRW, the Einstein-Straus model led to the original Swiss cheese model of a lumpy inhomogeneous universe (see e.g. [20]).

Bonnor's question, that 1996 afternoon, raised a totally new issue for the Einstein-Straus models, namely whether spherical symmetry was a crucial ingredient of the model and, therefore, for the existence of time-invariant bounded systems embedded in a FLRW universe. Indeed, the question triggered research by Senovilla and Vera [59], that led to the result about the impossibility of the Einstein-Straus model in cylindrical symmetry. In turn, this important result was the origin of over fifteen years of research about the rigidity, in the sense of uniqueness, of the model. The aim of this paper is to review these results on rigidity both for exact models and from a perturbative perspective.

Crucial to this endeavour was the development of a general mathematical theory of spacetime matching [41] and of its perturbative version [4, 47, 35]. This allowed to achieve quite general results about the possibility of generalizing both the shape of the cavity and the cosmological setting of the original Einstein-Straus model. As an example, described below in some detail, uniqueness of the static Λ-vacuum spherical region embedded in a non-static FLRW cosmological model has been proved [34].

The scope of the Einstein-Straus model has been taken well beyond both the physical scale originally considered and the physical problems for which the model was conceived. In fact, the model has been used not only at the solar system scale, but also on galaxy [27] and galaxy clusters' scales [27, 54]. On the other hand, the vacuole of the (original) Einstein-Straus model has been replaced by other spherically symmetric geometries, generally Lemaître-Tolman-Bondi (LTB), also spherically shaped regions of Szekeres, in order to construct *"generalized" Einstein-Straus* models for describing extra-galactic scale and cosmic voids (we refer to the reviews in [30, 20]). Lumpy inhomogeneous cosmological models based on the generalized Einstein-Straus Swiss cheese models are being used in the search of possible explanations to the accelerated expansion of the Universe by the study of lensing effects at cosmic scales produced by the voids (see e.g. [20]).

So far, all these generalized Einstein-Straus (and the corresponding Swiss cheese) models have assumed spherically shaped inhomogeneities (voids). One of themes of the research we will review here is how far can one push the Einstein-Straus model towards non-spherical generalizations. The fundamental ingredient we want to keep is that the bound system remains stationary, so as to keep the absence of influence of the cosmic dynamics on astrophysical scales. We will use the term *Einstein-Straus problem* to the problem of finding the most general stationary regions (vacuum or not) one can embed in a realistic cosmological model in a broad sense.

The Einstein-Straus model has also been taken beyond the exact solutions' settings to include metric perturbations. Perturbation theory in General Relativity (GR) is a natural framework to study small departures from symmetric configurations and thus to perform stability analysis. For instance, it allows to include simultaneously density, rotational and gravitational wave perturbation modes into an, otherwise, spatially homogeneous and isotropic cosmological model. Most interestingly, it allows *a priori* to perturb independently the interior and exterior spacetimes as well as the matching boundary. Furthermore, although the three perturbation modes are decoupled on a FLRW background, they may couple at a matching boundary. In this context, an important question is how general can the perturbations be in each model.

Inherent to perturbation theory is the issue of gauge freedom. In perturbed spacetime matchings, this can be complicated by the fact that three independent perturbation gauges may be in use. For the sake of completeness, we include a short review of linearized matching, where these issues are discussed, see also [47] for further details, including the definition of the so-called *doubly gauge invariant variables*.

Perturbations in the FLRW background are customarily split in scalar, vector and tensor modes, and the later are generically viewed as cosmological gravitational waves. Given that the gravitational wave detectors are already active, and gravitational waves are expected to be detected within the next five years (see e.g. [6, 55] for recent accounts and the review [56]), it would be interesting to investigate their inclusion in the models. One now certainly has the necessary mathematical machinery to do so, and preliminary results indicate the possibility of having, for instance, a stationary axially symmetric vacuole embedded in an expanding cosmological model containing tensor modes [38]. Even more, the linearized matching links the rotational and tensor modes degrees of freedom in the perturbation variables [38].

This is therefore an interesting timing to revise the state-of-the-art and point out potentially interesting directions of research about the Einstein-Straus problem, in the sense pointed out above.

The plan of the review is the following. In Section 2, the Einstein-Straus model is briefly presented. Similar summaries with different degrees of detail can be found in many places in the literature, see e.g. [20]. We include it here for the sake of completeness and in order to fix our notation. Section 3 is devoted to describing in some detail the uniqueness results concerning both general static regions and stationary and axisymmetric regions (irrespective of any symmetry consideration and/or matter content) embeddable in a FLRW expanding cosmology. Section 3.1 is devoted to the static case. The main conclusion here is that the only static vacuum region that can be embedded to an expanding FLRW is a spherically shaped region of Schwarzschild (i.e. the Einstein-Straus model). Similar uniqueness results hold for other matter models, such as vacuum with cosmological constant. Section 3.2 deals, in turn, with the uniqueness results for stationary and axially symmetric regions in FLRW expanding universes. The main result is that the stationary region must, in fact, be static, so that the previous conclusions on static regions apply. The uniqueness result thus states that the only way of having a stationary and axially symmetric or static region in an expanding FLRW is the Einstein-Straus model. Following the uniqueness results, the robustness of the Einstein-Straus model is further analyzed by considering alternative exact cosmological models. In Section 4 the replacement of the FLRW region by more general anisotropic cosmologies, i.e. the Bianchi models, is studied for static locally cylindrically symmetric interiors, leading to severe restrictions and no-go results for reasonable evolving cosmologies. The final part of the paper is devoted to the generalization of the Einstein-Straus model from a perturbative perspective. After a brief overview of perturbative matching theory in Section 5, and the use of the Hodge decomposition on the sphere instead of the usual spherical harmonic decomposition in Section 6, the linearized matching between stationary and axisymmetric perturbations of Schwarzschild and general perturbed FLRW is reviewed in Section 7. We finish with some conclusions in Section 8, pointing out some ongoing research, prospects for future work on the perturbed Einstein-Straus model and its possible implications on the relationship between astrophysical bounded systems and cosmological dynamics in the form of cosmic gravitational waves.

2 The Einstein-Straus model

This model consists of a spherically symmetric (both in shape and intrinsic geometry) Ricci-flat region embedded in a FLRW universe without cosmological constant. Recall that two spacetimes can be matched across their boundary if and only if the first fundamental forms q and second fundamental forms K agree on the matching hypersurface. A consequence of this are the well-known Israel conditions, which restrict the jump of the energy-momentum tensor across the boundary. In the present context, they imply that the cosmological fluid must be dust and the vacuole must be comoving with the cosmological fluid. Writing the FLRW metric in cosmic time coordinates

$$g^{\text{RW}} = -dt^2 + a^2(t) g_{\mathcal{M}}, \qquad \text{with} \qquad g_{\mathcal{M}} = dR^2 + \Sigma^2(R,\epsilon) d\Omega^2, \tag{1}$$

where $\epsilon = \{-1, 0, +1\}$, $\Sigma'^2 = 1 - \epsilon\Sigma^2$ with prime denoting derivative with respect to R, in units $G = c = 1$ the Friedman equation reads

$$\dot{a}^2 + \epsilon = \frac{8\pi\rho_0}{3a},$$

where the dot denotes derivative with respect to t and ρ_0 is a constant such that the cosmological energy-density ρ satisfies $\rho = \rho_0/a^3$. The boundary of the vacuole can be parametrized by $\{t = t, R = R_0\}$ (we ignore the angular variables as they behave trivially, and use t both as spacetime coordinate and intrinsic coordinate on the hypersurface, the precise meaning will be clear from the context). For the matching one needs the induced metric q^{RW} and the second fundamental form K^{RW}. Using the outward unit normal $\boldsymbol{n}_{\text{RW}} = a(t)dR$ these objects read, with $\Sigma_c := \Sigma|_{R=R_0}$, and $\Sigma'_c := \Sigma'|_{R=R_0}$,

$$q^{\text{RW}} = -dt^2 + a^2(t)\Sigma_c^2 d\Omega^2, \qquad K^{\text{RW}} = a(t)\Sigma_c\Sigma'_c d\Omega^2.$$

From Birkhoff's theorem, the geometry of the vacuole is Kruskal. Assuming that the boundary is away from the Schwarzschild horizon (this happens sufficiently away from the big bang or big crunch) the interior metric can be written in Schwarzschild coordinates

$$g^{\text{Sch}} = -\left(1 - \frac{2m}{r}\right)^2 dT^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\Omega^2. \tag{2}$$

The boundary can be parametrized as $\{T = T_0(t), r = r_0(t)\}$ and, given the time inversion symmetry of Schwarzschild, we can assume $\dot{T}_0 > 0$ without loss of generality. The induced metric on the boundary is

$$q^{\text{Sch}} = -N^2(t)dt^2 + r_0^2(t)d\Omega^2, \qquad N^2 = \left(1 - \frac{2m}{r_0}\right)\dot{T}_0^2 - \frac{\dot{r}_0^2}{1 - \frac{2m}{r_0}}.$$

Using the unit normal $\boldsymbol{n}_{\rm Sch} = \frac{1}{N}(\dot{T}_0 dr - \dot{r}_0 dT)$ (note that the global sign of N is kept free at this stage), the second fundamental form is

$$K^{\rm Sch} = \frac{1}{N}\left(-\dot{T}_0\ddot{r}_0 + \ddot{T}_0\dot{r}_0 + \frac{3m\dot{r}_0^2\dot{T}_0}{r_0\left(r_0 - 2m\right)} - \frac{m}{r_0^2}\left(1 - \frac{2m}{r_0}\right)\dot{T}_0^3\right)dt^2 + \frac{\dot{T}_0\left(r_0 - 2m\right)}{N}d\Omega^2.$$

Equality of the t-component of the induced metric requires $N^2 = 1$. Then, equality of the angular parts of the first and second fundamental forms imposes $N = 1$ and

$$r_0(t) = \Sigma_c a(t), \qquad \dot{T}_0 = \frac{a(t)\Sigma_c\Sigma_c'}{a(t)\Sigma_c - 2m}. \tag{3}$$

A straightforward computation shows that the equality of the t-component of the induced metric and second fundamental forms are satisfied provided the values of ρ_0 and m are linked by

$$m = \frac{4\pi}{3}\rho_0\Sigma_c^3,$$

which has a clear interpretation in terms of (Misner-Sharp)-mass conservation. This is the Einstein-Straus model [19].

A natural generalization consists in adding a cosmological constant Λ both to the FLRW and to the interior part (originally considered in [25] and fully solved in [2]). The Israel matching conditions on the energy-momentum tensor now impose the FLRW matter model to be dust with Λ, so that the Friedman equation is now

$$\dot{a}^2 + \epsilon = \frac{8\pi\rho_0}{3a} + \frac{\Lambda}{3}a^2.$$

By Birkhoff's theorem, the interior metric is the Kottler solution (also known as "Schwarzschild-(anti) de Sitter"), which away from the horizons is

$$g^{\mathcal{K}} = -\left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right)^2 dT^2 + \frac{dr^2}{1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}} + r^2 d\Omega^2.$$

In this case, the matching conditions are

$$r_0(t) = \Sigma_c a(t), \qquad \dot{T}_0 = \frac{a(t)\Sigma_c\Sigma_c'}{a(t)\Sigma_c - 2m - \frac{\Lambda}{3}\Sigma_c^3 a(t)^3}, \qquad m = \frac{4\pi}{3}\rho_0\Sigma_c^3.$$

We emphasize that any matching of two spacetimes immediately leads to a complementary matching, at least locally, where the "interior" and "exterior" regions on each spacetime reverse their roles (see [21] for details). In the matching above, this leads to the Oppenheimer-Snyder collapse model [53].

3 Rigidity of the Einstein-Straus model

The Einstein-Straus model is such that, for a given total mass inside the vacuole and a given energy density in the cosmological background, the radius of the static region is uniquely fixed. This already poses difficulties for the model since it is often the case that the size of the vacuole does not match the observed sizes of clustered matter in the universe, such as stars or galaxies. This fact indicates that the Einstein-Straus model may be lacking flexibility to accommodate the various situations present in cosmology (cf. [58] and [10]). In fact, the Einstein-Straus vacuole was found to be radially unstable in a certain sense [30]. The other main restriction of the model is its exact spherical symmetry. It is clear that vacuoles in the universe are not exactly spherically symmetric so a natural question is how robust is the model to non-spherical generalizations.

The first thing to consider is which fundamental ingredients of the model should be kept. The main motivation of the Einstein-Straus model was its ability to combine cosmological expansion at large scales with no observable effects on the local physics. Thus, the fundamental ingredient to keep is the absence of influence of the cosmic expansion inside the region. The simplest and most natural way to achieve this is imposing that the interior geometry is stationary, because then no dynamical effects whatsoever from the surrounding evolving cosmology would affect the local physics. Among stationary interiors, the simplest case corresponds to static situations, so it is natural to start with this case (note that the Einstein-Straus model is itself static).

