

# Bisimilarity and refinement for hybrid(ised) logics

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The complexity of modern software systems entails the need for reconfiguration mechanisms governing the dynamic evolution of their execution configurations in response to both external stimulus or internal performance measures. Formally, such systems may be represented by transition systems whose nodes correspond to the different configurations they may assume. Therefore, each node is endowed with, for example, an algebra, or a first-order structure, to precisely characterise the semantics of the services provided in the corresponding configuration.

Hybrid logics, which add to the modal description of transition structures the ability to refer to specific states, offer a generic framework to approach the specification and design of this sort of systems. Therefore, the quest for suitable notions of equivalence and refinement between models of hybrid logic specifications becomes fundamental to any design discipline adopting this perspective. This paper contributes to this effort from a distinctive point of view: instead of focussing on a specific hybrid logic, the paper introduces notions of bisimilarity and refinement for hybridised logics, i.e. standard specification logics (e.g. propositional, equational, fuzzy, etc) to which modal and hybrid features were added in a systematic way.

## 1 Introduction

The qualifier *reconfigurable* is used for software systems which behave differently in different modes of operation (often called *configurations*) and commute between them along their lifetime. Formally, such different behaviours can be modelled by imposing additional structure upon states in a transition system expressing the overall system's dynamics. This path has been explored in the authors' recent work [MFMB11] on a specification methodology for reconfigurable systems. The basic insight is that, starting from a classical state-machine specification, each state, regarded as a possible system's configurations, is equipped with a rich mathematical structure to describing its functionality. Technically, specifications become structured state-machines, states denoting algebras or first order structures, rather than sets.

A specification for this sort of system, as discussed in [MFMB11], should be able to make assertions both about the transition dynamics and, locally, about each particular configuration. This leads to the adoption of hybrid logic [Bra10] as the specification *lingua franca* for the envisaged methodology.

However, because specific problems may require specific logics to describe their configurations (e.g. , equational, first-order, fuzzy, etc.), our approach is rooted on very general grounds. Instead of choosing a particular version of hybrid logic, we start by choosing a specific logic for expressing requirements at the configuration (static) level. This is later taken as the *base* logic on top of which the characteristic features of hybrid logic, both at the level of syntax (i.e. modalities, nominals, etc.) and of the semantics (i.e. possible worlds), are developed. This process is called *hybridisation* and was characterised in [MMDB11, DM] as well as in the first author's forthcoming PhD thesis [Madar]. To be completely general, the approach to hybridisation is framed in the context of the institution theory of Goguen and Burstall [GB92, Dia08], each logic (base and hybridised) treated abstractly as an *institution*.

In this context, the quest for suitable notions of *equivalence* and *refinement* between models of hybridised logic specifications becomes fundamental to the envisaged design methodology. Such is the purpose of the present paper. Its contribution is a characterisation of bisimilarity and refinement for hybridised logics which requires a form of *elementary equivalence* [Hod97] between bisimilar states, as a generic formulation of the usual informal requirement that *truth remains invariant*. Clearly what *elementary equivalent* means in each case boils down to the way the satisfaction relation is defined for the base logic used to specify the semantics of local configurations.

The choice of similarity and bisimilarity to base refinement and equivalence of (models of) reconfigurable systems seems quite standard as a fine grained approach to observational methods for systems comparison. The notion of bisimulation and the associated conductive proof method, which is now pervasive in Computer Science, originated in concurrency theory due to the seminal work of David Park [Par81] and R. Milner in the quest for an appropriate definition of observational equivalence for communicating processes. But the concept also arose independently in modal logic as a refinement of notions of homomorphism between algebraic models. In the sequel the concept is revisited for models of hybridised logics adding up to the design methodology mentioned above.

The paper is organized as follows: Section 2 recalls institutions as abstract characterisations of logics and provides a brief, and simplified, overview of the hybridization method proposed in [MMDB11, DM]. This forms the context for the paper’s contribution. Then, Section 3 introduces a general notion of bisimulation for hybridised logics and characterizes the preservation of logic satisfaction under it. Section 4 follows a similar path but focussing on refinement as witnessed by a simulation relation.

## 2 Background

### 2.1 Institutions

An *institution* is a category theoretic formalisation<sup>1</sup> of a logical system, encompassing syntax, semantics and satisfaction. The concept was put forward by Goguen and Burstall, in the end of the seventies, in order to “*formalise the formal notion of logical systems*”, in response to the “*population explosion among the logical systems used in Computing Science*” [GB92].

The universal character of institutions proved effective and resilient as witnessed by the wide number of logics formalised in this framework. Examples range from the usual logics in classical mathematical logic (propositional, equational, first order, etc.), to the ones underlying specification and programming languages or used for describing particular systems from different domains. Well-known examples include *probabilistic logics* [BKI02], *quantum logics* [CMSS06], *hidden and observational logics* [BD94, BH06], *coalgebraic logics* [CÔ6], as well as logics for reasoning about *process algebras* [MR07], *functional* [ST12, SM09] and *imperative programming languages* [ST12].

The theory of institutions (see [Dia08] for a extensive account) was motivated by the need to abstract from the particular details of each individual logic and characterise generic issues, such as satisfaction and combination of logics, in very general terms. In Computer Science, this lead to the development of a solid *institution-independent specification theory*, on which, structuring and parameterisation mechanisms, required to scale up software specification methods, are defined ‘once and for all’, irrespective of the concrete logic used in each application domain. The definition is recalled below (e.g., [GB92, Dia08]) and illustrated with a few examples to which we return later in the paper.

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<sup>1</sup>The language of category theory [Lan71] is used to set the scene for institutions; categories, however, play no role in the paper’s contribution.

**Definition 2.1 (Institution)** An institution

$$\mathcal{I} = (\text{Sign}^{\mathcal{I}}, \text{Sen}^{\mathcal{I}}, \text{Mod}^{\mathcal{I}}, (\models_{\Sigma}^{\mathcal{I}})_{\Sigma \in |\text{Sign}^{\mathcal{I}}|})$$

consists of

- a category  $\text{Sign}^{\mathcal{I}}$  whose objects are called signatures and arrows signature morphisms;
- a functor  $\text{Sen}^{\mathcal{I}} : \text{Sign}^{\mathcal{I}} \rightarrow \text{Set}$  giving for each signature a set whose elements are called sentences over that signature;
- a functor  $\text{Mod}^{\mathcal{I}} : (\text{Sign}^{\mathcal{I}})^{op} \rightarrow \text{CAT}$ , giving for each signature  $\Sigma$  a category whose objects are called  $\Sigma$ -models, and whose arrows are called  $\Sigma$ -(model) homomorphisms; each arrow  $\varphi : \Sigma \rightarrow \Sigma' \in \text{Sign}^{\mathcal{I}}$ , (i.e.,  $\varphi : \Sigma' \rightarrow \Sigma \in (\text{Sign}^{\mathcal{I}})^{op}$ ) is mapped into a functor  $\text{Mod}^{\mathcal{I}}(\varphi) : \text{Mod}^{\mathcal{I}}(\Sigma') \rightarrow \text{Mod}^{\mathcal{I}}(\Sigma)$  called a reduct functor, whose effect is to cast a model of  $\Sigma'$  as a model of  $\Sigma$ ;
- a relation  $\models_{\Sigma}^{\mathcal{I}} \subseteq |\text{Mod}^{\mathcal{I}}(\Sigma)| \times \text{Sen}^{\mathcal{I}}(\Sigma)$  for each  $\Sigma \in |\text{Sign}^{\mathcal{I}}|$ , called the satisfaction relation,

such that for each morphism  $\varphi : \Sigma \rightarrow \Sigma' \in \text{Sign}^{\mathcal{I}}$ , the satisfaction condition

$$M' \models_{\Sigma'}^{\mathcal{I}} \text{Sen}^{\mathcal{I}}(\varphi)(\rho) \text{ iff } \text{Mod}^{\mathcal{I}}(\varphi)(M') \models_{\Sigma}^{\mathcal{I}} \rho \quad (1)$$

holds for each  $M' \in |\text{Mod}^{\mathcal{I}}(\Sigma')|$  and  $\rho \in \text{Sen}^{\mathcal{I}}(\Sigma)$ . Graphically,

$$\begin{array}{ccc} \Sigma & \text{Mod}^{\mathcal{I}}(\Sigma) & \xrightarrow{\models_{\Sigma}^{\mathcal{I}}} \text{Sen}^{\mathcal{I}}(\Sigma) \\ \varphi \downarrow & \text{Mod}^{\mathcal{I}}(\varphi) \uparrow & \downarrow \text{Sen}^{\mathcal{I}}(\varphi) \\ \Sigma' & \text{Mod}^{\mathcal{I}}(\Sigma') & \xrightarrow{\models_{\Sigma'}^{\mathcal{I}}} \text{Sen}^{\mathcal{I}}(\Sigma') \end{array}$$

**Example 2.1 (Propositional Logic)** A signature  $\text{Prop} \in |\text{Sign}^{PL}|$  is a set of propositional variables symbols and a signature morphism is just a function  $\varphi : \text{Prop} \rightarrow \text{Prop}'$ . Therefore,  $\text{Sign}^{PL}$  coincides with the category  $\text{Set}$ .

