Pascal's triangle and other number triangles in Clifford Analysis

G. Tomaz[*,†], M. I. Falcão[*,**] and H. R. Malonek[*,‡]

[*]*Center for Research and Development in Mathematics and Applications, University of Aveiro, Portugal*
[†]*Instituto Politécnico da Guarda, Portugal*
[**]*Department of Mathematics and Applications, University of Minho, Portugal*
[‡]*Department of Mathematics, University of Aveiro, Portugal*

Abstract. The recent introduction of generalized Appell sequences in the framework of Clifford Analysis solved an open question about a suitable construction of power-like monogenic polynomials as generalizations of the integer powers of a complex variable. The deep connection between Appell sequences and Pascal's triangle called also attention to other number triangles and, at the same time, to the construction of generalized Pascal matrices. Both aspects are considered in this communication.

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SOME HISTORICAL REMARKS AS INTRODUCTION

In his famous *Projet et Essais pour arriver à quelque certitude pour finir une bonne partie des disputes et pour avancer l'art d' inventer (1686)*, G.W. Leibniz pointed out that

> ....I found out that through numbers one can represent all types of true facts and their consequences....

One could speculate, if his opinion was influenced by B. Pascal's *Traité du triangle arithmétique, avec quelques autres petits traitez sur la mesme matière*, published about twenty years before 1665 in Paris, or not. But the admiration of Jacob Bernoulli for Pascal's triangle is doubtless, since he wrote in his important work *Ars Conjectandi* (posthumely published in 1713), which is widely regarded as the founding work of probability theory:

> This Table has clearly admirable and extraordinary properties, far beyond what I have already shown of the mystery of combinations hiding within it, it is known to those skilled in the more hidden parts of geometry that the most important secrets of all the rest of mathematics lie concealed within it.

If we continue to follow the traces of numbers in history, it is plain to see that a big part of the work of W.R. Hamilton and H. G. Grassmann in retrospect seems to be guided by the spirit of Leibniz, since it was *searching for a new understanding about the nature of numbers and, consequently, the creation of new concepts for arithmetical goals in Physics and Geometry by algebraic approaches.*

Today Clifford Analysis is the result of the work of several generations of mathematicians who were "standing on the shoulders of those giants" like Hamilton and Grassmann. Not surprising, that among the subjects treated so far in the framework of Clifford Analysis, also Pascal's triangle occurs confirming the words of Leibniz and Bernoulli. In this contribution we try to show, that starting from a subject which had its origin in the analysis of one real or complex variable, we arrive to properties of polynomials with values in non-commutative algebras, that are expressed in terms of binomial coefficients, but in a way that reflects very well the non-commutativity behind.

Our starting point are Appell sequences of polynomials in which the elements of Pascal's triangle naturally occur. In 1880 Appell introduced in [1] sequences of polynomials of one real variable $\{p_k(x)\}_{k\geq 0}$ having the property that p_k has exactly degree k and its derivative is a k-multiple of p_{k-1}, $k=1,\ldots$. From the Appell-property, we get that all polynomials of an Appell sequence have a form which involves the binomial coefficients, namely

$$p_k(x) = \sum_{s=0}^{k} \binom{k}{s} c_{k-s} x^s. \tag{1}$$

This fact that the general expression of the summand is equal to $\binom{k}{s}c_{k-s}x^s$ suggests to speak about an *exponential expansion* of $p_k(x)$ or that $p_k(x)$ obeys a binomial-type theorem (cf. [2]). Another elegant expression of the sequence $\{p_k(x)\}_{k\geq 0}$ is in matrix form, where Pascal's triangle occurs as a lower triangular matrix. More precisely, consider the vectors $p(x) = [p_0(x)\ \ p_1(x)\ \cdots\ p_m(x)]^T$ and $\xi(x) = [1\ \ x\ \cdots\ x^m]^T$. According to (1) we have $p(x) = M\xi(x)$, where M is the non-singular lower triangular matrix

$$M = [M_{ij}], \quad \text{where} \quad M_{ij} = \binom{i}{j}c_{i-j}, \quad 0 \leq j \leq i \leq m, \tag{2}$$

which can be easily written as the product of a lower triangular Pascal matrix $P = [P_{ij}]$ by a lower triangular coefficient matrix $C = [C_{ij}]$, where $P_{ij} = \binom{i}{j}$ and $C_{ij} = c_{i-j}$, $0 \leq j \leq i \leq m$. It is obvious that the constant terms of $p_k(x)$, $k = 0, 1, \ldots, m$ are immediately obtained from $p(0) = PC\xi(0) = [c_0\ \ c_1\ \cdots\ c_m]^T$.