The question is then how rigid or flexible is the possibility of having stationary/static regions embedded in an otherwise expanding FLRW universe. Ideally, one would like to make no further assumptions and find the most general model with these properties. The matter model inside may also be kept arbitrary, and see what are the possibilities allowed by the coexistence of a stationary/static region inside an expanding FLRW universe. This coexistence has been sometimes called the Einstein-Straus *problem* in the literature (see e.g. [2]). The first indication that the Einstein-Straus model might be very rigid came from a seminal work by Senovilla and Vera [59] who considered the possibility of matching a static and cylindrically symmetric region (with no restriction on the matter model) with a FLRW dynamical cosmology. The matching hypersurface was taken to be *locally* a cylinder, in the sense of being tangent in an open set to the two (commuting) generators of two spatial local isometries. No global consideration was needed. The result was a no-go theorem: no such model exists.

With the impossibility of generalizing the Einstein-Straus model to a cylindrical setting, it became of interest to study the problem in as much generality as possible. The static case was treated in complete generality in [33], [34] and it is by now well-understood. The more complicated stationary situation has been studied [52] under the additional assumptions of axisymmetry and a group action orthogonally transitive. The motivation to study this simplified problem lies in the fact that one expects equilibrium configurations to also exhibit an axial symmetry. It is worth to mention that one step in the black hole uniqueness theorems corresponds to showing that the domain of outer communications must be axially symmetric [26].

We devote the following Section 3.1 to describing the main results in the static setting, and Section 3.2 to review the uniqueness results in the stationary and axially symmetric setting.

3.1 Uniqueness results in the static case

This case was first considered in [33] under the additional assumption of axial symmetry, and one extra technical assumption relating cosmic and static times on the matching hypersurfaces. Both assumptions were dropped in [34] where a satisfactory uniqueness result for static region in FLRW was obtained.

The setup consists on a spacetime $(\mathcal{V}, g)$ composed by two regions $(\mathcal{W}^{\text{ST}}, g^{\text{ST}})$ and $(\mathcal{W}^{\text{RW}}, g^{\text{RW}})$ matched in absence of surface layers across their boundaries, denoted by Ω^{ST} and Ω^{RW} respectively, which once identified conform to a hypersurface Ω in $(\mathcal{V}, g)$. The region $(\mathcal{W}^{\text{ST}}, g^{\text{ST}})$ is strictly static, i.e. admits a Killing vector ξ which is timelike and orthogonal to hypersurfaces everywhere. $(\mathcal{W}^{\text{RW}}, g^{\text{RW}})$ is a codimension-zero submanifold with smooth boundary Ω^{RW} of the FLRW spacetime $(\mathcal{V}^{\text{RW}}, g^{\text{RW}})$, by which we mean the manifold $\mathcal{V}^{\text{RW}} = I \times \mathcal{M}$, where $I \subset \mathbb{R}$ is an open interval, $\mathcal{M}$ is either $\mathbb{E}^3$ $(\epsilon = 0)$, $\mathbb{S}^3$ $(\epsilon = 1)$ or $\mathbb{H}^3$ $(\epsilon = -1)$ and the FLRW metric g^{RW} takes the form (1). We call any coordinate system $\{R, \theta, \phi\}$ in which the metric takes this form a *spherical coordinate system*. Note that since $(\mathcal{M}, g_{\mathcal{M}})$ is homogeneous, there exist spherical coordinate systems centered at any point $p \in \mathcal{M}$, and this will be used below.

The function $a(t)$ is positive and smooth (in fact C^3 suffices). We assume that $\dot{a}$ does not vanish on any open set (this excludes uninteresting situations where the FRLW does not evolve). Define the "geometric" energy-density ρ^{RW} and pressure p^{RW} by

$$8\pi\rho^{\text{RW}} := 3(\dot{a}^2 + \epsilon)/a^2, \qquad 8\pi p^{\text{RW}} := (-2a\ddot{a} - \dot{a}^2 - \epsilon)/a^2, \tag{4}$$

so that, if the spacetime has a cosmological constant Λ, the energy-density and pressure of the cosmic fluid is $\rho = \rho^{\text{RW}} - \frac{\Lambda}{8\pi}$, $p = p^{\text{RW}} + \frac{\Lambda}{8\pi}$. We make the assumption that $\rho^{\text{RW}} + p^{\text{RW}} \neq 0$ so that we do have a non-trivial cosmic fluid (this allows us to define t unambiguously). Concerning the boundary Ω^{RW} it is assumed to be connected (this is irrelevant because the matching conditions are local, the assumption is made merely for notational convenience), and nowhere tangential to a hypersurface of constant cosmic time t. This assumption is physically reasonable and automatically satisfied if the boundary is causal. In fact, dropping this assumption would only make the presentation more involved, but would not spoil any of the results (see [34] for a discussion). Finally, we assume that Ω^{RW} is spatially compact. A sufficient condition for this is the "energy condition" $\rho^{\text{RW}} \geq 0$, see [33], and this is in fact the assumption made in [33]. However, it can be proved that spatial compactness suffices for the validity of all the results below. Note finally, that spatial compactness is indeed an assumption: allowing for non-compact boundaries, additional configurations not covered by the uniqueness results do arise, as shown in [45] (see also references therein), where configurations with planar and hyperbolic symmetries were found and analized. It would be interesting to analyze how far can one extend uniqueness without any compactness assumption on the boundary. Nevertheless, for the purposes of the Einstein-Straus problem, compactness is a completely natural assumption, as we want the local physics unaffected by the cosmological expansion be spatially confined.

By a detailed analysis of the matching conditions, the following restrictions on the boundary Ω^{RW} are obtained [34]. First of all, the intersection S_t^{RW} of Ω^{RW} with a hypersurface of constant cosmic time t is a sphere. More precisely, for any $t \in I$, there exists a point $c(t) \in \mathcal{M}$ so that S_t^{RW} is a coordinate sphere of radius $R(t)$ in a spherical coordinate system centered at $c(t)$. The radius $R(t)$ is restricted to satisfy the bound

$$\Sigma'^2 - \Sigma^2 \dot{a}^2|_{R=R(t)} > 0. \tag{5}$$

This inequality means that the surface S_t^{RW} is non-trapped (i.e. has a mean curvature vector spacelike everywhere). The necessity of this condition can be understood from the fact that no closed spacelike surface in a static spacetime can have a future (or past) causal and not-identically zero mean curvature vector [42]. The spherical symmetry of the surface S_t^{RW} and the fact that the matching conditions force the mean curvature vector to be continuous across the matching hypersurface implies that S_t^{RW} must be non-trapped, which is precisely (5). Note also that if the static Killing vector ξ admits Killing horizons and hence changes causal character, then the bound (5) is no longer necessary. This behaviour occurs in the Einstein-Straus model when $a(t)$ is sufficiently small so that $\Sigma_c a(t) = 2m$. The breakdown of the ODE (3) in the Einstein-Straus model is just a manifestation of the breakdown of the the static coordinate system in the interior region. In Kruskal coordinates, the matching would continue across $\Sigma_c a(t) = 2m$ without problem. Something similar would occur in the general setting if we allowed the static Killing to change causal character.

Returning to the shape of Ω^{RW}, the center point $c(t)$ follows a geodesic in $(\mathcal{M}, g_{\mathcal{M}})$ (which may degenerate to a point). The parameter t is not in general an affine parameter for the geodesic, so the center $c(t)$ is a priori allowed to move at any speed (in fact the trajectory can have turning points along the curve). In order to describe the matching hypersurface, choose any point c_0 along this geodesic and let c_0' be its tangent vector. We can choose a spherical coordinate system $\{\hat{R}, \hat{\theta}, \hat{\phi}\}$ centered at c_0 so that the axis of symmetry $\hat{\theta} = 0$ is along the tangent vector c_0'. In this coordinate system, the center $c(t)$ will have coordinates

$$c(t) = \{\hat{R} = \sigma(t), |\cos\hat{\theta}| = 1\}$$

(the value of $\hat{\theta}$ can be 0 or π depending on whether $c(t)$ lies after or before c_0 along the geodesic). The function $\sigma(t)$ describes the motion of $c(t)$ along the curve. The relationship between the spherical coordinate system $\{\hat{R}, \hat{\theta}, \hat{\phi}\}$ centered at c_0 and a spherical coordinate system $\{R, \theta, \phi\}$ centered at $c(t)$ with parallel axis (i.e. with coincident lines $|\cos\theta| = 1$ and $|\cos\hat{\theta}| = 1$) is easily found to be

$$\left.\begin{array}{rl} \Sigma(\hat{R})\sin\hat{\theta} & = \Sigma(R)\sin\theta \\ \Sigma(\hat{R})\cos\hat{\theta} & = \Sigma'(\sigma(t))\Sigma(R)\cos\theta + \Sigma(\sigma(t))\Sigma'(R) \\ \hat{\phi} & = \phi \end{array}\right\}. \tag{6}$$

Thus, the matching hypersurface Ω^{RW} in the spherical coordinates centered at c_0 can be parametrized by coordinates t, θ, ϕ as

$$\left.\begin{array}{rl} \Sigma(\hat{R}(t,\theta)) & = \sqrt{\sin^2\theta\ \Sigma^2|_{R(t)} + \left(\cos\theta\ \Sigma'|_{\sigma(t)}\Sigma|_{R(t)} + \Sigma|_{\sigma(t)}\Sigma'|_{R(t)}\right)^2} \\ \mathrm{cotan}(\hat{\theta}(t,\theta)) & = \mathrm{cotan}(\theta)\ \Sigma'|_{\sigma(t)} + \frac{\Sigma|_{\sigma(t)}\Sigma'|_{R(t)}}{\sin\theta\ \Sigma|_{R(t)}} \\ \hat{\phi} & = \phi \end{array}\right\},$$

which is a rather complicated form for the matching hypersurface. A useful alternative is to use a coordinate system $\{t, R, \theta, \phi\}$ in $(\mathcal{W}^{\mathrm{RW}}, g^{\mathrm{RW}})$ such that, for each t, $\{R, \theta, \phi\}$ is the spherical coordinate system centered at $c(t)$. Applying the coordinate transformation (6) to the metric

$$g^{\mathrm{RW}} = -dt^2 + a^2(t)\left(d\hat{R}^2 + \Sigma^2(\hat{R})\left(d\hat{\theta}^2 + \sin^2\hat{\theta}d\hat{\phi}^2\right)\right)$$

gives

$$\begin{aligned} ds^2 = &- dt^2 + a^2(t)\Big[(dR + f(t)\cos\theta dt)^2 + (\Sigma(R)d\theta - f(t)\Sigma'(R)\sin\theta dt)^2 + \\ &+\ \Sigma^2(R)\sin^2\theta d\phi^2\Big]. \end{aligned} \tag{7}$$

where $f(t) = \dot{\sigma}(t)$. The explicit calculation leading to (7) is somewhat long, a much more elegant method of obtaining this form of the metric is discussed in the Appendix of [34]. In these coordinates, the matching hypersurface $\Omega^{\rm RW}$ is simply $\{R = R(t)\}$.

The matching conditions in the FLRW part are supplemented with a differential equation relating the trajectory of the center $\sigma(t)$ with the radius $R(t)$. To write it down, define a function $\beta(t)$ via

$$\tanh\beta(t) = \epsilon_1 \frac{\Sigma\dot{a}}{\Sigma'}\bigg|_{R=R(t)}, \tag{8}$$

where $\epsilon_1 = \pm 1$ depending on whether the FLRW region to be matched is $R > R(t)$ or $R < R(t)$ (more specifically, the manifold with boundary $\mathcal{W}^{\rm RW}$ is $\{\epsilon_1 R \geq \epsilon_1 R(t)\}$). Note that $\beta(t)$ is well-defined because of (5). The ODE relating $R(t)$ and $\sigma(t)$ is conveniently written using an auxiliary function $\Delta(t) \neq 0$ as the following pair of differential equations

$$\dot{f} = \left(\frac{2\epsilon_1 \dot{a}\dot{R}}{\tanh\beta} + \frac{\dot{\Delta}}{\Delta} - \frac{2\cosh\beta}{\sinh\beta}\dot{\beta} \right) f,$$

$$f^2 \left.\frac{\Sigma'}{\Sigma}\right|_{R=R(t)} - \ddot{R} + \frac{\epsilon_1 \dot{a}\dot{R}^2}{\tanh\beta} + \frac{\dot{\Delta}}{\Delta}\left(\dot{R} + \frac{\epsilon_1}{a\tanh\beta} \right) - 2\left(\frac{\epsilon_1}{a} + \frac{\dot{R}}{\tanh\beta} \right)\dot{\beta} = 0. \tag{9}$$

In summary, the static domain embedded in the FLRW regions consists of a sphere with time dependent radius $R(t)$ and with center moving across the FLRW spacetime along a geodesic. Speed along the geodesic and radius $R(t)$ are linked by (9). The model is not spherically symmetric because the center of the sphere is allowed to move. However, it is very close to spherically symmetric and it turns out that the center of the sphere must be at rest for several relevant matter models in the static domain, as we discuss next.