Functor  $\text{Mod}$  maps each signature  $\text{Prop}$  to the category  $\text{Mod}^{PL}(\text{Prop})$  and each signature morphism  $\varphi$  to the reduct functor  $\text{Mod}^{PL}(\varphi)$ . Objects of  $\text{Mod}^{PL}(\text{Prop})$  are functions  $M : \text{Prop} \rightarrow \{\top, \perp\}$  and, its morphisms, functions  $h : \text{Prop} \rightarrow \text{Prop}$  such that  $M(p) = M'(h(p))$ . Given a signature morphism  $\varphi : \text{Prop} \rightarrow \text{Prop}'$ , the reduct of a model  $M' \in |\text{Mod}^{PL}(\text{Prop}')|$ , say  $M = \text{Mod}^{PL}(\varphi)(M')$  is defined, for each  $p \in \text{Prop}$ , as  $M(p) = M'(\varphi(p))$ .

The sentences functor maps each signature  $\text{Prop}$  to the set of propositional sentences  $\text{Sen}^{PL}(\text{Prop})$  and each morphism  $\varphi : \text{Prop} \rightarrow \text{Prop}'$  to the sentences' translation  $\text{Sen}^{PL}(\varphi) : \text{Sen}^{PL}(\text{Prop}) \rightarrow \text{Sen}^{PL}(\text{Prop}')$ . The set  $\text{Sen}^{PL}(\text{Prop})$  is the usual set of propositional formulae defined by the grammar

$$\rho ::= p \mid \rho \vee \rho \mid \rho \wedge \rho \mid \rho \Rightarrow \rho \mid \neg \rho$$

for  $p \in \text{Prop}$ . The translation of a sentence  $\text{Sen}^{PL}(\varphi)(\rho)$  is obtained by replacing each proposition of  $\rho$  by the respective  $\varphi$ -image.

Finally, for each  $\text{Prop} \in \text{Sen}^{PL}$ , the satisfaction relation  $\models_{\text{Prop}}^{PL}$  is defined as usual:

- $M \models_{\text{Prop}}^{PL} p$  iff  $M(p) = \top$ , for any  $p \in \text{Prop}$ ;
- $M \models_{\text{Prop}}^{PL} \rho \vee \rho'$  iff  $M \models_{\text{Prop}}^{PL} \rho$  or  $M \models_{\text{Prop}}^{PL} \rho'$ ,

and similarly for the other connectives.

**Example 2.2 (Equational logic)** Signatures in the institution  $EQ$  of equational logic are pairs  $(S, F)$  where  $S$  is a set of sort symbols and  $F = \{F_{\underline{ar} \rightarrow s} \mid \underline{ar} \in S^*, s \in S\}$  is a family of sets of operation symbols indexed by arities  $\underline{ar}$  (for the arguments) and sorts  $s$  (for the results). Signature morphisms map both components in a compatible way: they consist of pairs  $\varphi = (\varphi^{\text{st}}, \varphi^{\text{op}}) : (S, F) \rightarrow (S', F')$ , where  $\varphi^{\text{st}} : S \rightarrow S'$  is a function, and  $\varphi^{\text{op}} = \{\varphi_{\underline{ar} \rightarrow s}^{\text{op}} : F_{\underline{ar} \rightarrow s} \rightarrow F'_{\varphi^{\text{st}}(\underline{ar}) \rightarrow \varphi^{\text{st}}(s)} \mid \underline{ar} \in S^*, s \in S\}$  a family of functions mapping operations symbols respecting arities.

A model  $M$  for a signature  $(S, F)$  is an algebra interpreting each sort symbol  $s$  as a carrier set  $M_s$  and each operation symbol  $\sigma \in F_{\underline{ar} \rightarrow s}$  as a function  $M_\sigma : M_{\underline{ar}} \rightarrow M_s$ , where  $M_{\underline{ar}}$  is the product of the arguments' carriers. Model morphism are homomorphisms of algebras, i.e.,  $S$ -indexed families of functions  $\{h_s : M_s \rightarrow M'_s \mid s \in S\}$  such that for any  $m \in M_{\underline{ar}}$ , and for each  $\sigma \in F_{\underline{ar} \rightarrow s}$ ,  $h_s(M_\sigma(m)) = M'_s(h_{\underline{ar}}(m))$ . For each signature morphism  $\varphi$ , the reduct of a model  $M'$ , say  $M = \text{Mod}^{EQ}(\varphi)(M')$  is defined by  $(M)_x = M'_{\varphi(x)}$  for each sort and function symbol  $x$  from the domain signature of  $\varphi$ . The models functor maps signatures to categories of algebras and signature morphisms to the respective reduct functors.

Sentences are universal quantified equations  $(\forall X)t = t'$ . Sentence translations along a signature morphism  $\varphi : (S, F) \rightarrow (S', F')$ , i.e.,  $\text{Sen}^{EQ}(\varphi) : \text{Sen}^{EQ}(S, F) \rightarrow \text{Sen}^{EQ}(S', F')$ , replace symbols of  $(S, F)$  by the respective  $\varphi$ -images in  $(S', F')$ . The sentences functor maps each signature to the set of first-order sentences and each signature morphism to the respective sentences translation. The satisfaction relation is the usual Tarskian satisfaction defined recursively on the structure of the sentences as follows:

- $M \models_{(S, F)} t = t'$  when  $M_t = M_{t'}$ , where  $M_t$  denotes the interpretation of the  $(S, F)$ -term  $t$  in  $M$  defined recursively by  $M_{\sigma(t_1, \dots, t_n)} = M_\sigma(M_{t_1}, \dots, M_{t_n})$ .
- $M \models_{(S, F)} (\forall X)\rho$  when  $M' \models_{(S, F+X)} \rho$  for any  $(S, F+X)$ -expansion  $M'$  of  $M$ .

**Example 2.3 (Propositional Fuzzy Logic)** Multi-valued logics [Got01] generalise classic logics by replacing, as its truth domain, the 2-element Boolean algebra, by larger sets structured as complete residuate lattices. They were originally formalised as institutions in [ACEGG90] (but see also [Dia11] for a recent reference).

Residuate lattices are tuples  $L = (\mathbf{L}, \leq, \wedge, \vee, \top, \perp, \otimes)$ , where

- $(\mathbf{L}, \wedge, \vee, \top, \perp)$  is a lattice ordered by  $\leq$ , with carrier  $\mathbf{L}$ , with (binary) infimum ( $\wedge$ ) and supremum ( $\vee$ ), and biggest and smallest elements  $\top$  and  $\perp$ ;
- $\otimes$  is an associative binary operation such for any elements  $x, y, z \in L$ :
  - $x \otimes \top = \top \otimes x = x$ ;
  - $y \leq z$  implies that  $(x \otimes y) \leq (x \otimes z)$ ;
  - there exists an element  $x \Rightarrow z$  such that
 
$$y \leq (x \Rightarrow z) \text{ iff } x \otimes y \leq z.$$

The residuate lattice  $L$  is complete if any subset  $S \subseteq L$  has infimum and supremum denoted by  $\bigwedge S$  and  $\bigvee S$ , respectively.

Given a complete residuate lattice  $L$ , the institution  $MVL_L$  is defined as follows.

- $MVL_L$ -signature are PL-signatures.
- Sentences of  $MVL_L$  consist of pairs  $(\rho, p)$  where  $p$  is an element of  $L$  and  $\rho$  is defined as a PL-sentence over the set of connectives  $\{\Rightarrow, \vee, \top, \perp, \otimes\}$ .
- A  $MVL_L$ -model  $M$  is a function  $M : F\text{Prop} \rightarrow L$ .