A highly appraised paper on the Pascal matrix and its relatives is the article [3], where the authors highlighted the relation of the Pascal matrix with other special matrices to obtain matrix representations for Bernoulli and Bernstein polynomials, for example. But, in fact, the Pascal matrix is not the most important basic matrix that allows to develop a general method for studying Appell polynomials from a matrix point of view. The Pascal matrix occurs as constructive tool only in some of the classical Appell sequences. Nevertheless, the properties of the Appell polynomials allow to obtain a structure based on another matrix, the infinite creation matrix (see [3]) $H_\infty = [H_{ij}]$, a subdiagonal matrix whose nonzero entries are the integers

$$H_{j+1,j} = j+1,\ j = 0, 1, \ldots.$$

The simplicity of the structure of H is obvious. In addition, its restriction to the order $m+1$ leads to a nilpotent matrix of degree $m+1$. The Pascal matrix occurs in several situations, since $e^H = P$ (see [3]). Moreover, a similar relation is true in some generalization of the Pascal matrices which contain also integer powers of x. Such generalized Pascal matrix has the form $P(x) = e^{xH}$. Notice that for $x = 0$, $x = 1$ and $x = -1$, the generalized Pascal matrix reduces to the identity matrix, P and P^{-1}, respectively.

In fact, from Theorem 1 mentioned in the last section follows that all matrix representations of different kind of Appell polynomials are based on the same H (and in the hypercomplex case on its suitably defined hypercomplex counterpart, which we present also in the last section). This implies an unifying character through creation matrices of the very briefly presented approach. Moreover, the creation matrix H is also a useful tool to obtain the matrix M which transforms the vector $\xi(x)$ into the vector $p(x)$ of the Appell polynomials.

Finally, we notice that Appell sequences of one variable can also be obtained by means of a generating function $F(t,x)$ which is obtained by multiplication of two power series

$$F(t,X) = f(t)\,e^{xt} = \sum_{n=0}^{+\infty} p_n(x)\,\frac{t^n}{n!}, \text{ where } f(t) = \sum_{n=0}^{+\infty} c_n\,\frac{t^n}{n!}, \text{ and } e^{xt} = \sum_{n=0}^{+\infty} x^n\,\frac{t^n}{n!}, \text{ with } f(0) \neq 0. \tag{3}$$

Thus, if $f(t)$ is any convergent power series on the whole real line with the Taylor expansion given as in (3), the Appell polynomials $\{p_k(x)\}_{k\geq 0}$ are generated by the exponential generating function $F(t,x)$. By appropriately choosing the function $f(t)$, many - but not all - of the classical Appell polynomials can be derived. In particular, the expression of functions $f(t)$ for Bernoulli and Euler polynomials can be found in [4]. [1]

BASIC NOTATIONS USED IN CLIFFORD ANALYSIS

Let $\{e_1, e_2, \cdots, e_n\}$ be an orthonormal base of the Euclidean vector space $\mathbb{R}^n$ with a product according to the multiplication rules $e_k e_l + e_l e_k = -2\delta_{kl},\ k,l = 1, \cdots, n$, where δ_{kl} is the Kronecker symbol. This non-commutative product generates the 2^n-dimensional Clifford algebra $C\ell_{0,n}$ over $\mathbb{R}$. Considering the subset $\mathscr{A} := \text{span}_{\mathbb{R}}\{1, e_1, \ldots, e_n\}$ of $C\ell_{0,n}$, the real vector space $\mathbb{R}^{n+1}$ can be embedded in $\mathscr{A}$ by the identification of each element $(x_0, x_1, \cdots, x_n) \in \mathbb{R}^{n+1}$ with the *paravector* $x \in \mathscr{A}$. Here, x_0 and $\underline{x} = e_1x_1 + \cdots + e_nx_n$ are, the so-called, scalar and vector parts of the paravector $x \in \mathscr{A}_n$. The conjugate of x is $\bar{x} = x_0 - \underline{x}$ and the norm $|x|$ of x is defined by $|x| = \sqrt{x\bar{x}}$. In what follows we consider $C\ell_{0,n}$-valued functions defined in some open subset $\Omega \subset \mathbb{R}^{n+1}$, i.e., functions of the form

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[1] Generalized Bernoulli, Euler and Laguerre polynomials in the context of Clifford Analysis obtained by suitably defined hypercomplex generating functions can be found in [5], [6] and [7].