An important consequence of the matching procedure (cf. Lemma 1 in [34]) is that the Killing vector field ξ is everywhere transverse to the matching hypersurface $\Omega^{\rm ST}$ in the static region. Combining this with the fact that the static geometry is invariant along ξ it follows that the static metric in the spacetime region obtained by dragging $\Omega^{\rm ST}$ with the static Killing vector can be fully determined in terms of hypersurface geometry of $\Omega^{\rm ST}$. Since, in turn, the geometry on $\Omega^{\rm ST}$ is related to the geometry of $\Omega^{\rm RW}$ via the matching conditions, it follows that the spacetime geometry of the static region becomes completely determined [34] in a neighbourhood of its matching hypersurface in terms of the FLRW geometry and the functions $R(t), \sigma(t)$ and $\Delta(t)$ (see Theorem 1 in [34] for details). Specifically, there exist coordinates $\{T, t, \theta, \phi\}$ so that the metric $g^{\rm ST}$ takes the form (note that t is a spacelike coordinate in the static domain)

$$g^{\rm ST} = -\frac{(\cosh\beta + \mu\sinh\beta)^2}{\Delta^2} dT^2 + (\mu\cosh\beta + \sinh\beta)^2 dt^2 +$$
$$+ a^2(t)\Sigma^2(R(t)) \left[\left(d\theta - f(t) \left.\frac{\Sigma'}{\Sigma}\right|_{R=R(t)} \sin\theta dt \right)^2 + \sin^2\theta d\phi^2 \right], \tag{10}$$

where $\mu := \epsilon a(t)(\dot{R}(t) + f(t)\cos\theta)$. The matching hypersurface $\Omega^{\rm ST}$ is defined by the embedding $\{t, \theta, \phi\} \to \{T = T(t), t, \theta, \phi\}$, where $T(t)$ satisfies $\dot{T}(t) = \Delta(t)$ and the portion of the static spacetime to be matched to the exterior region is $\{T \geq T(t)\}$. The metric (10) is foliated by round spheres $\{T = \text{const}, t = \text{const}\}$ but it is not spherically symmetric in general (unless $f(t) = 0$ and $R(t) = 0$). To complete the picture, we review the energy-momentum tensor in the static part. Introduce two one-forms and one symmetric two-tensor h by

$$\theta^0 = \frac{(\cosh\beta + \mu\sinh\beta)}{\Delta} dT, \qquad \theta^1 = (\mu\cosh\beta + \sinh\beta)\, dt,$$
$$h = a^2(t)\Sigma^2(R(t)) \left[\left(d\theta - f(t) \left. \frac{\Sigma'}{\Sigma} \right|_{R=R(t)} \sin\theta dt \right)^2 + \sin^2\theta d\phi^2 \right],$$

The Einstein tensor of $(\mathcal{W}^{\text{ST}}, g^{\text{ST}})$ is (cf. Proposition 1 in [34])

$$\frac{1}{8\pi} G^{\text{ST}} = \rho^{\text{ST}} \theta^0 \otimes \theta^0 + p_r^{\text{ST}} \theta^1 \otimes \theta^1 + p_t^{\text{ST}} h$$

where ρ^{ST}, p_r^{ST} and p_t^{ST} read

$$\begin{aligned}
\rho^{\text{ST}} &= \frac{\rho^{\text{RW}}\mu - p^{\text{RW}}\tanh\beta}{\mu + \tanh\beta}, \\
\rho^{\text{ST}} + p_r^{\text{ST}} &= \frac{\left(\rho^{\text{RW}} + p^{\text{RW}}\right)\mu}{(\mu\sinh\beta + \cosh\beta)(\sinh\beta + \mu\cosh\beta)}, \qquad (11)\\
p_t^{\text{ST}} &= \frac{3p^{\text{RW}} - \rho^{\text{RW}}}{6} + \frac{\tanh\beta\left(2f^2\sin^2\theta - \Sigma^2(R(t))(\rho + p)(\mu^2 - 1)\right)}{16\pi\Sigma^2(R(t))(\mu + \tanh\beta)(1 + \mu\tanh\beta)} + \\
&+ \frac{\left[-\ddot{\beta} + \frac{\dot{\Delta}}{\Delta}\left(\frac{\dot{a}}{a}\frac{1}{\tanh\beta} + \dot{\beta}\right) - \frac{2\dot{\beta}\dot{a}}{a}\left(1 + \frac{\mu}{\tanh\beta}\right) + \mu\left(\frac{\ddot{a}}{a} + \frac{\mu\dot{a}^2}{a^2\tanh\beta}\right)\right]}{8\pi\cosh^2\beta\,(\mu + \tanh\beta)(1 + \mu\tanh\beta)},
\end{aligned}$$

and ρ^{RW} and p^{RW} were defined in (4). Several consequences of (11) can be drawn [33, 34]. Concerning the uniqueness of the Einstein-Straus model, under the assumption $\rho^{\text{ST}} + p^{\text{ST}} = 0$ (which includes vacuum with or without cosmological constant or a non-singular electromagnetic field) it follows that $\mu = 0$ and hence $f(t) = 0$ and $R(t) = 0$. The second equation in (9) gives (with an appropriate but still completely general choice of integration constant),

$$\dot{T}(t) = \Delta(t) = \frac{1}{\Sigma'_c}\cosh^2\beta(t) = \frac{\Sigma'_c}{\Sigma'^2_c - \Sigma_c^2\dot{a}^2},$$

where (8) and the definitions of Σ_c and Σ'_c in Section 2 have been used. The static metric simplifies to

$$g^{\text{ST}} = -\left(\Sigma'^2_c - \Sigma_c^2\dot{a}^2\right) dT^2 + \frac{\Sigma_c^2\dot{a}^2}{\Sigma'^2_c - \Sigma_c^2\dot{a}^2} dt^2 + \Sigma_c^2 a^2(t) d\Omega^2.$$

From this metric, uniqueness of the Einstein-Straus model as the unique static vacuum (with or without cosmological constant) region embedded in a FRLW cosmological model follows easily. So, static vacuoles in a FLRW model must be spherically symmetric both in shape and interior geometry.

It is an open problem to analyze whether there are any physically realistic matter models in the interior static region for which the motion of the static domain inside FLRW has interesting properties. Note that a priori nothing prevents the geodesic $c(t)$ from being spacelike, so the motion of the static domain can in principle be superluminal for the cosmic observers. This is of course reminiscent to the superluminal warp drive discovered by Alcubierre [1].

3.2 Uniqueness results in the stationary and axisymmetric case

The study of axially symmetric equilibrium regions in FLRW universe was dealt with in [52], and, in short, the main result found was that those stationary regions must, in fact, be static, and therefore the results of the previous section apply.

With the same definitions and assumptions concerning the FLRW region $(\mathcal{W}^{\text{RW}}, g^{\text{RW}})$ and its boundary Ω^{RW} as in the previous section, we assume now that the interior region $(\mathcal{W}^{\text{SX}}, g^{\text{SX}})$ is (strictly) stationary and axisymmetric. More specifically, we demand that (i) the spacetime admits a two-dimensional group of isometries G_2 acting simply-transitively on timelike surfaces T_2 and containing a (spacelike) cyclic subgroup, so that $G_2 = \mathbb{R} \times \mathbb{S}^1$, and (ii) that the set of fixed points of the cyclic group is not empty. Consequences of the definition are that the G_2 group has to be Abelian [17, 14, 3], and that the set of fixed points must form a timelike two-surface [40], which is the axis. The axial Killing η is then intrinsically defined by normalizing it demanding $\partial_\alpha \eta^2 \partial^\alpha \eta^2 / 4\eta^2 \to 1$ at the axis. See also [61] and [13]. In addition, we demand that the isometry group is orthogonally transitively (OT), i.e. that the two planes orthogonal to the orbits of the isometry group are surface forming. This assumption is also known as the "circularity condition", and in many cases of interest it follows as a consequence of the Einstein field equations. Indeed, the G_2 on T_2 group must act orthogonally transitively in a region that intersects the axis of symmetry whenever the Ricci tensor has an invariant 2-plane spanned by the tangents to the orbits of the G_2 on T_2 group [16]. By the Einstein field equations, this includes Λ-term type matter (i.e. vacuum with or without cosmological constant), perfect fluids without convective motions, and also stationary and axisymmetric electrovacuum [61].

An OT stationary and axisymmetric spacetime $(\mathcal{V}^{\text{SX}}, g^{\text{SX}})$ is locally characterized by the existence of a coordinate system $\{T, \Phi, x^M\}$ $(M, N, ... = 2, 3)$ in which the line-element for the metric g^{SX} outside the axis reads [61]

$$g^{\text{SX}} = -e^{2U} (dT + A d\Phi)^2 + e^{-2U} W^2 d\Phi^2 + g_{MN} dx^M dx^N, \tag{12}$$

where U, A, W and g_{MN} are functions of x^M, the axial Killing vector field is given by $\eta = \partial_\Phi$, and a timelike (future-pointing) Killing vector field is given by $\xi = \partial_T$. Although useful for the sake of clarity, the use of coordinates is not essential for the results below, which only depend on the intrinsic geometric properties of of $(\mathcal{W}^{\text{SX}}, g^{\text{SX}})$.

As before, no specific matter content is assumed in the stationary and axisymmetric region. Regarding the matching hypersurface Ω, besides those in the previous Section 3.1, we make the only extra assumption that it preserves the axial symmetry [62] of $(\mathcal{W}^{\text{SX}}, g^{\text{SX}})$ and of $(\mathcal{W}^{\text{RW}}, g^{\text{RW}})$. This means that Ω^{SX} is assumed to be invariant under the axial symmetry of $(\mathcal{W}^{\text{SX}}, g^{\text{SX}})$ and that there is an axial Killing vector η_{RW} in the Killing algebra of $(\mathcal{W}^{\text{RW}}, g^{\text{RW}})$ tangent to Ω^{RW}.

With these assumptions at hand, in [52], it is proven, first, that the stationary (timelike) Killing vector field ξ is nowhere tangent to Ω^{SX}. As explained in the previous section, this serves in particular to construct a neighbourhood of Ω^{SX} by dragging it along the orbits of ξ, in which the geometry is thus fully determined by the information in Ω^{SX}. The main result in [52] is that if a OT stationary and axisymmetric region $(\mathcal{W}^{\text{SX}}, g^{\text{SX}})$ can be matched to a FLRW region through a hypersurface Ω^{SX} preserving the axial symmetry, then the region $(\mathcal{W}^{\text{SX}}, g^{\text{SX}})$ must be, in fact, static on a neighbourhood of Ω^{SX}.

All in all, in that neighbourhood of the matching hypersurface the OT stationary axisymmetric region $(\mathcal{W}^{\text{SX}}, g^{\text{SX}})$ thus becomes a static axisymmetric region $(\mathcal{W}^{\text{ST}}, g^{\text{ST}})$, and therefore the results of the previous Section 3.1 (cf. [34]) apply.

So far, no explicit condition on the matter content of the stationary region has been used. When conditions on the matter content on the (OT) stationary and axisymmetric (and hence static) region are imposed, the functions that determine the matching hypersurface and the static geometry are determined by the results reviewed in Section 3.1.

In particular, and for completeness, let us consider a vacuum (with or without a cosmological constant) stationary and axisymmetric region $(\mathcal{W}^{\text{SX}}, g^{\text{SX}})$ matched to FLRW preserving the axial symmetry. As mentioned above, a *vacuum* matter content forces the axial symmetry and stationary group G_2 on T_2 to act orthogonal transitively. On the other hand, a region of FLRW $(\mathcal{W}^{\text{RW}}, g^{\text{RW}})$ matched to a vacuum region must satisfy the assumptions made and, in particular, have a causal Ω^{RW} (in fact tangent to the fluid flow). The above result implies then that the vacuum region must be static. The results of Section 3.1 thus apply, and imply, in turn, that the whole region (not just its boundary) has to be spherically symmetric, and hence Schwarzschild.

To sum up, this means that the only stationary and axially symmetric vacuum region that can be matched to FLRW is a spherically symmetric piece of Schwarzschild. This constitutes still another uniqueness result of the Einstein-Straus model when the vacuum region lies inside FLRW.