- For any  $M \in \text{Mod}^{MVL_L}(FProp)$  and for any  $(\rho, p) \in \text{Sen}^{MVL_L}(FProp)$  the satisfaction relation is

$$M \models_{FProp}^{MVL_L} (\rho, p) \text{ iff } p \leq (M \models \rho)$$

where  $M \models \rho$  is inductively defined as follows:

- for any proposition  $p \in FProp$ ,  $(M \models p) = M(p)$ ;
- $(M \models \top) = \top$ ;
- $(M \models \perp) = \perp$ ;
- $(M \models \rho_1 \star \rho_2) = (M \models \rho_1) \star (M \models \rho_2)$ , for  $\star \in \{\vee, \Rightarrow, \otimes\}$ ;

This institution captures many multi-valued logics in the literature. For instance, taking  $L$  as the Łukasiewicz arithmetic lattice over the closed interval  $[0, 1]$ , where  $x \otimes y = 1 - \max\{0, x + y - 1\}$  (and  $x \Rightarrow y = \min\{1, 1 - x + y\}$ ), yields the standard propositional fuzzy logic.

## 2.2 Brief overview on the hybridisation method

Having recalled the notion of an institution, we shall now briefly review the core of the *hybridisation* method mentioned in the introduction and proposed in [MMDB11, DM]. We concentrate in a simplified version, i.e., quantifier-free and non-constrained, of the general method. The method enriches a base (arbitrary) institution  $\mathcal{I} = (\text{Sign}^{\mathcal{I}}, \text{Sen}^{\mathcal{I}}, \text{Mod}^{\mathcal{I}}, (\models_{\Sigma}^{\mathcal{I}})_{\Sigma \in |\text{Sign}^{\mathcal{I}}|})$  with hybrid logic features and the corresponding Kripke semantics. The result is still an institution,  $\mathcal{H}\mathcal{I}$ , called the *hybridisation of  $\mathcal{I}$* .

*The category of  $\mathcal{H}\mathcal{I}$ -signatures.* First of all the base signature is enriched with nominals and polyadic modalities. Therefore, the category of  $\mathcal{I}$ -hybrid signatures, denoted by  $\text{Sign}^{\mathcal{H}\mathcal{I}}$ , is defined as the direct (cartesian) product of categories:

$$\text{Sign}^{\mathcal{H}\mathcal{I}} = \text{Sign}^{\mathcal{I}} \times \text{Sign}^{REL}.$$

Thus, signatures are triples  $(\Sigma, \text{Nom}, \Lambda)$ , where  $\Sigma \in |\text{Sign}^{\mathcal{I}}|$  and, in the *REL*-signature  $(\text{Nom}, \Lambda)$ ,  $\text{Nom}$  is a set of constants called *nominals* and  $\Lambda$  is a set of relational symbols called *modalities*;  $\Lambda_n$  stands for the set of modalities of arity  $n$ . Morphisms  $\varphi \in \text{Sign}^{\mathcal{H}\mathcal{I}}((\Sigma, \text{Nom}, \Lambda), (\Sigma', \text{Nom}', \Lambda'))$  are triples  $\varphi = (\varphi_{\text{Sig}}, \varphi_{\text{Nom}}, \varphi_{\text{MS}})$  where  $\varphi_{\text{Sig}} \in \text{Sign}^{\mathcal{I}}(\Sigma, \Sigma')$ ,  $\varphi_{\text{Nom}} : \text{Nom} \rightarrow \text{Nom}'$  is a function and  $\varphi_{\text{MS}} = (\varphi_n : \Lambda_n \rightarrow \Lambda'_n)_{n \in \mathbb{N}}$  a  $\mathbb{N}$ -family of functions mapping nominals and  $n$ -ary-modality symbols, respectively.

*$\mathcal{H}\mathcal{I}$ -sentences functor.* The second step is to enrich the base sentences accordingly. The sentences of the base institution and the nominals are taken as atoms and composed with the boolean connectives, modalities, and satisfaction operators as follows:  $\text{Sen}^{\mathcal{H}\mathcal{I}}(\Sigma, \text{Nom}, \Lambda)$  is the least set such that

- $\text{Nom} \subseteq \text{Sen}^{\mathcal{H}\mathcal{I}}(\Delta)$ ;
- $\text{Sen}^{\mathcal{I}}(\Sigma) \subseteq \text{Sen}^{\mathcal{H}\mathcal{I}}(\Delta)$ ;
- $\rho \star \rho' \in \text{Sen}^{\mathcal{H}\mathcal{I}}(\Delta)$  for any  $\rho, \rho' \in \text{Sen}^{\mathcal{H}\mathcal{I}}(\Delta)$  and any  $\star \in \{\vee, \wedge, \Rightarrow\}$ ,
- $\neg \rho \in \text{Sen}^{\mathcal{H}\mathcal{I}}(\Delta)$ , for any  $\rho \in \text{Sen}^{\mathcal{H}\mathcal{I}}(\Delta)$ ,
- $@_i \rho \in \text{Sen}^{\mathcal{H}\mathcal{I}}(\Delta)$  for any  $\rho \in \text{Sen}^{\mathcal{H}\mathcal{I}}(\Delta)$  and  $i \in \text{Nom}$ ;
- $[\lambda](\rho_1, \dots, \rho_n), \langle \lambda \rangle(\rho_1, \dots, \rho_n) \in \text{Sen}^{\mathcal{H}\mathcal{I}}(\Delta)$ , for any  $\lambda \in \Lambda_{n+1}, \rho_i \in \text{Sen}^{\mathcal{H}\mathcal{I}}(\Delta), i \in \{1, \dots, n\}$ .

Given a  $\mathcal{H}\mathcal{I}$ -signature morphism  $\varphi = (\varphi_{\text{Sig}}, \varphi_{\text{Nom}}, \varphi_{\text{MS}}) : (\Sigma, \text{Nom}, \Lambda) \rightarrow (\Sigma', \text{Nom}', \Lambda')$ , the translation of sentences  $\text{Sen}^{\mathcal{H}\mathcal{I}}(\varphi)$  is defined as follows:

- $\text{Sen}^{\mathcal{H}\mathcal{I}}(\varphi)(\rho) = \text{Sen}^{\mathcal{I}}(\varphi_{\text{Sig}})(\rho)$  for any  $\rho \in \text{Sen}^{\mathcal{I}}(\Sigma)$ ;
- $\text{Sen}^{\mathcal{H}\mathcal{I}}(\varphi)(i) = \varphi_{\text{Nom}}(i)$ ;
- $\text{Sen}^{\mathcal{H}\mathcal{I}}(\varphi)(\neg \rho) = \neg \text{Sen}^{\mathcal{H}\mathcal{I}}(\varphi)(\rho)$ ;

- $\text{Sen}^{\mathcal{H}\mathcal{I}}(\varphi)(\rho \star \rho') = \text{Sen}^{\mathcal{H}\mathcal{I}}(\varphi)(\rho) \star \text{Sen}^{\mathcal{H}\mathcal{I}}(\varphi)(\rho')$ ,  $\star \in \{\vee, \wedge, \Rightarrow\}$ ;
- $\text{Sen}^{\mathcal{H}\mathcal{I}}(\varphi)(@_i \rho) = @_i \text{Sen}^{\mathcal{H}\mathcal{I}}(\varphi)(\rho)$ ;
- $\text{Sen}^{\mathcal{H}\mathcal{I}}(\varphi)([\lambda](\rho_1, \dots, \rho_n)) = [\varphi_{\text{MS}}(\lambda)](\text{Sen}^{\mathcal{H}\mathcal{I}}(\varphi)(\rho_1), \dots, \text{Sen}^{\mathcal{H}\mathcal{I}}(\varphi)(\rho_n))$ ;
- $\text{Sen}^{\mathcal{H}\mathcal{I}}(\varphi)(\langle \lambda \rangle(\rho_1, \dots, \rho_n)) = \langle \varphi_{\text{MS}}(\lambda) \rangle(\text{Sen}^{\mathcal{H}\mathcal{I}}(\varphi)(\rho_1), \dots, \text{Sen}^{\mathcal{H}\mathcal{I}}(\varphi)(\rho_n))$ .