$f(z)=\sum_A f_A(z)e_A$, where $f_A(z)$ are real valued. We suppose that f is hypercomplex differentiable in Ω in the sense of [8] and [9], i.e. has a uniquely defined areolar derivative f' in each point of Ω. Then f is real differentiable (even real analytic) and f' can be expressed in terms of the partial derivatives with respect to x_k as $f'=\frac{1}{2}(\partial_0-\partial_{\underline{x}})f$, where $\partial_0:=\frac{\partial}{\partial x_0}$, $\quad \partial_{\underline{x}}:=e_1\frac{\partial}{\partial x_1}+\cdots+e_n\frac{\partial}{\partial x_n}$. If now $\overline{\partial}:=\frac{1}{2}(\partial_0+\partial_{\underline{x}})$ is the usual generalized Cauchy-Riemann differential operator, then, obviously $f'=\partial f$. Since in [8] it has been shown that a hypercomplex differentiable function belongs to the kernel of $\overline{\partial}$, i.e. satisfies the property $\overline{\partial}f=0$ (f is a *monogenic function* in the sense of Clifford Analysis), then it follows that in fact $f'=\partial_0 f$ like in the complex case.

APPELL POLYNOMIALS AND NUMBER TRIANGLES IN CLIFFORD ANALYSIS

The first atempt to generalize the Appell-property to higher dimensional cases, was made in [10], through the definition of the so-called *standard homogeneous Appell polynomials*,

$$\mathscr{P}_k^n(x)=\sum_{s=0}^{k}T_s^k(n)\,x^{k-s}\bar{x}^s,\ x\in\mathbb{R}^{n+1},\qquad \text{where}\qquad T_s^k(n)=\binom{k}{s}\frac{(\frac{n+1}{2})_{(k-s)}(\frac{n-1}{2})_{(s)}}{n_{(k)}}. \tag{4}$$

The elements of the Pascal's triangle are present in the above expression but, caused by the non-commutativity of the Clifford Algebra, the corresponding number triangle formed by the rational coefficients $T_s^k(n)$ is not symmetric. Several properties of this number triangle can be found in [11]. It is not difficult to obtain another closely related triangle, namely the triangle with the elements

$$I_s^k(n):=\alpha_k(n)T_s^k(n),\ k=0,1,\ldots,\ s=0,1,\ldots,k,\qquad \text{where}\qquad \alpha_k(n):=2^k\prod_{i=0}^{[\frac{k-1}{2}]}(n+2i).$$

Among several interesting properties, we can prove that all elements $I_s^k(n)$, $n\geq 1$, are non-negative integers. This fact is illustrated below, where we present the first five lines of the triangle.

$$\begin{array}{lllll}
\mathbf{1} & & & & \\
\mathbf{1}\,(n+1) & \mathbf{1}\,(n-1) & & & \\
\mathbf{1}\,(n+3) & \mathbf{2}\,(n-1) & \mathbf{1}\,(n-1) & & \\
\mathbf{1}\,(n+3)(n+5) & \mathbf{3}\,(n-1)(n+3) & \mathbf{3}\,(n-1)(n+1) & \mathbf{1}\,(n-1)(n+3) & \\
\mathbf{1}\,(n+5)(n+7) & \mathbf{4}\,(n-1)(n+5) & \mathbf{6}\,(n-1)(n+1) & \mathbf{4}\,(n-1)(n+1) & \ldots
\end{array}$$

More details about different representations of Appell sequences in the hypercomplex context and contributions of other authors to this subject can be found in [12, 13, 14, 15, 16].

A BLOCK VERSION OF THE CREATION MATRIX H

The definition of a block version of the creation matrix H, mentioned in the Introduction, is the crucial idea for obtaining new matrix representations of polynomials in several real or complex variables as well as in several hypercomplex variables. The bridge to Appell sequences is then based on the block version of a theorem, which we here for brief mention only in the one variable version.

Theorem 1 *Let H be the creation matrix of order $m+1$. If $G(x,t)=f(t)e^{xt}$ is the generating function of an Appell sequence $\{p_n(x)\}_{n\geq 0}$, then $f(H)=M$.*

Here the matrix M is the *transfer matrix*, mentioned in (2).