The complementary result, when the vacuum region lies outside FLRW, constitutes a uniqueness result of the Oppenheimer-Snyder model [53]. It is worth noticing that this result can also be interpreted as a no-go result for the possible interiors of Kerr. Indeed, this result states that an axially symmetric region of (an evolving) FLRW, irrespective of its relative rotation with the exterior, cannot be the source of a stationary and axisymmetric vacuum region, in particular, Kerr.

4 Robustness of Einstein-Straus: Generalized exact cosmologies

In the previous sections, the robustness of the Einstein-Straus model has been discussed by considering generalizations of the interior vacuole and keeping the FLRW model as the exterior cosmological model. The question arises as to what happens if these symmetry assumptions concerning the exterior metric are relaxed. There are two different ways to study departures from the spherical FLRW: either perturbatively or using exact solutions. In this section, we concentrate on the later possibility and, in particular, we review the results found in [46] for spatially homogeneous (but anisotropic) cosmologies.

A step in the direction of generalising the exterior was taken by Bonnor, who considered the embedding of a Schwarzschild region in an expanding spherically symmetric inhomogeneous Lemaître-Tolman-Bondi (LTB) exterior. He found that such a matching is possible in general, and that it allows the average density of the Schwarzschild interior to be chosen independently of the exterior LTB density [12]. So, clearly, if the spherical symmetry is kept, the exterior can be readily generalized to the case of inhomogeneous dust cosmologies. An interesting review about physical aspects of spherically symmetric Einstein-Straus and McVittie type models can be found in [15].

While keeping the spherical symmetry of the cavity allows for straightforward generalizations of Einstein-Straus, breaking this symmetry brings in unexpected complications, as we have seen in the previous section. The same seems to hold if we consider exact generalizations of the FLRW cylindrical symmetry exterior to spatially homogeneous but anisotropic spacetimes. This problem was considered in [46] and we summarize it next:

Consider the problem of matching, across a hypersurface Ω which is spatially a topological sphere[1], of an interior locally cylindrically symmetric (LCS) static spacetime $(\mathcal{W}^{\text{ST}}, g^{\text{ST}})$, which represents a spatially compact region, to a spatially homogeneous anisotropic exterior $(\mathcal{W}^{\text{HOM}}, g^{\text{HOM}})$ having a Lie group G_3 acting on S_3 surfaces. Once more, we make no assumptions *a priori* on the matter contents, so the main results are purely geometric. Only at the end of the section, assumptions on the matter content will lead to a final no-go result.

We wish to preserve globally the axial symmetry and thus we need to impose first the existence of a group of cyclic symmetries in the exterior region. This, combined with the assumption of a spacelike spherical topological shaped[2] Ω^{ST}, implies the existence of an exterior axis. Thus, the cyclic symmetry in the exterior must really be an axial symmetry (see the discussion in Section 3.2). Moreover, we require that the cylindrical symmetry is preserved (in the sense of [62]) at least on a non-empty open subset of Ω. This has the consequence that $(\mathcal{W}^{\text{HOM}}, g^{\text{HOM}})$ must admit a G_4 on S_3 group of isometries, so that it is locally rotationally symmetric (LRS). In [46], it is shown how all the LRS spatially homogeneous metrics can be written in the form adapted to the two commuting Killing vectors defining the cylindrical symmetry ∂_φ and ∂_z, where ∂_φ is axial with axis at $r = 0$:

$$g^{\text{HOM}} = -dt^2 + b^2(t)dr^2 - 2\epsilon r b^2(t) dr dz + \hat{C}^2 d\varphi^2 + 2\hat{E} dz d\varphi + \hat{D}^2 dz^2 \tag{13}$$

with

$$\begin{aligned} \hat{C}^2 &= b^2(t)\Sigma^2(r,k) + na^2(t)(F(r,k)+k)^2, \\ \hat{D}^2 &= a^2(t) + \epsilon r^2 b^2(t), \quad \hat{E} = na^2(t)(F(r,k)+k), \end{aligned} \tag{14}$$

where the functions Σ and F are given by

$$\Sigma(r,k) = \begin{cases} \sin r, & k=+1 \\ r, & k=0 \\ \sinh r, & k=-1 \end{cases} \quad \text{and} \quad F(r,k) = \begin{cases} -\cos r, & k=+1 \\ r^2/2, & k=0 \\ \cosh r, & k=-1, \end{cases}$$

and where ϵ and n are given such that $\epsilon = 0,1;\;\; n = 0,1;\;\; \epsilon n = \epsilon k = 0$[3]. The metrics are classified according to these constants in Table 1.

The matching conditions are then investigated and a crucial step was to observe that they lead to the following necessary relations involving only exterior metric functions:

$$\hat{D}_{,t}\hat{C}_{,r} - \hat{D}_{,r}\hat{C}_{,t} \overset{\Omega}{=} 0, \quad \hat{E}_{,t}\hat{D}_{,r} - \hat{E}_{,r}\hat{D}_{,t} \overset{\Omega}{=} 0, \quad \hat{E}_{,t}\hat{C}_{,r} - \hat{E}_{,r}\hat{C}_{,t} \overset{\Omega}{=} 0, \tag{15}$$

[1] The assumption made in [46] is that the region $(\mathcal{W}^{\text{ST}}, g^{\text{ST}})$ was simply connected, in order to force Ω^{ST} being spatially a topological sphere. We prefer to impose here the assumption directly on Ω^{ST} and leave the possibility of having more general interiors.

[2] This implies the existence of north and south poles on Ω^{ST} where the axial killing vector in the interior vanishes and, therefore, also the generator of the cyclic symmetry in the exterior region on Ω^{HOM} by construction.

[3] Let us note that the case $\epsilon = n = 0$ with $k = 1$ is special, since it corresponds to the Kantowski-Sachs (KS) class of metrics, which do not admit a G_3 on S_3 subgroup [61]. It was included in the study for completeness.

Bianchi types	ϵ	n	k
I, VII_0	0	0	0
III	0	0	-1
IX	0	1	1
II	0	1	0
VIII,III	0	1	-1
V,VII_h	1	0	0

Table 1: Classification of possible G_3 on S_3 subgroup types according to the values of $\{\epsilon, k, n\}$ for the metric given by (13).

which, for non-static exteriors, imply $n = 0$ and thus exclude Bianchi types II, III, VIII and IX.

By inserting (15) back in the matching conditions one is able to prove a series of results that lead to the following conclusion: The only expanding spatially homogeneous spacetimes which can be matched to a locally cylindrically symmetric static interior region preserving the (cylindrical) symmetry, across a non-spacelike hypersurface which is spatially a topological sphere, are given by

$$ds^2 = -dt^2 + \beta^2 dz^2 + b^2(t) \left[(dr - \epsilon r dz)^2 + \Sigma^2(r,k) d\varphi^2 \right], \tag{16}$$

where β is constant. This metric for $k = 1$ belongs to the Kantowski-Sachs class, when $k = -1$ admits a G_3 on S_3 of Bianchi type III, and when $k = 0$ of Bianchi types I,V,VII_0,VII_h.

The metrics (16) are very special, partly due to the fact that condition $a_{,t} = 0$ (see (13)-(14)) imposes a strong constraint, implying that there cannot be any time (nor space) evolution along the direction orthogonal to the orbits of the subgroup G_3 on S_2 of the LRS. There are also constraints imposed through the matching in the interior region and the interested reader can find those in [46]. Note that the no-go result found in [59] is trivially recovered, since FLRW is included in the LRS class for $b(t) = a(t)$, and the above would imply a static FLRW metric.

If one specifies a particular class of matter fields, then one gets further constraints on the cosmological dynamics. For example, if the matter in the static region is not specified but the dynamical (cosmological) region is assumed to contain a perfect fluid with pressure p and energy-density ρ satisfying the dominant energy condition everywhere, then $\epsilon = 0$ necessarily, which corresponds to a stiff fluid equation of state $\rho = p = \alpha^2/(4t^2(\alpha - kt)^2)$, with $b(t) = \sqrt{\alpha t - kt^2}$, where $\alpha > 0$. On the other hand, if the interior is vacuum then the exterior must be also vacuum.

The overall no-go results, in this case, can be seen in two ways: either as a consequence of the assumption that the interior metric is static and cylindrically symmetric, which seems to prohibit time dependence along one direction, or as a consequence of the particular exterior metrics we are considering, which are homogeneous then prohibiting the coefficients along this direction to be space dependent. The perturbative approach described in the next sections can help to clarify this question.

To conclude this section we remark that there exists an example of a Einstein-Straus model with exact non-spherical inhomogeneous cosmologies. This is provided by the Szekeres dust solution which has no Killing vectors, in general, but contains intrinsic symmetries on 2-spaces of constant curvature: The Szekeres solution is divided into class

I, which generalizes the LTB solution having non-concentric spheres of constant mass, and class II which includes the Kantowski-Sachs solution. Class I solutions have been proved to be interiors to the Schwarzschild solution [9] and this result has been generalized to include the cosmological constant [31, 45]. As mentioned before, one can invert the roles of the two spacetimes involved and hence construct an Einstein-Straus type model with a Schwarzschild or Kottler cavity within a class I Szekeres' cosmology. Let us remark that the Szekeres class has been used recently in Swiss cheese models as interiors to FLRW (see [7] and references therein).

Class II Szekeres dust metrics are less known, but contain curious inhomogeneous solutions with cylindrical symmetry [60]. In this case, it seems harder to be able to get a physically reasonable Einstein-Straus model considering what we have described above.

5 Brief overview of perturbative matching theory

A perturbed spacetime consists of a symmetric two-covariant tensor (the "perturbation metric") defined on a fixed spacetime (the "background"). From a structural point of view, spacetime perturbation theory is a gauge theory in the sense that many perturbation metrics describe the same physical situation (i.e. they are gauge related). The underlying geometrical reason for this gauge freedom can be understood from the following intuitive picture of perturbation theory. We imagine a one-parameter family of spacetimes $(\mathcal{V}_\varepsilon, g_\varepsilon)$ such that all the manifolds are diffeomorphic to each other. This allows one to pull back g_ε onto a single manifold in the family (say $\mathcal{V}_0 := \mathcal{V}_{\varepsilon=0}$) and work with a one-parameter family of metrics on a single manifold. If all the construction is smooth in ε, derivatives with respect to this parameter can be taken. The perturbation metric $g^{(1)}$ is simply the derivative at $\varepsilon = 0$ of this family of metrics. However, the identification of points in the different manifolds (the diffeomorphism above) is highly non-unique. Any other choice of identification would lead to a different, but geometrically equivalent, perturbation metric. This is the gauge freedom of the theory. Intuitively, it is clear that the gauge freedom will consist of a vector field on the background, because this measures the shift of the new identification with respect to the previous one, and an initial direction (in ε) is all what is required to compute derivatives with respect to ε at $\varepsilon = 0$.

When two spacetimes with boundary are matched, an identification of the boundaries is required. As already mentioned before, if the boundaries are nowhere null, the matching conditions require the equality of the induced metric and second fundamental form (with appropriate choices of orientation). To compare the tensors it is necessary to pull them back to a single manifold and this is done via the identification of boundaries. Contrarily than before, the matching theory is strongly dependent on the identification of the boundaries. In fact, the matching conditions demand the *existence* of one such identification for which the first and second fundamental forms agree.