*$\mathcal{H}\mathcal{I}$ -models functor.* Models of the hybridised logic  $\mathcal{H}\mathcal{I}$  can be regarded as  $(\Lambda)$ -Kripke structures whose worlds are  $\mathcal{I}$ -models. Formally  $(\Sigma, \text{Nom}, \Lambda)$ -models are pairs  $(M, W)$  where

- $W$  is a  $(\text{Nom}, \Lambda)$ -model in  $REL$ ;
- $M$  is a function  $|W| \rightarrow |\text{Mod}^{\mathcal{I}}(\Sigma)|$ .

In each world  $(M, W)$ ,  $\{W_n \mid n \in \text{Nom}\}$  provides interpretations for *nominals* in  $\text{Nom}$ , whereas relations  $\{W_\lambda \mid \lambda \in \Lambda_n, n \in \omega\}$  interpret *modalities*  $\Lambda$ . We denote  $M(w)$  simply by  $M_w$ . The reduct definition is lifted from the base institution: the reduct of a  $\Delta'$ -model  $(M', W')$  along a signature morphism  $\varphi = (\varphi_{\text{Sig}}, \varphi_{\text{Nom}}, \varphi_{\text{MS}}) : \Delta \rightarrow \Delta'$ , denoted by  $\text{Mod}^{\mathcal{H}\mathcal{I}}(\varphi)(M', W')$ , is the  $\Delta$ -model  $(M, W)$  such that

- $W$  is the  $(\varphi_{\text{Nom}}, \varphi_{\text{MS}})$ -reduct of  $W'$ ; i.e.
  - $|W| = |W'|$ ;
  - for any  $n \in \text{Nom}$ ,  $W_n = W'_{\varphi_{\text{Nom}}(n)}$ ;
  - for any  $\lambda \in \Lambda$ ,  $W_\lambda = W'_{\varphi_{\text{MS}}(\lambda)}$ ;
- for any  $w \in |W|$ ,  $M_w = \text{Mod}^{\mathcal{I}}(\varphi_{\text{Sig}})(M'_w)$ .

*The Satisfaction Relation.* Let  $(\Sigma, \text{Nom}, \Lambda) \in |\text{Sign}^{\mathcal{H}\mathcal{I}}|$  and  $(M, W) \in |\text{Mod}^{\mathcal{H}\mathcal{I}}(\Sigma, \text{Nom}, \Lambda)|$ . For any  $w \in |W|$  we define:

- $(M, W) \models^w \rho$  iff  $M_w \models^{\mathcal{I}} \rho$ ; when  $\rho \in \text{Sen}^{\mathcal{I}}(\Sigma)$ ,
- $(M, W) \models^w i$  iff  $W_i = w$ ; when  $i \in \text{Nom}$ ,
- $(M, W) \models^w \rho \vee \rho'$  iff  $(M, W) \models^w \rho$  or  $(M, W) \models^w \rho'$ ,
- $(M, W) \models^w \rho \wedge \rho'$  iff  $(M, W) \models^w \rho$  and  $(M, W) \models^w \rho'$ ,
- $(M, W) \models^w \rho \Rightarrow \rho'$  iff  $(M, W) \models^w \rho$  implies that  $(M, W) \models^w \rho'$ ,
- $(M, W) \models^w \neg \rho$  iff  $(M, W) \not\models^w \rho$ ,
- $(M, W) \models^w [\lambda](\xi_1, \dots, \xi_n)$  iff for any  $(w, w_1, \dots, w_n) \in W_\lambda$  we have that  $(M, W) \models^{w_i} \xi_i$  for some  $1 \leq i \leq n$ .
- $(M, W) \models^w \langle \lambda \rangle(\xi_1, \dots, \xi_n)$  iff there exists  $(w, w_1, \dots, w_n) \in W_\lambda$  such that and  $(M, W) \models^{w_i} \xi_i$  for any  $1 \leq i \leq n$ .
- $(M, W) \models^w @_j \rho$  iff  $(M, W) \models^{w_j} \rho$ ,

We write  $(M, W) \models \rho$  iff  $(M, W) \models^w \rho$  for any  $w \in |W|$ .

As expected  $\mathcal{H}\mathcal{I}$  is itself an institution:

**Theorem 2.1 ([MMDB11])** *Let  $\Delta = (\Sigma, \text{Nom}, \Lambda)$  and  $\Delta' = (\Sigma', \text{Nom}', \Lambda')$  be two  $\mathcal{H}\mathcal{I}$ -signatures and  $\varphi : \Delta \rightarrow \Delta'$  a morphism of signatures. For any  $\rho \in \text{Sen}^{\mathcal{H}\mathcal{I}}(\Delta)$ ,  $(M', W') \in |\text{Mod}^{\mathcal{C}}(\Delta')|$ , and  $w \in |W|$ ,*

$$\text{Mod}^{\mathcal{H}\mathcal{I}}(\varphi)(M', W') \models^w \rho \text{ iff } (M', W') \models^w \text{Sen}^{\mathcal{H}\mathcal{I}}(\varphi)(\rho).$$

Let us illustrate the method by applying it to the three institutions described above.

**Example 2.4** ( $\mathcal{H}PL$ ) *The hybridisation of the propositional logic institution  $PL$  is an institution where signatures are triples  $(Prop, \text{Nom}, \Lambda)$  and sentences are generated by*

$$\rho ::= \rho_0 \mid i \mid @_i \rho \mid \rho \odot \rho \mid \neg \rho \mid \langle \lambda \rangle (\rho, \dots, \rho) \mid [\lambda] (\rho, \dots, \rho) \quad (2)$$

where  $\rho_0 \in \text{Sen}^{PL}(Prop)$ ,  $i \in \text{Nom}$ ,  $\lambda \in \Lambda_n$  and  $\odot = \{\vee, \wedge, \Rightarrow\}$ . Note there is a double level of connectives in the sentences: the one coming from base  $PL$ -sentences and another introduced by the hybridisation process. However, they “semantically collapse” and, hence, no distinction between them needs to be done (see [DM] for details). A  $(Prop, \text{Nom}, \Lambda)$ -model is a pair  $(M, W)$ , where  $W$  is a transition structure with a set of worlds  $|W|$ . Constants  $W_i, i \in \text{Nom}$  stand for the named worlds and  $(n+1)$ -ary relations  $W_\lambda, \lambda \in \Lambda_n$  are the accessibility relations characterising the structure. For each world  $w \in |W|$ ,  $M(w)$  is a (local)  $PL$ -model, assigning propositions in  $Prop$  to the world  $w$ .

Restricting the signatures to those with just a single unary modality (i.e., where  $\Lambda_1 = \{\lambda\}$  and  $\Lambda_n = \emptyset$  for the remaining  $n \neq 1$ ), results in the usual institution for classical hybrid propositional logic [Bra10].

**Example 2.5** ( $\mathcal{H}MVL_L$ ) *The institution obtained through the hybridization of  $MVL_L$ , for a fixed  $L$ , is similar to the  $\mathcal{H}PL$  institution defined above, but for two aspects,*

- sentences are defined as in (2) but considering  $MVL$   $FPop$ -sentences  $(\rho_0, p)$  as atomic;
- to each world  $w \in |W|$  is associated a function assigning to each proposition its value in  $L$ .

It is interesting to note that expressivity increases even if one restricts to the case of a (one-world) standard semantics. For instance, differently from the base case where each sentence is tagged by a  $L$ -value, one may now deal with more structured expressions involving several  $L$ -values, as in, for example,  $(\rho, p) \wedge (\rho', p')$ .