The block version $\mathbb{H}$ that we now present allows the generalization of the Pascal matrix and is in the core of an unified approach to matrix representations of hypercomplex Appell sequences (see [7])

Definition 1 *The matrix* $\mathbb{H} = [\mathbb{H}_{sr}]$, *where*

$$\mathbb{H}_{sr} = \begin{cases} H & , s = r \\ sI & , s = r+1 \\ O & , \textit{otherwise}, \quad (s,r = 0,\ldots,n), \end{cases}$$

I and O are, respectively, the identity matrix and the null matrix of order $n+1$, *is called block creation matrix of order* $(n+1)(n+1)$.

This matrix is such that $\mathbb{H}^k = \mathscr{O}$, $k > 2n$ ($\mathscr{O}$ is the null block matrix of order $(n+1)(n+1)$). We convention that $\mathbb{H}^0 = \mathscr{I}$, where $\mathscr{I} = [E_0\ E_1\ \cdots\ E_n]$ and $E_s = [O\ \ \cdots\ I\ \cdots\ O]^T$ is a block vector of dimension $(n+1)(n+1) \times (n+1)$, with I at s^{th} row-block, and $E_s = O$ whenever $s > n$. Such block vectors satisfy the orthogonality property $E_s^T E_r = \delta_{sr} I$. The matrix $\mathbb{H}$ has properties similar to those of H, namely

$$\begin{aligned} &(E_s)^T \mathbb{H} E_r = \mathbb{H}_{sr} \\ &\mathbb{H} E_r = (r+1)E_{r+1} + E_r H \\ &\mathbb{H}^k E_r = \sum_{\alpha=0}^{k} \binom{k}{\alpha} \frac{(r+k-\alpha)!}{r!} E_{r+k-\alpha} H^{\alpha}. \end{aligned}$$

These properties allow to prove that $e^{\mathbb{H}} = \mathscr{P}$ is the block matrix of order $(n+1)(n+1)$ defined by

$$\mathscr{P}_{sr} = \begin{cases} \binom{s}{r} P & , s \geq r \\ O & , \text{otherwise}, \ (s,r = 0,\ldots,n), \end{cases} \tag{5}$$

where P is the ordinary Pascal matrix of order $n+1$.

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REFERENCES

1. P. Appell, *Ann. Sci. École Norm. Sup.* **9**, 119–144 (1880).
2. B. C. Carlson: *J. Math. Anal. Appl.*, **3**2 543–558 (1970).
3. L. Aceto, and D. Trigiante, *Amer. Math. Monthly* **108** 232–245 (2001).
4. E. D. Rainville, *Special Functions, Chelsea Publishing Company, New York,*, 1960.
5. H. R. Malonek, and G. Tomaz, *Discrete Appl. Math.* **157**, 838–847 (2009).
6. H. R. Malonek, and G. Tomaz, *Int. J. Pure Appl. Math.* **44** (3), 447–465 (2008).
7. H. R. Malonek, and G. Tomaz, *Computational Science and Its Applications-ICCSA 2011 (LNCS 6784, Part III) Springer-Verlag, Berlin,* 261–270 (2011).
8. K. Gürlebeck, and H. Malonek, *Complex Variables Theory Appl.* **39**, 199–228 (1999).
9. H. Malonek, *Complex Variables Theory Appl.* **14**, 25–33 (1990).
10. M. I. Falcão, and H. Malonek, *Generalized exponentials through Appell sets in* $\mathbb{R}^{n+1}$ *and Bessel functions,* in *AIP Conference Proceedings*, edited by T. E. Simos, G. Psihoyios, and C. Tsitouras, 2007, vol. 936, pp. 738–741.
11. M.I. Falcão and H.R. Malonek. *Opuscula Mathematica*, accepted for publication, 2012.
12. I. Cação, and H. Malonek, *On Complete Sets of Hypercomplex Appell Polynomials,* in *AIP Conference Proceedings*, edited by T. E. Simos, G. Psihoyios, and C. Tsitouras, 2008, vol. 1048, pp. 647–650.
13. S. Bock, and K. Gürlebeck, *Math. Methods Appl. Sci.* **33**, 394–411 (2010).
14. I. Cação, M. I. Falcão, and H. Malonek, *Math. Comput. Model.* **53**, 1084–1094 (2011).
15. I. Cação, M. I. Falcão, and H. Malonek, *Comput. Methods Funct. Theory* **12**, 371–391 (2012).
16. H. Malonek, and M. I. Falcão, *Advances in Applied Clifford Algebras*, (2012) doi:10.1007/s00006-012-0361-5.