Assume now that we are studying perturbation theory on a background spacetime constructed from the matching of two spacetimes. The question then arises of what are the conditions that the metric perturbation tensors on each side must satisfy to have a perturbed matching spacetime. This issue is somewhat more involved than one may think a priori and it was solved in a complete manner for the first time by Battye and Carter [4] and independently by Mukohyama [47] (this has been extended to second order in [35]). Previous attempts [24, 23, 43] did not take into account all the subtleties of the interplay between two completely different gauge freedoms inherent to this problem. Indeed, in the picture above of perturbation theory in terms of a collection of spacetimes $(\mathcal{V}_\varepsilon, g_\varepsilon)$, each one of them arises now as the matching of two spacetimes with boundary $\mathcal{W}^{\pm}_{\varepsilon}$ across their respective boundaries $\Omega^{\pm}_{\varepsilon}$. For better visualization, assume that each one of $\mathcal{W}^{\pm}_{\varepsilon}$ is a submanifold with boundary of a larger boundary-less manifold $\widehat{\mathcal{W}}^{\pm}_{\varepsilon}$ and assume that the $\{\widehat{\mathcal{W}}^{+}_{\varepsilon}\}$ manifolds are identified among themselves (say with $\widehat{\mathcal{W}}^{+}_{0} := \widehat{\mathcal{W}}^{+}_{\varepsilon=0}$) via an $\varepsilon-$dependent diffeomorphism. The hypersurface Ω^{+}_{ε} projects down to $\widehat{\mathcal{W}}^{+}_{0}$ as a hypersurface $\widehat{\Omega}^{+}_{\varepsilon}$. Now we have a collection of hypersurfaces in one single manifold, and one can think of taking $\varepsilon-$derivatives of geometric quantities intrinsic to the hypersurface. The important point is that, given a point $p \in \Omega^{+}_{\varepsilon=0}$, we do not know how this point maps into $\widehat{\Omega}^{+}_{\varepsilon}$. For that, it is necessary to prescribe *first* how p is mapped into Ω^{+}_{ε}. The identification of $\{\Omega^{+}_{\varepsilon}\}$ among themselves is an additional gauge freedom. It is fully independent of the standard gauge freedom in perturbation theory (called "spacetime gauge freedom" from now on) and is referred to as *hypersurface gauge freedom* [47]. The composition of both identifications gives, as ε varies, and for any $p \in \Omega^{+}_{\varepsilon=0}$, a path $\gamma_p(\varepsilon)$ in $\widehat{\mathcal{W}}^{+}_{0}$ starting at p. Since everything is smooth in ε, the tangent vector to this path at $\varepsilon = 0$ defines a vector field Z^{+} on $\Omega^{+}_{0} := \Omega^{+}_{\varepsilon=0} (= \widehat{\Omega}_{\varepsilon=0})$. This vector field is not necessarily tangent (nor normal) to Ω^{+}_{0} and it depends on both gauge freedoms. A schematic figure for the definition of Z^{+} and how it depends on the gauges is given in Figure 5. If we let $n^{(0)}_{+}$ be a unit normal vector to Ω^{+}_{0}, we can decompose $Z^{+} = Q^{+} n^{(0)}_{+} + T^{+}$, where T^{+} is tangent to Ω^{+}_{0}. From the discussion above, it should be clear that Q^{+} is independent of the hypersurface gauge while T^{+} strongly depends on it. In fact, it can always be made zero by an appropriate choice of gauge. However, doing this is not usually a good idea because the matching has two regions and, at each value of ε, the matching requires an identification between Ω^{+}_{ε} and Ω^{-}_{ε}. After a choice of hypersurface gauge to identify Ω^{+}_{ε} with Ω^{+}_{0} we have no freedom left to choose a hypersurface gauge to identity Ω^{-}_{ε} and Ω^{-}_{0}. The matching conditions will tell us how this identification must be done. So, had we chosen $T^{+} = 0$, we would still have to leave T^{-} free and let the linearized matching theory determine its value. Further details on the double gauge freedom of linearized perturbation theory can be found in the paper of Mukohyama [47] and in [37], where the issue is discussed in depth including a critical analysis of previous attempts to formulate a consistent perturbative matching theory.

After this brief discussion on gauge issues for linearized matching theory, let us describe the actual perturbative matching conditions (for details see [4, 47, 35]). The matching theory involves the equality of the first and second fundamental forms of the boundaries. To compare them they are pulled-back into a single boundary via the identification. In perturbative matching theory everything occurs on the abstract hypersurface Ω_0 diffeomorphic to Ω^{+}_{0} and Ω^{-}_{0} of the background spacetime. On $\Omega^{\pm}_{0}$ we attach two vector fields $Z^{\pm} = Q^{\pm} n^{(0)}_{\pm} + T^{\pm}$ whose geometric meaning has been discussed above. They are not know a priori, firstly, because of the hypersurface gauge freedom, and secondly, because the identification of the boundaries Ω^{+}_{ε} and Ω^{-}_{ε} is not known a priori. To the unknowns $Z^{\pm}$ we add four symmetric tensors $q^{(1)}_{\pm}$ and $K^{(1)}_{\pm}$ intrinsic to Ω_0 which arise as the $\varepsilon-$derivative at $\varepsilon = 0$ of the first fundamental form $q^{\pm}_{\varepsilon}$ and second fundamental form $K^{\pm}_{\varepsilon}$ of $\Omega^{\pm}_{\varepsilon}$. These tensors are intrinsic to $\Omega^{\pm}_{\varepsilon}$, so before taking ε-derivatives they must be pulled back onto Ω_0 via the hypersurface gauges (on each side). Thus, $q^{(1)}_{\pm}$ and $K^{(1)}_{\pm}$ are hypersurface gauge-dependent by construction. On the other hand, their construction is fully independent of

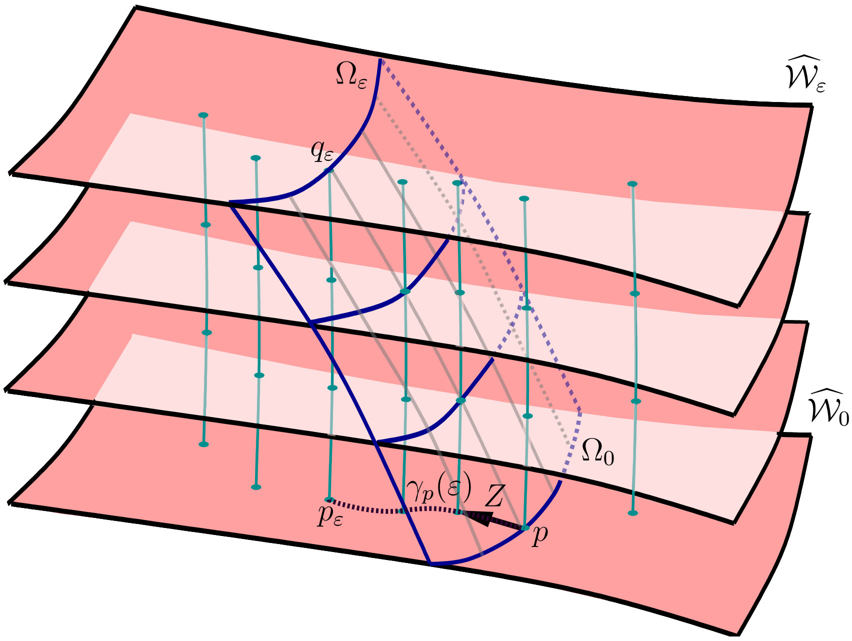

Figure 1: The different spacetimes $\widehat{\mathcal{W}}_\varepsilon$ are represented by horizontal sheets, with $\widehat{\mathcal{W}}_0$ at the bottom. On each $\widehat{\mathcal{W}}_\varepsilon$ there is a hypersurface Ω_ε. The hypersurfaces Ω_ε for all ε span a manifold transverse to $\mathcal{V}_\varepsilon$ (the blue sheet in the figure). The choice of spacetime gauge is represented by big dots on each $\widehat{\mathcal{W}}_\varepsilon$ linked by green curves, while the choice of hypersurface gauge is represented by the grey curves on the blue sheet. The point p is mapped first to q_ε through the hypersurface gauge and back to $p_\varepsilon \in \widehat{\mathcal{W}}_0$ through the spacetime gauge. The points p_ε define on $\widehat{\mathcal{W}}_0$ the path $\gamma_p(\varepsilon)$. The vector Z is the tangent of $\gamma_p(\varepsilon)$ at p.

the spacetime gauge freedom.

In order to write down their explicit expression, let $g^{(1)\pm}$ be the perturbed metric (i.e. the fundamental unknown in metric perturbation theory) on each side of the background spacetime. Let also $\Psi_0^\pm : \Omega_0 \to \mathcal{W}_0^\pm$ be the embedding of the matching hypersurface on each region of the background spacetime. Let y^i $(i, j, \ldots = 1, \ldots, n-1)$ be a local coordinate system on Ω_0 and define tangent vectors $e_i^\pm = \Psi^\pm_{0\star}(\partial_{y^i})$. There are also unique (up to orientation) unit one-forms $\boldsymbol{n}_\pm^{(0)}$ normal to the boundaries. We choose them so that the corresponding vector $n_+^{(0)}$ points towards $\mathcal{W}_0^+$ and $n_-^{(0)}$ points outside of $\mathcal{W}_0^-$ or viceversa. The first and second fundamental forms of the background are simply $q^{(0)\pm} := \Psi_0^{\pm\star}(g^{(0)\pm})$, $\quad K^{(0)\pm} := \Psi_0^{\pm\star}(\nabla^\pm \boldsymbol{n}_\pm^{(0)})$, where $\nabla^\pm$ is the covariant derivative in $(\mathcal{W}_0^\pm, g^{(0)\pm})$. Given that the background configuration is already composed of the matching of $\mathcal{V}_0^+$ and $\mathcal{V}_0^-$ through $\Omega_0^+ := \Omega_0$, we already have $q^{(0)+} = q^{(0)-}$ and $K^{(0)+} = K^{(0)-}$. Then $q^{(1)\pm}$ and $K^{(1)\pm}$ are defined as follows [47]

$$\begin{aligned} q_{ij}^{(1)\pm} &= \mathcal{L}_{T^\pm} q_{ij}^{(0)\pm} + 2Q^\pm K_{ij}^{(0)\pm} + e_i^{\pm\alpha} e_j^{\pm\beta} g_{\alpha\beta}^{(1)\pm}, \qquad (17) \\ K_{ij}^{(1)\pm} &= \mathcal{L}_{T^\pm} K_{ij}^{(0)\pm} - \sigma D_i D_j Q^\pm \\ &+ Q^\pm(-n_\pm^{(0)\mu} n_\pm^{(0)\nu} R_{\alpha\mu\beta\nu}^{(0)\pm} e_i^{\pm\alpha} e_j^{\pm\beta} + K_{il}^{(0)\pm} K^{(0)}{}^{l}{}_{j}{}^{\pm}) \\ &+ \frac{\sigma}{2} g_{\alpha\beta}^{(1)\pm} n_\pm^{(0)\alpha} n_\pm^{(0)\beta} K_{ij}^{(0)\pm} - n_\pm^{(0)}{}_\mu S_{\alpha\beta}^{(1)\pm\mu} e_i^{\pm\alpha} e_j^{\pm\beta}, \qquad (18) \end{aligned}$$

where $\sigma := g^{(0)\pm}(n^{(0)}_{\pm}, n^{(0)}_{\pm})$, D is the covariant derivative of $(\Omega_0, q^{(0)\pm})$, $R^{(0)\pm}_{\alpha\mu\beta\nu}$ is the Riemann tensor of $(\mathcal{W}^{\pm}_0, g^{(0)\pm})$ and $S^{(1)}{}^{\pm\alpha}_{\beta\gamma} := \frac{1}{2}(\nabla^{\pm}_{\beta}\, g^{(1)}{}^{\pm\alpha}_{\gamma} + \nabla^{\pm}_{\gamma}\, g^{(1)}{}^{\pm\alpha}_{\beta} - \nabla^{\pm\,\alpha}\, g^{(1)}{}^{\pm}_{\beta\gamma})$. The first order matching conditions (in the absence of shells) require the equalities

$$q^{(1)+} = q^{(1)-}, \qquad\qquad K^{(1)+} = K^{(1)-}. \tag{19}$$

We emphasize that $Q^{\pm}$ and $T^{\pm}$ are a priori unknown quantities and fulfilling the matching conditions requires *showing* that two vectors $Z^{\pm}$ exist such that (19) are satisfied. The spacetime gauge freedom can be exploited to fix either or both vectors $Z^{\pm}$ a priori, but this should be avoided (or at least carefully analyzed) if additional spacetime gauge choices are made, in order not to restrict a priori the possible matchings. Regarding the hypersurface gauge, this can be used to fix one of the vectors T^+ or T^-, but not both. Note also that the linearized matching conditions are, by construction, spacetime gauge invariant because, as discussed above, the tensors $q^{(1)\pm}$, $K^{(1)\pm}$ are necessarily spacetime gauge invariant. In fact, it is straightforward to check explicitly that the right-hands sides of (17) and (18) are spacetime gauge invariant (the individual terms are not, and it is precisely the spacetime gauge dependence in $Z^{\pm}$ which makes these objects spacetime gauge invariant). Moreover, the set of conditions (19) are hypersurface gauge invariant, provided the background is properly matched, since, as shown in [47], under such a hypersurface gauge transformation given by the vector ζ in Ω_0, $q^{(1)}$ transforms as $q^{(1)} + \mathcal{L}_{\zeta}\, q^{(0)}$, and similarly for $K^{(1)}$.

6 Spherical symmetry: Hodge decomposition

After the previous summary on linearized matching, in this section we introduce the second main ingredient for the linearized Einstein-Straus model reviewed in the following sections: the Hodge decomposition on the sphere [38].

In order to exploit the underlying spherical symmetry of the background configuration it is common practice to decompose the perturbations, and their related objects and equations, in terms of scalar, vector and tensor harmonics on the sphere. That was the procedure used in the seminal work on perturbations around spherical matched background configurations, due to Gerlach and Sengupta (GS) in [23] and [24], revisited and improved by Martín-García and Gundlach in [43].