**Example 2.6** ( $\mathcal{H}EQ$ ) *Signatures of  $\mathcal{H}EQ$  are triples  $((S, F), \text{Nom}, \Lambda)$  and the sentences are defined as in (2) but taking  $(S, F)$ -equations  $(\forall X)t = t'$  as atomic base sentences. Models are Kripke structures with a (local)- $(S, F)$ -algebra per world. This institution is a suitable framework to specify reconfigurable system in a “configurations-as-worlds” perspective: distinct configurations are modelled by distinct algebras; and reconfigurations expressed by transitions (c.f. [MFMB11, Madar]). Clearly, in this sort of specifications interfaces are given equationally, based on  $EQ$ -signatures. Nominals identify the “relevant” configurations and reconfigurations amount to state transitions. Therefore, one resorts to equations tagged with the satisfaction operators to specify the configurations, plain equations to specify global properties of the system and the modal features to specify its reconfigurability dynamics.*

### 3 Bisimulation for hybridised Logics

Having briefly reviewed what an institution is and how, through a systematic process, one may introduce in an arbitrary logic both modalities and nominals to explicitly refer to states in a specification, we may now focus on the paper’s specific contribution. Our starting point is a method to specify reconfigurable software as transition systems whose states represent particular configurations. They can themselves be an algebraic specification, a relation structure or even another, local transition system. Such two-staged specifications are common in the Software Engineering practice (see, e.g., Gurevich’s Abstract State Machines [BS03]); the originality of our method lies in its genericity: whatever logic is found useful to specify each concrete configuration, a method is offered to compute its hybrid counterpart. In this setting, this section and the following one seek for suitable notions of equivalence and refinement for this kind of specifications. Naturally, such notions should also be parametric on the base logic used, i.e., on the language in which the specifications of each concrete configuration are written. The price to pay

is, of course, some extra notation and the use of a generic framework — that of *institutions* — in which concepts can be formulated and results proved once and for all.

As the external layer of a reconfigurable system specification is that of a transition system, it is natural to resort to suitable formulations of *bisimilarity* and *similarity* to capture equivalence and refinement, respectively. The precise characterisation of such notions at the high level of abstraction chosen, is, in fact, the paper’s contribution.

Intuitively a bisimulation relates worlds which exhibit the “same” (observable) information and preserves this property along transitions. Thus, to define a general notion of bisimulation over Kripke structures whose states are models of whatever base logic was chosen for specifications, we have to make precise what the “same” information actually means. For example, if the system’s configurations are specified by *equations*, as abstract data types, to establish that two such configurations are bisimilar will certainly require that each specification generates the same variety. Actually, in this case, they are essentially the same data type. In the more general setting of this paper the base logic is a parameter and we have to deal with its hybridised version  $\mathcal{H}\mathcal{S}$ . Our proposal is, thus, to resort to the broad notion of *elementary equivalence* (e.g.[Hod97]), and add to the bisimulation definition the requirement that local configurations, i.e., local  $\mathcal{S}$ -models related by a bisimulation be *elementarily equivalent*. Formally,

**Definition 3.1** *Let  $M, M' \in \text{Mod}^{\mathcal{S}}(\Sigma)$  and  $\text{Sen}'$  be a subfunctor of  $\text{Sen}^{\mathcal{S}}$ . Models  $M$  and  $M'$  are elementarily equivalent with respect to sentences in  $\text{Sen}'(\Sigma)$ , in symbols  $M \equiv_{\text{Sen}'}^{\text{Sen}^{\mathcal{S}}} M'$ , if for any  $\rho \in \text{Sen}'(\Sigma)$*

$$M \models^{\mathcal{S}} \rho \text{ iff } M' \models^{\mathcal{S}} \rho. \quad (3)$$

Under the institution theory motto — *truth is invariant under change of notation* — we write  $M \equiv_{\varphi}^{\text{Sen}'}$   $M'$  whenever  $M \equiv_{\text{Sen}'}^{\text{Mod}^{\mathcal{S}}(\varphi)}(M')$  for a given  $\varphi \in \text{Sign}^{\mathcal{S}}(\Sigma, \Sigma')$ ,  $M \in \text{Mod}^{\mathcal{S}}(\Sigma)$  and  $M' \in \text{Mod}^{\mathcal{S}}(\Sigma')$ . Models  $M$  and  $M'$  are said to be  $\varphi, \text{Sen}'$ -elementarily equivalent.

Resorting to the satisfaction condition in  $\mathcal{S}$ , the following characterisation of  $\varphi, \text{Sen}'$ -elementary equivalence pops out:

**Corollary 3.1**  *$M \equiv_{\varphi}^{\text{Sen}'}$   $M'$  iff, for any  $\rho \in \text{Sen}'(\Sigma)$ ,  $M \models_{\Sigma}^{\mathcal{S}} \rho \Leftrightarrow M' \models_{\Sigma}^{\mathcal{S}} \text{Sign}^{\mathcal{S}}(\varphi)(\rho)$ .*

If only an implication  $\Rightarrow$  holds in the right hand side of the above equivalence we write  $M \gg_{\varphi}^{\text{Sen}'}$   $M'$ . Note the role of  $\varphi$  above: as a signature morphism it captures the possible *change of notation* from a specification to another. For example it may cater for renaming propositions in Ex. 3.1 or signature components in Ex. 3.2. However, its pertinence becomes clearer in refinement situations, as discussed in the next section. There it may accommodate many forms of interface enrichment or adaptation (e.g. through the introduction of auxilliary operations).

Let us now define bisimulation in this general setting.

**Definition 3.2** *Let  $\mathcal{H}\mathcal{S}$  be the hybridization of the institution  $\mathcal{S}$  and  $\varphi \in \text{Sign}^{\mathcal{H}\mathcal{S}}(\Delta, \Delta')$  a signature morphism. Let  $\text{Sen}'$  be a subfunctor of  $\text{Sen}^{\mathcal{S}}$ . A  $\varphi, \text{Sen}'$ -bisimulation between models  $(M, W) \in \text{Mod}^{\mathcal{H}\mathcal{S}}(\Delta)$  and  $(M', W') \in \text{Mod}^{\mathcal{H}\mathcal{S}}(\Delta')$  is a non-empty relation  $B_{\varphi} \subseteq |W| \times |W'|$  such that*

- (i) *for any  $wB_{\varphi}w'$ , and for any  $i \in \text{Nom}$ ,  $W_i = w$  iff  $W'_{\varphi_{\text{Nom}}(i)} = w'$ .*
- (ii) *for any  $wB_{\varphi}w'$ ,  $M_w \equiv_{\varphi_{\text{Sig}}}^{\text{Sen}'} M'_{w'}$ .*
- (iii) *for any  $i \in \text{Nom}$ ,  $W_i B_{\varphi} W'_{\varphi_{\text{Nom}}(i)}$ .*
- (iv) *For any  $\lambda \in \Lambda_n$ , if  $(w, w_1, \dots, w_n) \in W_{\lambda}$  and  $wB_{\varphi}w'$ , then for each  $k \in \{1, \dots, n\}$  there is a  $w'_k \in |W'|$  such that  $w_k B_{\varphi} w'_k$  and  $(w', w'_1, \dots, w'_n) \in W'_{\varphi_{\text{MS}}(\lambda)}$ .*



(v) For any  $\lambda \in \Lambda_n$  if  $(w', w'_1, \dots, w'_n) \in W'_{\varphi_{\text{MS}}(\lambda)}$  and  $wB_\varphi w'$ , then for each  $k \in \{1, \dots, n\}$  there is a  $w_k \in |W|$ , such that  $w_k B_\varphi w'_k$  and  $(w, w_1, \dots, w_n) \in W_\lambda$ .

The following result establishes that, for quantifier-free hybridisations, the (local)-hybrid satisfaction  $\models^{\mathcal{H}\mathcal{I}}$  is invariant under  $\varphi, \text{Sen}$ -bisimulations:

**Theorem 3.1** *Let  $\mathcal{H}\mathcal{I}$  be a quantifier-free hybridization of the institution  $\mathcal{I}$  and  $\varphi \in \text{Sign}^{\mathcal{H}\mathcal{I}}(\Delta, \Delta')$  a signature morphism. Let  $B_\varphi \subseteq |W| \times |W'|$  be a  $\varphi, \text{Sen}$ -bisimulation. Then, for any  $wB_\varphi w'$  and for any  $\rho \in \text{Sen}^{\mathcal{H}\mathcal{I}}(\Delta)$ ,*

$$(M, W) \models^w \rho \text{ iff } (M', W') \models^{w'} \text{Sen}^{\mathcal{H}\mathcal{I}}(\varphi)(\rho). \quad (4)$$

**Proof.** The proof is by induction on the structure of the sentences.

1.  $\rho = i$  for some  $i \in \text{Nom}$ :

$$\begin{aligned} & (M, W) \models^w i \\ \Leftrightarrow & \quad \{ \text{defn. of } \models^w \} \\ & W_i = w \\ \Leftrightarrow & \quad \{ \text{(i) of Defn 3.2} \} \\ & W'_{\varphi(i)} = w' \\ \Leftrightarrow & \quad \{ \text{defn. of } \models^{w'} \} \\ & (M', W') \models^{w'} \varphi_{\text{Nom}}(i) \\ \Leftrightarrow & \quad \{ \text{defn of } \text{Sen}^{\mathcal{H}\mathcal{I}}(\varphi) \} \\ & (M', W') \models^{w'} \text{Sen}^{\mathcal{H}\mathcal{I}}(\varphi)(i) \end{aligned}$$

2.  $\rho \in \text{Sen}^{\mathcal{I}}(\Sigma)$ :

$$\begin{aligned} & (M, W) \models^w \rho \\ \Leftrightarrow & \quad \{ \text{defn. of } \models^w \} \\ & M_w \models^{\mathcal{I}} \rho \\ \Leftrightarrow & \quad \{ \text{by hypothesis } M_w \equiv_{\varphi_{\text{Sig}}} M'_{w'} + \text{Cor 3.1} \} \\ & M'_{w'} \models \text{Sen}^{\mathcal{I}}(\varphi_{\text{Sig}})(\rho) \\ \Leftrightarrow & \quad \{ \text{defn. of } \models^{w'} \} \\ & (M', W') \models^{w'} \text{Sen}^{\mathcal{I}}(\varphi_{\text{Sig}})(\rho) \\ \Leftrightarrow & \quad \{ \text{defn of } \text{Sen}^{\mathcal{H}\mathcal{I}}(\varphi) \} \\ & (M', W') \models^{w'} \text{Sen}^{\mathcal{H}\mathcal{I}}(\varphi)(\rho) \end{aligned}$$

3.  $\rho = \xi \vee \xi'$  for some  $\xi, \xi' \in \text{Sen}^{\mathcal{H}\mathcal{I}}(\Delta)$ :

$$\begin{aligned} & (M, W) \models^w \xi \vee \xi' \\ \Leftrightarrow & \quad \{ \text{defn. of } \models^w \} \\ & (M, W) \models^w \xi \text{ or } (M, W) \models^w \xi' \\ \Leftrightarrow & \quad \{ \text{I.H.} \} \end{aligned}$$

$$\begin{aligned}
& (M', W') \models^{w'} \text{Sen}^{\mathcal{H}, \mathcal{I}}(\varphi)(\xi) \text{ or} \\
& (M', W') \models^{w'} \text{Sen}^{\mathcal{H}, \mathcal{I}}(\varphi)(\xi') \\
\Leftrightarrow & \quad \{ \text{defn. of } \models^w \} \\
& (M', W') \models^{w'} \text{Sen}^{\mathcal{H}, \mathcal{I}}(\varphi)(\xi \vee \xi')
\end{aligned}$$

The proofs for cases  $\rho = \xi \wedge \xi'$ ,  $\rho = \xi \Rightarrow \xi'$ ,  $\rho = \neg\xi$ , etc. are analogous.

4.  $\rho = [\lambda](\xi_1, \dots, \xi_n)$  for some  $\xi_1, \dots, \xi_n \in \text{Sen}^{\mathcal{H}, \mathcal{I}}(\Delta)$ ,  $\lambda \in \Lambda_{n+1}$ :

$$\begin{aligned}
& (M, W) \models^w [\lambda](\xi_1, \dots, \xi_n) \\
\Leftrightarrow & \quad \{ \text{defn. of } \models^w \} \\
& \text{for any } (w, w_1, \dots, w_n) \in W_\lambda \text{ there is some } k \in \{1, \dots, n\} \\
& \text{such that } (M, W) \models^{w_k} \xi_k \\
\Leftrightarrow & \quad \{ * \} \\
& \text{for any } (w', w'_1, \dots, w'_n) \in W'_{\varphi_{\text{MS}}(\lambda)} \text{ there is some} \\
& p \in \{1, \dots, n\} \text{ such that } (M', W') \models^{w'_p} \text{Sen}^{\mathcal{H}, \mathcal{I}}(\varphi)(\xi_p) \\
\Leftrightarrow & \quad \{ \text{defn. of } \models^{w'} \} \\
& (M', W') \models^{w'} [\varphi_{\text{MS}}(\lambda)](\text{Sen}^{\mathcal{H}, \mathcal{I}}(\varphi)(\xi_1), \dots, \text{Sen}^{\mathcal{H}, \mathcal{I}}(\varphi)(\xi_n)) \\
\Leftrightarrow & \quad \{ \text{defn. of } \text{Sen}^{\mathcal{H}, \mathcal{I}}(\varphi) \} \\
& (M', W') \models^{w'} \text{Sen}^{\mathcal{H}, \mathcal{I}}(\varphi)([\lambda](\xi_1, \dots, \xi_n))
\end{aligned}$$

For the step marked with \* we proceed as follows. Supposing  $(w', w'_1, \dots, w'_n) \in W'_{\varphi_{\text{MS}}(\lambda)}$  with  $wB_{\varphi}w'$ , we have by clause (v) of Defn. 3.2 that there are  $w_k$ , with  $k \in \{1, \dots, n\}$ , such that  $(w, w_1, \dots, w_n) \in W_\lambda$ . By hypothesis,  $(M, W) \models^{w_p} \xi_p$  for some  $p \in \{1, \dots, n\}$ . Moreover, by I.H.  $(M', W') \models^{w'_p} \text{Sen}^{\mathcal{H}, \mathcal{I}}(\varphi)(\xi_p)$ . Clause (iv) of Defn. 3.2 entails the converse implication. The proof for sentences of form  $\rho = \langle \lambda \rangle(\xi_1, \dots, \xi_n)$  is analogous.

5.  $\rho = @_i \xi$  for some  $\xi \in \text{Sen}^{\mathcal{H}, \mathcal{I}}(\Delta)$  and  $i \in \text{Nom}$ :

$$\begin{aligned}
& (M, W) \models^w @_i \xi \\
\Leftrightarrow & \quad \{ \text{defn. of } \models^w \} \\
& (M, W) \models^{W_i} \xi \\
\Leftrightarrow & \quad \{ \text{I.H. and clause (iii) of Defn 3.2} \} \\
& (M', W') \models^{W'_{\varphi_{\text{Nom}}(i)}} \text{Sen}^{\mathcal{H}, \mathcal{I}}(\varphi)(\xi) \\
\Leftrightarrow & \quad \{ \text{defn. of } \models^w \} \\
& (M', W') \models^w @_{\varphi_{\text{Nom}}(i)} \text{Sen}^{\mathcal{H}, \mathcal{I}}(\varphi)(\xi) \\
\Leftrightarrow & \quad \{ \text{defn. of } \text{Sen}^{\mathcal{H}, \mathcal{I}}(\varphi) \} \\
& (M', W') \models^w \text{Sen}^{\mathcal{H}, \mathcal{I}}(\varphi)(@_i \xi)
\end{aligned}$$

□

As direct consequence of the previous theorem we get the following characterisation of the preservation of (global) satisfaction,  $\models^{\mathcal{H}\mathcal{I}}$ , under  $\varphi$ -bisimilarity:

**Corollary 3.2** *On the conditions of Theorem 3.1, let  $(M, W) \rightleftharpoons_{\varphi} (M', W')$  witnessed by a total and surjective bisimulation. Then,*

$$(M, W) \models^{\mathcal{H}\mathcal{I}} \rho \text{ iff } (M', W') \models^{\mathcal{H}\mathcal{I}} \text{Sen}^{\mathcal{H}\mathcal{I}}(\varphi)(\rho). \quad (5)$$

**Example 3.1 (Bisimulation in  $\mathcal{H}PL$ )** *Let us instantiate Defn. 3.2 for the  $\mathcal{H}PL$  case (cf. Ex. 2.1), considering  $\varphi = id$  and  $\text{Sen}' = \text{Sen}^{\mathcal{I}}$ . A bisimulation  $B$  is such that  $(M, W)B(M', W')$ , for any two models  $(M, W), (M', W') \in |\text{Mod}^{\mathcal{H}PL}(P, \text{Nom}, \{\lambda\})|$ , if*