The aim in [38] was to use an alternative method, based on the Hodge decomposition on the sphere in terms of scalars, in order to avoid the need to deal with infinite series of objects. In particular, the whole set of matching conditions for the linearized Einstein-Straus model was presented in [38] as a finite number of equations involving scalars that depend on the three coordinates in the matching hypersurface Ω_0, in contrast with an infinite number of equations for an infinite set of functions of one variable. It is clear that one can always go from the Hodge scalars to the spherical harmonics decomposition in a straightforward way. However, it is not always easy to rewrite the infinite number of expressions appearing in a spectral decomposition in terms of Hodge scalars.

Consider the round unit metric $\Omega_{AB}dx^A dx^B = d\vartheta^2 + \sin^2\vartheta d\varphi^2$, with η_{AB} and D_A denoting the corresponding volume form and covariant derivative respectively, and $(\star dG)_A = \eta^C{}_A D_C G$ the Hodge dual with respect to Ω_{AB}. Let us recall that the usual Hodge decomposition on $(\mathbb{S}^2, \Omega_{AB})$ states that any one-form V_A can be canonically decomposed as $V_A = D_A F + (\star dG)_A$, where F and G are functions on $\mathbb{S}^2$, and any symmetric tensor T_{AB}

as $T_{AB} = D_A U_B + D_B U_A + H\Omega_{AB}$, for some U_A on $\mathbb{S}^2$, which can be in turn decomposed in terms of scalars.

Based on this, it is convenient to define the following two functionals. Given three scalars $\mathcal{X}_{tr}, \mathcal{X}_1$ and $\mathcal{X}_2$ on $(\mathbb{S}^2, \Omega_{AB})$ we define the functional one form $V_A(\mathcal{X}_1, \mathcal{X}_2)$ as

$$V_A(\mathcal{X}_1, \mathcal{X}_2) = D_A \mathcal{X}_1 + (\star d\mathcal{X}_2)_A,$$

and the functional symmetric tensor $T_{AB}(\mathcal{X}_{tr}, \mathcal{X}_1, \mathcal{X}_2)$ as

$$T_{AB}(\mathcal{X}_{tr}, \mathcal{X}_1, \mathcal{X}_2) = D_A V_B(\mathcal{X}_1, \mathcal{X}_2) + D_B V_A(\mathcal{X}_1, \mathcal{X}_2) + \mathcal{X}_{tr}\Omega_{AB}.$$

Let us recall that the decomposition defines these $\mathcal{X}$'s on $\mathbb{S}^2$ up to the kernels of the operators V_A and T_{AB}. We allowed for the appearance of all these kernels in [38], where their relevance (or their lack of) was already discussed. In order to avoid spurious information and present a more concise review –and also to ease the translation and comparison with the quantities used in the previous literature in terms of the harmonic decompositions,– we use the reformulation already presented in [39] where the Hodge decomposition is, in fact, unique.

Indeed, in order to fix the Hodge decomposition uniquely we define a *canonical dual decomposition* by demanding that the functions $\mathcal{X}_1, \mathcal{X}_2$ in $V_A(\mathcal{X}_1, \mathcal{X}_2)$ are always orthogonal (in the L^2 sense on $\mathbb{S}^2$) to 1, and the functions $\mathcal{X}_1, \mathcal{X}_2$ in $T_{AB}(\mathcal{X}_{tr}, \mathcal{X}_1, \mathcal{X}_2)$ are orthogonal to 1 and to the $l = 1$ spherical harmonics. Schematically, we may use

$$W_A \overset{\mathbb{S}^2}{\to} \mathcal{X}_1, \mathcal{X}_2 \qquad \text{to indicate} \qquad W_A = V_A(\mathcal{X}_1, \mathcal{X}_2)$$

and

$$W_{AB} \overset{\mathbb{S}^2}{\to} \mathcal{X}_{tr}, \mathcal{X}_1, \mathcal{X}_2 \qquad \text{when} \qquad W_{AB} = T_{AB}(\mathcal{X}_{tr}, \mathcal{X}_1, \mathcal{X}_2).$$

We will use the following notation: given any function f on $\mathbb{S}^2$ we define

$$f_{||0} := Y_0 \int_{\mathbb{S}^2} f\, Y_0 d\Omega^2, \qquad f_{||1} := Y_1 \int_{\mathbb{S}^2} f\, Y_1 d\Omega^2 \left(= \sum_m Y_1^m \int_{\mathbb{S}^2} f\, Y_1^m d\Omega^2 \right),$$

so that $f - f_{||0}$ is orthogonal to the $l = 0$ harmonics Y_0 and $f - f_{||1}$ is orthogonal to the $l = 1$ harmonics Y_1^m.

Note finally that the Hodge decomposition in terms of scalars involves two types of objects depending on their behaviour under reflection on the sphere. The scalars with subscripts 1 and *tr* remain unchanged under reflection, and are typically called longitudinal, even or polar quantities, while those with subscripts 2 change, and correspond to the transversal, odd or axial quantities.

Let us consider now the general spherically symmetric spacetime $\mathcal{V} = M^2 \times \mathbb{S}^2$ with metric $g_{\alpha\beta} = \omega_{IJ} \oplus r^2 \Omega_{AB}$, so that (M^2, ω_{IJ}) is a 2-dim Lorentzian space and $r > 0$ a function on M^2. The dual in (M^2, ω_{IJ}) will be indicated by $*$ and the covariant derivative by ∇.

We can now proceed to decompose any one-form (vector) or symmetric two-tensor on $\mathcal{V}$ by first taking the part orthogonal to the sphere and then apply the Hodge canonical decomposition to the part tangent to the sphere. In particular, given a normalized timelike one-form u_α orthogonal to the spheres, its corresponding one-form u_I on (M^2, ω_{IJ}) (defined by $u_\alpha = (u_I, 0)$) can be used to construct a convenient orthonormal basis $\{u_I, m_I\}$

so that $\omega_{IJ} = -u_I u_J + m_I m_J$ (this is $u_I u^I = -1$ and $m_I := *u_I$), and consider then the one-form on $\mathcal{V}$ defined by $m_\alpha := (m_I, 0)$. Given any vector V_I we will simply denote by V_u and V_m the contractions $u^I V_I$ and $m^I V_I$ respectively.

We apply now this decomposition to encode the objects that will describe the (first order) perturbation of a background consisting of two spherically symmetric regions $(\mathcal{W}^+, g^{(0)+})$ and $(\mathcal{W}^-, g^{(0)-})$ matched across corresponding spherically symmetric boundaries Ω_0^+ and Ω_0^-. At each side $\pm$ (we avoid the use of $\pm$ just now for clarity) the metric perturbation tensor $g^{(1)}_{\alpha\beta}$ gets thus decomposed as

$$\begin{aligned} & g^{(1)}_{IJ} = \mathcal{Z}_{IJ}, \\ & g^{(1)}_{IA} \stackrel{\mathbb{S}^2}{\to} \mathcal{Z}_{I1}, \mathcal{Z}_{I2}, \\ & g^{(1)}_{AB} \stackrel{\mathbb{S}^2}{\to} \mathcal{Z}^{\mathbb{S}^2}_{tr}, \mathcal{Z}^{\mathbb{S}^2}_{1}, \mathcal{Z}^{\mathbb{S}^2}_{2}, \end{aligned} \tag{20}$$

where $\mathcal{Z}_{I1}$ and $\mathcal{Z}_{I2}$ are two one-forms defined on M^2, and analogously for any symmetric tensor. The deformation vector Z_α, defined on M at points on Ω, is decomposed as

$$Z_\alpha \to \ \{Z_I \to Q, T\} \oplus \{Z_A \stackrel{\mathbb{S}^2}{\to} \mathcal{T}_1, \mathcal{T}_2\},$$

whereby the part Z_I gets decomposed, in turn, onto the normal and tangential parts to Ω_0, Q and T respectively. Given the (first order) perturbation tensor and the deformation vector at either $\pm$ side of the matching hypersurface, one can calculate the symmetric tensors $q^{(1)}_{ij}$ and $K^{(1)}_{ij}$, i.e. the "perturbed first and second fundamental forms", using (17)-(18). Recalling now that $q^{(1)}_{ij}$ and $K^{(1)}_{ij}$ are defined on $(\Omega_0, q^{(0)}_{ij})$ and that Ω_0 at either side are tangent to the spheres $\{\theta, \phi\}$, let us denote by λ the parameter that follows the direction on Ω_0 orthogonal to the spheres in order to decompose $q^{(1)}_{ij}$ and $K^{(1)}_{ij}$ into $q^{(1)}_{\lambda\lambda}$, $K^{(1)}_{\lambda\lambda}$, plus

$$q^{(1)}_{\lambda A} \stackrel{\mathbb{S}^2}{\to} F^q, G^q, \qquad q^{(1)}_{AB} \stackrel{\mathbb{S}^2}{\to} H^q, P^q, R^q, \tag{21}$$

and

$$K^{(1)}_{\lambda A} \stackrel{\mathbb{S}^2}{\to} F^k, G^k, \qquad K^{(1)}_{AB} \stackrel{\mathbb{S}^2}{\to} H^k, P^k, R^k. \tag{22}$$

Note that all these functions are scalars on the sphere that depend only on λ. The first order matching conditions (19) are therefore equivalent to

$$\begin{aligned} & q^{(1)+}_{\lambda\lambda} = q^{(1)-}_{\lambda\lambda}, \quad F^q_+ = F^q_-, \quad G^q_+ = G^q_-, \quad H^q_+ = H^q_-, \quad P^q_+ = P^q_-, \quad R^q_+ = R^q_-, \\ & K^{(1)+}_{\lambda\lambda} = K^{(1)-}_{\lambda\lambda}, \quad F^k_+ = F^k_-, \quad G^k_+ = G^k_-, \quad H^k_+ = H^k_-, \quad P^k_+ = P^k_-, \quad R^k_+ = R^k_- \end{aligned} \tag{23}$$

Except for the simplification of the kernels by the canonical decomposition, these are the linearized matching conditions presented in [38].

6.1 Gerlach and Sengupta (GS) 2+2 formalism

Let us emphasize again that the conditions (19), and thus (23), concern quantities defined on Ω_0 and are therefore independent on the coordinates used in $\mathcal{W}^+$ and $\mathcal{W}^-$ for their computation. These quantities are thus, on the one hand, spacetime gauge independent by construction. Moreover, as discussed in Section 5, the *equations* are also hypersurface gauge independent. All this makes it unnecessary the use of gauge invariant quantities in order to establish the perturbed matching conditions. Having said that, however, the use of gauge independent quantities turns out to be convenient in the end, mostly when one eventually wants to impose the Einstein field equations. As shown in the works [23], [24], [43], the use of spacetime gauge invariants is very convenient in order to combine the Einstein field equations with the perturbed matching conditions.

One can proceed by constructing gauge invariants in terms of Hodge scalars using analogous expressions to those in the harmonic decomposition constructed in [24, 23, 43].

Let us now concentrate on the odd (axial) sector. The odd gauge invariant quantities are encoded in the vector [36] (cf. [24])

$$\mathcal{K}_I := \mathcal{Z}_{I2} - \nabla_I \mathcal{Z}_2^{\mathbb{S}^2} + 2\mathcal{Z}_2^{\mathbb{S}^2} r^{-1} \nabla_I r.$$

Note that $\mathcal{K}_I$, as defined above, contains $l = 1$ harmonics, from $\mathcal{Z}_{I2}$, but only the $l \geq 2$ sector is gauge invariant. In other words, the part of $\mathcal{K}_I$ orthogonal to Y^1 (i.e. $\mathcal{K}_I - \mathcal{K}_{I||1}$ in the notation above) is the gauge invariant part. Once the orthonormal basis $\{u_I, m_I\}$ has been identified at both sides, the odd sector of the linearized matching, which corresponds to the set of equations

$$G^q_+ = G^q_-, \qquad R^q_+ = R^q_-, \qquad G^k_+ = G^k_-, \qquad R^k_+ = R^k_-, \tag{24}$$

in (23), is equivalent in the $l \geq 2$ sector (the part orthogonal to $l = 0$ and $l = 1$) to [36] (cf. [24])

$$\mathcal{K}^+_u \overset{\Omega_0}{=} \mathcal{K}^-_u, \qquad \mathcal{K}^+_m \overset{\Omega_0}{=} \mathcal{K}^-_m, \qquad *d(r^{-2}\boldsymbol{\mathcal{K}}^+) \overset{\Omega_0}{=} *d(r^{-2}\boldsymbol{\mathcal{K}}^-) \tag{25}$$

plus an equation for $\mathcal{T}_2^+ - \mathcal{T}_2^-$.

7 Linearized Einstein-Straus model: matching conditions

We are ready to consider the linearized matching of the perturbed Schwarzschild and FLRW spacetimes, as it was analyzed in [38]. Take the Einstein-Straus model as described in Section 2: the FLRW geometry in cosmic time coordinates (1), the Schwarzschild in standard coordinates (2), and the background matching hypersurface Ω_0 described by $\Omega_0^{\text{RW}} : \{t = t, R = R_0\}$ and $\Omega_0^{\text{ST}} : \{T = T_0(t), r = r_0(t)\}$ respectively, where $T_0(t)$ and $r_0(t)$ satisfy (3) and the angular part is again ignored.