- (i) *for any  $i \in \text{Nom}$ ,  $wBw'$ ,  $w = W_i$  iff  $w' = W'_i$ ;*
- (ii)  *$M_w \equiv M'_{w'}$ , i.e., bisimilar states satisfy the same sentences;*
- (iii) *for any  $i \in \text{Nom}$ ,  $W_iBW'_i$ ;*
- (iv) *for any  $(w, w_1) \in W_{\lambda}$  with  $wBw'$ , there is a  $w'_1 \in |W'|$  such that  $w_1Bw'_1$  and  $(w_1, w'_1) \in W'_{\lambda}$ ;*
- (v) *for any  $(w', w'_1) \in W'_{\lambda}$  with  $wBw'$ , there is a  $w_1 \in |W|$  such that  $w_1Bw'_1$  and  $(w_1, w'_1) \in W_{\lambda}$ ;*

*Note that condition (ii) is equivalent to say that bisimilar states have assigned the same set of propositions (for any  $p \in P$ ,  $M_w(p) = \top$  iff  $M'_{w'}(p) = \top$ ). As expected, this definition corresponds exactly to standard bisimulation for propositional hybrid logic (see, e.g. [Cat05, Defn 4.1.1]).*

The definition of bisimulation computed in the previous example, can also capture the case of propositional modal logic: just consider pure modal signatures (i.e., with an empty set of nominals), as condition (i) is trivially satisfied. Moreover, instantiating Theorem 3.1 we get the classical result about preservation of modal truth by bisimulation.

**Example 3.2 (Bisimulation for  $\mathcal{H}EQ$ )** *Consider now the instantiation of 3.2 for  $\mathcal{H}EQ$  (cf. Ex 2.6). All one has to do is to replace condition (iv) in Defn 3.2 by its instantiation for algebras: two algebras are elementarily equivalent if the respective generated varieties coincides [Grü79].*

## 4 Refinements for generic hybridised logics

Let us come back to the general case of a reconfigurable system described by a set of configurations and a transition structure entailing changes from one to another. If equivalence of specifications of such systems corresponds to a notion of bisimilarity in which bisimilar configurations are enforced to be elementary equivalent, a *refinement* relation corresponds to *similarity*. This entails, on the one hand, preservation (but not reflection) of transitions, i.e., of reconfiguration steps, from the abstract to the concrete system. And, on the other hand, at each local configuration, preservation of the original properties along local refinement. Formally,

**Definition 4.1** *Let  $\mathcal{H}\mathcal{I}$  be the hybridisation of an institution  $\mathcal{I}$ ,  $\varphi \in \text{Sign}^{\mathcal{H}\mathcal{I}}(\Delta, \Delta')$  a signature morphism and  $\text{Sen}'$  a subfunctor of  $\text{Sen}^{\mathcal{I}}$ . A  $\varphi, \text{Sen}'$ -refinement of  $(M, W) \in \text{Mod}^{\mathcal{H}\mathcal{I}}(\Delta)$  by  $(M', W') \in \text{Mod}^{\mathcal{H}\mathcal{I}}(\Delta')$  consists of a non-empty relation  $R_{\varphi}^{\text{Sen}'} \subseteq |W| \times |W'|$  such that, for any  $wR_{\varphi}^{\text{Sen}'}w'$ ,*

$$(f.i) \text{ for any } i \in \text{Nom}, \text{ if } W_i = w \text{ then } W'_{\varphi_{\text{Nom}}(i)} = w'.$$

$$(f.ii) M_w \gg_{\varphi}^{\text{Sen}'} M'_{w'}.$$

(f.iii) for any  $i \in \text{Nom}$ ,  $W_i \mathbf{R}_\varphi^{\text{Sen}'} W'_{\varphi_{\text{Nom}}(i)}$ .

(f.iv) For any  $\lambda \in \Lambda_n$ , if  $(w, w_1, \dots, w_n) \in W_\lambda$  then for each  $k \in \{1, \dots, n\}$  there is a  $w'_k \in |W'|$  such that  $w_k \mathbf{R}_\varphi w'_k$  and  $(w', w'_1, \dots, w'_n) \in W'_{\varphi_{\text{MS}}(\lambda)}$ .

The question is, now, to see whether (hybrid) satisfaction is, or is not, preserved by refinement. On a first attempt, it is natural to accept a positive answer which, although intuitive, is wrong. Actually, not all hybrid sentences can be preserved along a refinement chain. Note on the proof of Th 3.1, that the preservation of hybrid satisfaction of sentences  $[\lambda](\xi_1, \dots, \xi_n)$  is entailed by condition (ii) of Defn 3.2, but the latter is stated on the opposite direction to refinement. As a simple counter-example, define a  $\mathbf{R}_\varphi$ -refinement from a  $\Delta$ -hybrid model  $(M, W)$  with  $|W| = \{w\}$  and  $W_\lambda = \emptyset$  for  $\lambda \in \Lambda_n$  to any other  $\Delta'$ -hybrid model  $(M', W')$  such that  $\text{Mod}^{\mathcal{H}\mathcal{S}}(\varphi_{\text{Sig}})(M'_{w'}) = M_w$  for some  $w' \in |W'|$ . Sentence  $[\lambda](\xi_1, \dots, \xi_n)$ , which trivially holds in the world  $w$  of  $(M, W)$ , may fail to be satisfied in the  $\mathbf{R}_\varphi$ -related world  $w'$  of  $(M', W')$ . Sentences like  $\neg\xi$  provide another counter-example. The reason is that, by hypothesis, preservation is only assumed on the refinement direction and, of course, non satisfaction in one direction, does not implies non satisfaction in the other. Therefore, differently from the bisimulations case, the preservation of the satisfaction under refinement does not hold for all the hybrid sentences universe. Actually, the ‘boxed’ and negated sentences are exactly the cases where it may fail.

Finally, a note regarding parameter  $\text{Sen}'$  in condition (f.ii). First of all note that the “unrestricted” implication of clause (f.ii) in Defn 4.1 is very strong: it often implies the converse implication as well. For instance, in  $\mathcal{H}PL$ , the condition holds iff  $M_w = \text{Mod}(\varphi)(M'_{w'})$ . In particular, an *id*-refinement implies the equality of realizations of related worlds (since, the implication “ $M_w \models^{PL} \neg p$  then  $M'_{w'} \models^{PL} \neg p$ ” is equivalent to the implication “ $M'_{w'} \models^{PL} p$  then  $M_w \models^{PL} p$ ”. Hence,  $M_w = M'_{w'}$ ). It seems reasonable to weaken this condition to yield a strict inclusion. One way to do this is to restrict the focus to a subset of the sentences in the base institution. In the example mentioned above this will correspond to exclude *PL* negations, which amounts to take as  $\text{Sen}'(\text{Prop})$  the set of propositional sentences without negations.

Given an institution  $\mathcal{S} = (\text{Sign}^{\mathcal{S}}, \text{Sen}^{\mathcal{S}}, \text{Mod}^{\mathcal{S}}, (\models_\Sigma)_{\Sigma \in |\text{Sign}^{\mathcal{S}}|})$  and a sentences subfunctor  $\text{Sen}' \subseteq \text{Sen}^{\mathcal{S}}$ , we denote by  $\mathcal{H}\mathcal{S}'$  the hybridisation of the institution  $\mathcal{S}' = (\text{Sign}^{\mathcal{S}}, \text{Sen}', \text{Mod}^{\mathcal{S}}, (\models_\Sigma)_{\Sigma \in |\text{Sign}^{\mathcal{S}}|})$ .