The orthonormal basis we take on the Lorentzian space orthogonal to the spheres is formed by $\boldsymbol{u} = dt$, $\boldsymbol{m} = adR$. Note that this choice corresponds to the tangent and normal forms to Ω_0 at Ω_0, respectively. More precisely, $\boldsymbol{m}$ is chosen to be $\boldsymbol{n^{(0)}}$ at points on Ω_0. On the Schwarzschild side we have $u|_{\Omega_0} = \dot{T}_0 \partial_T + \dot{r}_0 \partial_r$ and $\boldsymbol{m}|_{\Omega_0} := \boldsymbol{n^{(0)}} = -\dot{r}_0 dT + \dot{T}_0 dr$.

Take the first order perturbations of FLRW, in no specific gauge, formally decomposed into the usual scalar, vector and tensor (SVT) modes, i.e.[4] (Latin indices a, b, c are used for tensors on $(\mathcal{M}, g_{\mathcal{M}})$)

$$g^{(1)+}_{tt} = -2\Psi \qquad g^{(1)+}_{ta} = aW_a$$
$$g^{(1)+}_{ab} = a^2(-2\Phi\gamma_{ab} + \chi_{ab})$$

[4]Note that in [38], the FLRW geometry was written in conformal time, while proper time is used here.

with

$$W_a = \partial_a W + \tilde{W}_a, \qquad \chi_{ab} = (\nabla_a\nabla_b - \frac{1}{3}\gamma_{ab}\nabla^2)\chi + 2\nabla_{(a}Y_{b)} + \Pi_{ab}$$

satisfying the constraints

$$\nabla^a Y_a = \Pi_a{}^a = 0, \qquad \nabla^a \Pi_{ab} = 0, \qquad \nabla^a \tilde{W}_a = 0.$$

The canonical Hodge decomposition is then used to encode the part tangent to the spheres into $\mathbb{S}^2$ scalars in the following schematic way [38]:
vector

$$\tilde{W}_a \to \tilde{W}_R \oplus \{\tilde{W}_A \stackrel{\mathbb{S}^2}{\to} \mathcal{W}_1, \mathcal{W}_2\}, \qquad Y_a \to Y_R \oplus \{Y_A \stackrel{\mathbb{S}^2}{\to} \mathcal{Y}_1, \mathcal{Y}_2\},$$

tensor

$$\Pi_{RA} \stackrel{\mathbb{S}^2}{\to} \mathcal{Q}_1, \mathcal{Q}_2, \qquad \Pi_{AB} \stackrel{\mathbb{S}^2}{\to} \mathcal{H}, \mathcal{U}_1, \mathcal{U}_2.$$

All in all, encoding $g^{(1)+}$ using the SVT modes together with the Hodge decomposition leaves us with 15 SVT-Hodge quantities, corresponding to the scalar modes $\{\Psi, \Phi, W, \chi\}$, vector modes $\{\tilde{W}_R, \mathcal{W}_1, \mathcal{W}_2, Y_R, \mathcal{Y}_1, \mathcal{Y}_2\}$ and tensor modes $\{\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{H}, \mathcal{U}_1, \mathcal{U}_2\}$, all scalars on $\mathbb{S}^2$, not all independent due to the previous constraints, and, on the other hand, *not unique*. Consider, in particular, the only 4 scalars we have in the odd sector; vector modes $\{\mathcal{W}_2, \mathcal{Y}_2\}$ and tensor modes $\{\mathcal{Q}_2, \mathcal{U}_2\}$. As discussed in the previous section, only two gauge invariant quantities exist in the odd sector. These correspond to the two components of the gauge invariant odd vector (and for $l \geq 2$), $\mathcal{K}_I^+$, which given the above construction in terms of the SVT-Hodge quantities, read [39, 36]

$$\mathcal{K}_u^+ = a\left(\mathcal{W}_2 - a(\dot{\mathcal{U}}_2 + \dot{\mathcal{Y}}_2)\right), \qquad \mathcal{K}_m^+ = a\left(\mathcal{Q}_2 - \mathcal{U}_2{}' + 2\mathcal{U}_2\Sigma'\Sigma^{-1}\right).$$

Consider now the stationary and axially symmetric vacuum perturbations in the Weyl gauge. They can be described in terms of two functions $U^{(1)}(r,\theta)$ and $A^{(1)}(r,\theta)$, which correspond, basically, to the perturbation of the gravitational Newtonian potential and the rotational perturbation, respectively. The perturbation tensor reads [38]

$$\begin{aligned} g^{(1)\mathrm{Sch}} &= -2\left(1-\frac{2m}{r}\right)\left(U^{(1)}dt^2 + A^{(1)}dtd\phi\right) \\ &\quad -2r^2\sin^2\theta U^{(1)}d\phi^2 + 2\left(k^{(1)} - U^{(1)}\right)\left[\left(1-\frac{2m}{r}\right)^{-1}dr^2 + r^2 d\theta^2\right]. \end{aligned} \tag{26}$$

Note that, when using the full set of the Einstein field equations for vacuum, the function $k^{(1)}$ is determined up to quadratures once $U^{(1)}(r,\theta)$ and $A^{(1)}(r,\theta)$ are found. We stress, however, that the vacuum equations, although indicated, were not imposed in the perturbed Schwarzschild region (nor in the perturbed FLRW region) in [38], and therefore the results found there are purely geometric.

Instead of working with $A^{(1)}(r,\theta)$, it is convenient to use an auxiliary function, $\mathcal{G}$, defined by $A^{(1)} := \sin\theta\partial_\theta\mathcal{G}$. In terms of the Hodge decomposition of the perturbation tensor (20), applied to (26), we have $\mathcal{G} = -\left(1-\frac{2M}{r}\right)^{-1}\mathcal{Z}^-_{T2}$. Another auxiliary function $\mathcal{P}$ can be introduced for $k^{(1)}$.

The whole set of matching conditions (23) for the linearized Einstein-Straus model, both in the odd and even sectors, were found in [38] in terms of these functions $U^{(1)}(r,\theta)$, $\mathcal{G}$

and $\mathcal{P}$ in the Schwarzschild side together with the above 15 SVT-Hodge scalars describing the FLRW perturbation. The whole set is too long to be included here, but in order to review the main results in [38] only the odd sector of the linearized matching is needed.

The odd sector of the linearized matching (24) projected to the part orthogonal to the $l = 0$ and $l = 1$ harmonics can be rewritten as the following three relations [38]

$$\mathcal{W}_2 - a[\dot{\mathcal{U}}_2 + \dot{\mathcal{Y}}_2] \overset{\Omega_0}{=} -\mathcal{G}\Sigma_c' a^{-1}, \tag{27}$$

$$\mathcal{Q}_2 - \mathcal{U}_2{}' + 2\Sigma_c^{-1}\Sigma_c'\mathcal{U}_2 \overset{\Omega_0}{=} -\mathcal{G}a^{-1}\dot{a}\Sigma_c, \tag{28}$$

$$\mathcal{W}_2{}' - a\frac{d}{dt}[(\mathcal{U}_2{}' + \mathcal{Y}_2{}')|_{\Omega_0}] \overset{\Omega_0}{=} \mathcal{G}\frac{\Sigma_c^3 a\epsilon - 3M}{\Sigma_c^2 a^2} + \frac{\partial \mathcal{G}}{\partial r}(\Sigma_c^2\epsilon - 1), \tag{29}$$

plus an equation for the difference $\mathcal{T}_2^{\mathrm{RW}} - \mathcal{T}_2^{\mathrm{Sch}}$. The three equations above can be shown [36] to correspond indeed to (25), taking into account that $\boldsymbol{\mathcal{K}}^{\mathrm{Sch}} = -\left(1 - \frac{2M}{r}\right)\mathcal{G}dT$. These equations were presented in [38] in full, including the parts lying on the $l = 0$ and $l = 1$ harmonics. There, a series of kernels inherent to the usual Hodge decomposition where the responsible for the usual freedom found in the $l = 0$ and $l = 1$ sectors when using harmonic decompositions. By using the *canonical* Hodge decomposition introduced in Section 6, we can have a better control of that freedom, and understand its nature, getting rid of the spurious terms. Indeed, by doing that it can be shown [36] that the projection of the linearized matching on the $l = 1$ harmonics –on the $l = 0$ it is trivial, since all scalars in the odd sector are orthogonal to $l = 0$– gives

$$\begin{aligned}\dot{a}a^2\Sigma_c^2 \left[\mathcal{W}_2' - a\frac{d}{dt}\mathcal{Y}_2'|_{\Omega_0} - 2\Sigma_c^{-1}\Sigma_c'\left(\mathcal{W}_2 - a\frac{d}{dt}\mathcal{Y}_2\right)\right]_{||1} \\ \overset{\Omega_0}{=} a(2M - a\Sigma_c)\dot{\mathcal{G}}_{||1} + 2\dot{a}(a\Sigma_c - 3M)\mathcal{G}_{||1}\end{aligned} \tag{30}$$

while $(\mathcal{T}_2^{\mathrm{RW}} - \mathcal{T}_2^{\mathrm{Sch}})_{||1}$ is free.

The first consequence of the above equations is that if the FLRW remains unperturbed then the stationary region must be static in the range of variation of $r_0(t)$: equations (27)-(29) imply that the part of $\mathcal{G}$ orthogonal to $l = 1$ vanishes, and (30) implies that $\mathcal{G}_{||1} = Ca^3/(2M - a\Sigma_c)$, where C is a constant. Therefore $\mathcal{G} = Ca^3/(2M - a\Sigma_c)\cos\theta$, and thus[5]

$$A^{(1)}|_{\Omega_0} = C\sin^2\theta a^3/(2M - a\Sigma_c).$$

As shown in [38], this implies that the perturbed spacetime is static in the range of variation of $r_0(t)$.

This result generalizes that in [52] because now the matching hypersurface does not necessarily keep the axial symmetry. Therefore, the only way of having a stationary and axisymmetric vacuum arbitrarily shaped (at the linear level) region in FLRW is to have the Einstein-Straus model. Let us remark again that by the interior/exterior duality, this result also implies that a piece of FLRW, irrespective of its shape and its relative rotation with the exterior, cannot describe the interior of Kerr.

[5]This equation was obtained in [38] by deriving the necessary constraints for the aforementioned kernels.

7.1 Constraint on FLRW

Another interesting consequence of the matching conditions is that the combination of (27) and (28) produces one equation that involves only quantities in FLRW [38]

$$\frac{\dot{a}\Sigma_c}{a\Sigma_c'}\left(\mathcal{W}_2 - a[\dot{\mathcal{U}}_2 + \dot{\mathcal{Y}}_2]\right) \overset{\Omega_0}{=} \mathcal{Q}_2 - \mathcal{U}_2{}' + 2\Sigma_c^{-1}\Sigma_c'\mathcal{U}_2,$$

and thus constitutes a *constraint* in FLRW. Recalling that this equation is meant to be orthogonal to $l = 1$ (and vanishes identically if projected on $l = 0$), this constraint implies that if the perturbed FLRW contains vector modes with $l \geq 2$ harmonics on Ω_0, then it must contain also tensor modes there. Since, as we have just seen above, the existence of a rotation in the stationary region implies the existence of, at least, vector perturbations in FLRW, then both vector and tensor modes must exist on Ω_0.

It must be stressed that, as demonstrated in [5] (and references therein), there exist configurations of FLRW linear perturbations containing only vector perturbations which vanish identically inside a spherical surface. Such configurations are compatible with the results reviewed here, since that interior region is FLRW and the above constraints do not apply. A completely different matter is the embedding of a Schwarzschild spherical cavity (or a vacuum perturbation thereof) into any such model: the Schwarzschild cavity cannot reach the perturbed FLRW region, as otherwise the constraints above would require that tensor perturbations are also present (at least near the boundary of the Schwarzschild cavity).

There also exists a Einstein-Straus perturbative model by Chamorro [18] consisting on small rotation Kerr vacuole within a perturbed FLRW. Again, the constraint above does not apply to this model because there are no $l \geq 2$ modes there.

The fact that tensor modes must exist near Ω_0 once some rotation with $l \geq 2$, whatever small, exists in the stationary region, may indicate the existence of some kind of gravitational waves on FLRW near Ω_0. In order to analyze further this issue one needs to take the Einstein field equations into consideration. That is the purpose of our work in preparation [36], some result of which will appear in the proceedings of the ERE2012 meeting.