**Definition 4.2 (Sen'-Positive Existential sentences)** *The  $\text{Sen}'$ -positive existential sentences of a signature  $\Delta \in |\text{Sign}^{\mathcal{H}\mathcal{S}'}|$  are given by a subfunctor  $\text{Sen}'_+ \subseteq \text{Sen}^{\mathcal{H}\mathcal{S}'}$  defined inductively for each signature  $\Delta$  as  $\text{Sen}^{\mathcal{H}\mathcal{S}'}$  but excluding both negations and box modalities. For each signature morphism  $\varphi : \Delta \rightarrow \Delta'$ ,  $\text{Sen}'_+(\varphi)$  is the restriction of  $\text{Sen}^{\mathcal{H}\mathcal{S}'}$  to  $\text{Sen}'_+(\Delta)$ .*

**Theorem 4.1** *Let  $\mathcal{H}\mathcal{S}$  be the quantifier free hybridisation of an institution  $\mathcal{S}$ ,  $\text{Sen}'$  a subfunctor of  $\text{Sen}^{\mathcal{S}}$ ,  $\varphi \in \text{Sign}^{\mathcal{H}\mathcal{S}}(\Delta, \Delta')$  a signature morphism,  $\mathbf{R}_\varphi^{\text{Sen}'}$  a  $\varphi, \text{Sen}'$ -refinement relation and  $(M, W) \in \text{Mod}^{\mathcal{H}\mathcal{S}}(\Delta)$  and  $(M', W') \in \text{Mod}^{\mathcal{H}\mathcal{S}}(\Delta')$  two models such that  $(M', W')$  is a refinement of  $(M, W)$  witnessed by relation  $\mathbf{R}_\varphi^{\text{Sen}'}$ . Then, for any  $w \mathbf{R}_\varphi^{\text{Sen}'} w'$  and  $\rho \in \text{Sen}'_+(\Delta)$ ,*

$$(M, W) \models^w \rho \text{ implies that } (M', W') \models^{w'} \text{Sen}'_+(\varphi)(\rho).$$

**Proof.** The proof is by induction on the structure of the existential positive sentences and comes directly from the proof of Th 3.1, taking the right to left implication. Preservation of base sentences follows exactly the same proof since the IH is precisely about the  $\text{Sen}'$  sentences. What remains to be proved is the case  $\rho = \langle \lambda \rangle(\xi_1, \dots, \xi_n)$ . Thus,

$$\begin{aligned} (M, W) \models^w \langle \lambda \rangle(\xi_1, \dots, \xi_n) \\ \Leftrightarrow \quad \{ \text{defn. of } \models^w \} \end{aligned}$$

there exists  $(w, w_1, \dots, w_n) \in W_\lambda$   
 such that  $(M, W) \models^{w_k} \xi_k$  for any  $k \in \{1, \dots, n\}$   
 $\Rightarrow$  { By (f.iii), we have  $w_k R_\varphi w'_k$  for any  $k \in \{1, \dots, n\}$  + I.H. }  
 there exists  $(w', w'_1, \dots, w'_n) \in W'_{\varphi_{MS}(\lambda)}$   
 such that  $(M', W') \models^{w'_k} \xi_k$  for any  $k \in \{1, \dots, n\}$   
 $\Leftrightarrow$  { defn. of  $\models^{w'}$  }  
 $(M', W') \models^{w'} \langle \varphi_{MS}(\lambda) \rangle (\text{Sen}^{\mathcal{H}\mathcal{I}}(\varphi)(\xi_1), \dots, \text{Sen}^{\mathcal{H}\mathcal{I}}(\varphi)(\xi_n))$   
 $\Leftrightarrow$  { defn. of  $\text{Sen}^{\mathcal{H}\mathcal{I}}(\varphi)$  }  
 $(M', W') \models^{w'} \text{Sen}^{\mathcal{H}\mathcal{I}}(\varphi)(\langle \lambda \rangle (\xi_1, \dots, \xi_n))$

□

**Corollary 4.1** *In the conditions of Th 4.1, for any  $\rho \in \text{Sen}_+^{\mathcal{H}\mathcal{I}}(\Delta)$ , if  $R_\varphi$  is surjective, then*

$$(M, W) \models \rho \text{ implies that } (M', W') \models \text{Sen}^{\mathcal{H}\mathcal{I}}(\varphi)(\rho).$$

The following examples illustrate refinement situations in this setting.

**Example 4.1 (Refinement in  $\mathcal{H}MVL_L$ )** *Figure 4.1 illustrates an example of a  $\text{Sen}'$ -refinement in  $\mathcal{H}MVL_{L_4}$ , for  $L_4$  represented in Figure 4.1. Consider  $\text{Sen}' \subseteq \text{Sen}^{\mathcal{I}}$  restricting the base sentences to propositions, i.e.,  $\text{Sen}'(LProp) = \{(p, l) \mid p \in LProp \text{ and } l \in L_4\}$ . Conditions (f.i) and (f.iii) are obviously satisfied. In*

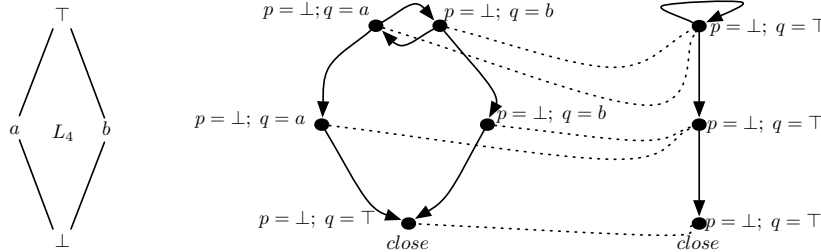
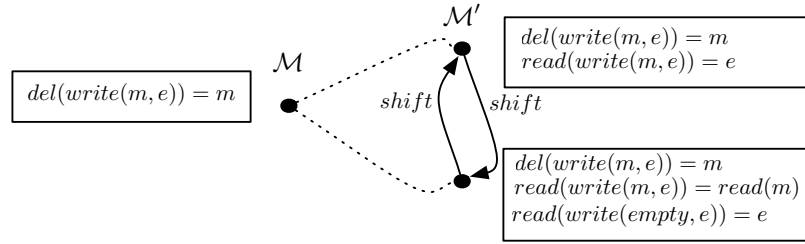


Figure 1: Refinement in  $\mathcal{H}MVL_L$

what concerns the verification of condition (f.ii) for which  $(p, l) \in \text{Sen}'(LProp)$ ,  $M_w \models_{LProp}^{MVL_{L_4}} (p, l) \Rightarrow M'_{w'} \models_{LProp}^{MVL_{L_4}} (p, l)$ , it is sufficient to be that,  $(M_w \models p) \leq (M'_{w'} \models p)$ ,  $p \in LProp$ .

**Example 4.2 (Refinement in  $\mathcal{H}EQ$ )** *Consider a store system abstractly modelled as the initial algebra  $A$  of the  $((S, F), \Gamma)$  where  $S = \{\text{mem}, \text{elem}\}$ ,  $F_{\text{mem} \times \text{elem} \rightarrow \text{mem}} = \{\text{write}\}$ ,  $F_{\text{mem} \rightarrow \text{mem}} = \{\text{del}\}$  and  $F_{\text{ar} \rightarrow s} = \emptyset$  otherwise and  $\Gamma = \{\text{del}(\text{write}(m, e)) = m\}$ . Suppose one intends to refine this structure into a read function configurable in two different modes: in one of them it reads the first element in the store, in the other the last. Reconfiguration between the two execution modes is enforced by an external event shift. Note that the abstract model can be seen as the  $((S, F), \emptyset, \{\text{shift}\})$ -hybrid model  $\mathcal{M} = (M, W)$ , taking  $|W| = \{\star\}$ ,  $W_{\text{shift}} = \emptyset$  and  $M_\star = A$ . Then, we take the inclusion morphism  $\varphi_{\text{Sig}} : (S, F) \hookrightarrow (S, F')$  where  $F'$*

Figure 2: Refinement in  $\mathcal{H}EQ$ 

extends  $F$  with  $F_{mem \rightarrow elem} = read$  and  $F_{mem} = \{empty\}$ . For the envisaged refinement let us consider the model  $\mathcal{M}' = (M', W')$  where  $W' = \{s_1, s_2\}$  and  $W'_{shift} = \{(s_1, s_2), (s_2, s_1)\}$  and where  $M_{s_1}$  and  $M_{s_2}$  are the initial algebras of the equations presented in Figure 4.2. It is not difficult to see that  $R = \{(\star, s_1), (\star, s_2)\}$  is a  $\varphi$ -refinement relation: conditions (f.i) and (f.iii) are trivially fulfilled and, condition (f.ii) is a direct consequence of properties representability of the initial models.

## 5 Conclusions

The paper introduced notions of equivalence and refinement between models of hybridised logic specifications, i.e. specifications formalised in hybridised versions of base logics used to describe a systems' possible configurations. The definition is parametric on precisely the base logic relevant for each application. Current work on this topic includes research on a full equivalence theorem, showing, in particular, in which cases  $\mathcal{H}\mathcal{I}$  logical equivalence entails bisimilarity. Another topic concerns the study of typical constructions on Kripke structures (e.g. bounded morphism images, substructures and disjoint unions) and their characterisation under bisimilarity and refinement.

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