8 Conclusions and outlook

This paper is concerned with the difficulties that the Einstein-Straus model encounters. A fundamental one refers to its high level of rigidity and the impossible generalization to non-spherical symmetry if the bound system is required to be time independent so as to retain the property that cosmic expansion does not affect the local systems. Moreover, if one views the model within the LTB class with a step function density profile, the model is unstable to perturbations [57], [32], cf. also the discussion in [30]. The rigidity result so far requires either stationarity and axial symmetry or staticity. An interesting open problem would be to relax the conditions and assume only stationarity.

Despite these difficulties, the Einstein-Straus model has played and plays a very important role in cosmology in different areas or research, most notably on the influence (or lack thereof) of the cosmic expansion on local systems, or in the problem of averaging in cosmology at least on an observational level (see e.g. [20]). The model is still widely used as textbook explanation of the lack of influence of the cosmic expansion on astrophysical systems and, in fact, there are not many known alternatives (a notable exception is the McVittie model [44], [28], [29] which is also spherically symmetric and mimics the geometry of Schwarzschild at small scales while approaching a FLRW model at long distances, and which has been studied thoroughly, see [15] and references therein). Concerning the use of the Einstein-Straus model on observational cosmology, mainly by studying lensing effects, its generalizations have systematically consisted in keeping spherical symmetry and allowing for some dynamics in the interior. The prominent example here consists of LTB regions inside a FLRW universe (see references in [20]).

An important ingredient for the rigidity of the Einstein-Straus model is the large symmetry of the FLRW background. It is therefore an interesting problem to analyze how the model gets modified in the presence of cosmic perturbations. In a conservative approach, one still wants to keep the main properties (stationarity of a region inside a cosmological model) as far as possible and analyze the possible departures from the model in more realistic situations. Moreover, by studying perturbed Einstein-Straus models one seeks going beyond the problem of the influence of cosmic expansion on local systems, and tackle the problem of the influence of general cosmic dynamics. Surprisingly, it turns out that the existence of static (stationary) regions does impose conditions on the cosmic perturbations, at least near the boundary. Whether this is a real effect or simply an indication that the interior region should not be kept stationary remains to be seen.

Another implication of the perturbed Einstein-Straus model is that extra care is required in the standard decomposition of metric perturbations in terms of scalar, vector and tensor modes. As discussed above, any rotation in the vacuole implies necessarily the presence of both vector and tensor modes in the cosmic perturbations. It is interesting to analyze whether these tensor modes could represent cosmic gravitational waves. Some preliminary results along these lines have already been presented in [39]. A detailed and more complete approach will appear elsewhere.

Another interesting future line of research is to allow for non-stationary perturbations in the vacuole and study the transmission of gravitational waves from the cosmic region to the bound system.

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References

[1] M. Alcubierre (1994) *The warp drive: hyper-fast travel within general relativity.* Class. Quantum Grav. **11** L73–L77.

[2] R. Balbinot, R. Bergamini and A. Comastri (1988) *Solution of the Einstein-Strauss problem with a Λ term.* Phys. Rev. D **38** 2415–2418.

[3] A. Barnes (2000) *A comment on a paper by Carot et al.* Class. Quantum Grav. **17** 2605–2609.

[4] R. A. Battye and B. Carter (2001) *Generic junction conditions in brane-world scenarios.* Phys. Lett. B **509** 331–336.

[5] J. Bicak, J. Katz and D. Lynden-Bell (2007) *Cosmological perturbation theory, instantaneous gauges, and local inertial frames.* Phys. Rev. D **76** 063501.

[6] M. A. Bizouard and M. A. Papa (2013) *Searching for gravitational waves with the LIGO and Virgo interferometers.* C. R. Phys. **14** 352–365.

[7] K. Bolejko (2009) *The Szekeres Swiss Cheese model and the CMB observations.* Gen. Rel. Grav. **41** 1737–1755.

[8] C. Bona and J. Stela (1987) *"Swiss cheese" models with pressure.* Phys. Rev. D **36** 2915–2918.

[9] W. B. Bonnor (1976) *Non-radiative solutions of Einstein's equations for dust.* Comm. Math. Phys. **51** 191–199.

[10] W. B. Bonnor (1996) *The cosmic expansion and local dynamics.* Mon. Not. Roy. Astr. Soc. **282** 1467–1469.

[11] W. B. Bonnor (1999) *Size of a hydrogen atom in the expanding universe.* Class. Quantum Grav. **16** 1313–1323.

[12] W. B. Bonnor (2000) *A generalization of the Einstein-Straus vacuole.* Class. Quantum Grav. **17** 2739–2748.

[13] J. Carot (2000) *Some developments on axial symmetry.* Class. Quantum Grav. **17** 2675–2690.

[14] J. Carot, J. M. M. Senovilla and R. Vera (1999) *On the definition of cylindrical symmetry.* Class. Quantum Grav. **16** 3025–3034.

[15] M. Carrera and D. Giulini (2010) *Influence of global cosmological expansion on local dynamics and kinematics.* Rev. Mod. Phys. **82** 169–208.

[16] B. Carter (1969) *Killing horizons and orthogonally transitive groups in space-time.* J. Math. Phys. **10** 70–81.

[17] B. Carter (1970) *The commutation property of a stationary, axisymmetric system.* Comm. Math. Phys. **17** 233–238.

[18] A. Chamorro (1988) *A Kerr cavity with a small rotation parameter embedded in Friedmann universes.* Gen. Rel. Grav. **20** 1309–1323.

[19] A. Einstein and E. G. Straus (1945) *The Influence of the Expansion of Space on the Gravitation Fields Surrounding the Individual Stars.* Rev. Mod. Phys. **17** 120–124.

[20] G. Ellis, R. Maartens and M. MacCallum, *Relativistic Cosmology* (Cambridge University Press 2012).

[21] F. Fayos, J. Senovilla and R. Torres (1996) *General matching of two spherically symmetric spacetimes.* Phys. Rev. D **54** 4862–4872.

[22] R. Gautreau (1984) *Imbedding a Schwarzschild mass into cosmology.* Phys. Rev. D **29** 198–206.

[23] U. H. Gerlach and U. K. Sengupta (1979) *Even parity junction conditions for perturbations on most general spherically symmetric spacetimes.* J. Math. Phys. **20** 2540–2546.

[24] U. H. Gerlach and U. K. Sengupta (1979) *Junction conditions for odd-parity perturbations on most general spherically symmetric space-times.* Phys. Rev. D **20** 3009–3014.

[25] C. Gilbert (1956) *The gravitational field of a star in the expanding universe.* Mon. Not. Roy. Astr. Soc. **116** 678–683.

[26] S. Hawking and G. Ellis, *The large scale structure of space-time.* Cambridge Monographs on Mathematical Physics (Cambridge University Press 1975).

[27] M. Ishak, W. Rindler, J. Dossett, J. Moldenhauer and C. Allison (2008) *A New Independent Limit on the Cosmological Constant/Dark Energy from the Relativistic Bending of Light by Galaxies and Clusters of Galaxies.* Mon. Not. Roy. Astr. Soc. **388** 1279–1283.

[28] G. Järnefelt (1940) *Zur relativistischen Perihelbewebung der Planetbahnen.* Ann. Acad. Soc. Sci. Fennicae Ser. A **55** Paper 3.

[29] G. Järnefelt (1942) *Das Einkrperproblem in dem sich ausdehnenden Raumeder Einstein-de Sitter'schen Welt.* Ann. Acad. Soc. Sci. Fennicae Ser. A **1** Paper 12.

[30] A. Krasiński, *Inhomogeneous cosmological models* (Cambridge University Press 1997).

[31] K. Lake (2000) *Gravitational collapse of dust with a cosmological constant.* Phys. Rev. D **62** 027301.

[32] K. Lake and R. Pim (1985) *Development of voids in the thin-wall approximation. I. General characteristics of spherical vacuum voids.* Astrophys. J. **298** 439–447.

[33] M. Mars (1998) *Axially symmetric Einstein-Straus models.* Phys. Rev. D **57** 3389–3400.

[34] M. Mars (2001) *On the uniqueness of the Einstein-Straus model.* Class. Quantum Grav. **18** 3645–3663.

[35] M. Mars (2005) *First- and second-order perturbations of hypersurfaces.* Class. Quantum Grav. **22** 3325–3347.

[36] M. Mars, F. C. Mena and R. Vera in preparation.

[37] M. Mars, F. C. Mena and R. Vera (2007) *Linear perturbations of matched spacetimes: the gauge problem and background symmetries.* Class. Quantum Grav. **24** 3673–3689.

[38] M. Mars, F. C. Mena and R. Vera (2008) *First order perturbations of the Einstein-Straus and Oppenheimer-Snyder models.* Phys. Rev. D **78** 084022.

[39] M. Mars, F. C. Mena and R. Vera (2013) *Cosmological gravitational waves and Einstein-Straus voids.* to appear in "Progress in Mathematical Relativity, Gravitation and Cosmology - Proceedings of the Spanish Relativity Meeting in Portugal (ERE2012)", Springer Proceedings in Mathematics, 2013.

[40] M. Mars and J. M. M. Senovilla (1999) *Axial symmetry and conformal Killing vectors.* Class. Quantum Grav. **10** 1633–1647.

[41] M. Mars and J. M. M. Senovilla (1999) *Geometry of general hypersurfaces in spacetime: junction conditions.* Class. Quantum Grav. **10** 1865–1897.

[42] M. Mars and J. M. M. Senovilla (2003) *Trapped surfaces and symmetries.* Class. Quantum Grav. **20** L293–L300.

[43] J. Martín-García and C. Gundlach (2001) *Gauge-invariant and coordinate-independent perturbations of stellar collapse. II. Matching to the exterior.* Phys. Rev. D **64** 024012.

[44] G. McVittie (1933) *The mass-particle in an expanding universe.* Mon. Not. Roy. Astr. Soc. **93** 325–339.

[45] F. Mena, J. Natário and P. Tod (2008) *Gravitational Collapse to Toroidal and Higher Genus Asymptotically AdS Black Holes.* Adv. Theor. Math. Phys. **12** 1163–1181.

[46] F. Mena, R. Tavakol and R. Vera (2002) *Generalization of the Einstein-Straus model to anisotropic settings.* Phys. Rev. D **66** 044004.

[47] S. Mukohyama (2000) *Perturbation of the junction condition and doubly gauge-invariant variables.* Class. Quantum Grav. **17** 4777–4798.

[48] B. Nolan (1993) *Sources for McVittie's mass particle in an expanding universe.* J. Math. Phys. **34** 178–185.

[49] B. C. Nolan (1998) *A point mass in an isotropic universe: Existence, uniqueness, and basic properties.* Phys. Rev. D **58** 064006.

[50] B. C. Nolan (1999) *A point mass in an isotropic universe: II. Global properties.* Class. Quantum Grav. **16** 1227–1254.

[51] B. C. Nolan (1999) *A point mass in an isotropic universe: III. The region $R \leq 2m$.* Class. Quantum Grav. **16** 3183–3191.

[52] B. C. Nolan and R. Vera (2005) *Axially symmetric equilibrium regions of Friedmann-Lemaître-Robertson-Walker universes.* Class. Quantum Grav. **22** 4031–4050.

[53] J. Oppenheimer and H. Snyder (1939) *On Continued Gravitational Contraction.* Phys. Rev. **56** 455–459.

[54] R. Plaga (2005) *Is the cluster of galaxies "Abell 194" surrounded by an Einstein-Straus vacuole?* Astron. Astrophys. **440** L41–L43.

[55] K. Riles (2013) *Gravitational Waves: Sources, Detectors and Searches.* Prog. Part. Nucl. Phys. **68** 1–54.

[56] B. S. Sathyaprakash and B. F. Schutz (2009) *Physics, Astrophysics and Cosmology with Gravitational Waves.* Living Rev. Relativity **12**.

[57] H. Sato, *Voids in the expanding universe.* In B. Bertotti, F. de Felice and P. A., eds., *General Relativity and Gravitation: Proceedings of the 10th International Conference on General Relativity and Gravitation* (D. Reidel and Co., Dordrecht 1984), pages 289–312.

[58] E. Schücking (1954) *Das Schwarzschildsche Linienelement und die Expansion des Weltalls.* Z. Phys. **137** 595–603.

[59] J. Senovilla and R. Vera (1997) *Impossibility of the Cylindrically Symmetric Einstein-Straus Model.* Phys. Rev. Lett. **78** 2284–2287.

[60] J. Senovilla and R. Vera (2000) *Cylindrically symmetric dust spacetime.* Class. Quantum Grav. **17** 2843–2846.

[61] H. Stephani, D. Kramer, M. A. H. MacCallum, C. Hoenselaers and E. Herlt, *Exact Solutions of Eisntein's Field Equations.* Cambridge Monographs on Mathematical Physics Series (Cambridge University Press 2003).

[62] R. Vera (2002) *Symmetry-preserving matchings.* Class. Quantum Grav. **19** 5249–5264.