On the non-linear stability of scalar field cosmologies

Artur Alho[♭], **Filipe C. Mena**[♭] **and Juan A. Valiente Kroon**[†]

[♭]Centro de Matemática, Universidade do Minho, 4710-057 Braga, Portugal

[†]School of Mathematical Sciences, Queen Mary, University of London, London E1 4NS, U.K.

E-mail: aalho@math.uminho.pt, fmena@math.uminho.pt, jav@maths.qmul.ac.uk

Abstract. We review recent work on the stability of flat spatially homogeneous and isotropic backgrounds with a self-interacting scalar field. First, we derive a first order quasi-linear symmetric hyperbolic system for the Einstein-nonlinear-scalar field system. Then, using the linearized system, we show how to obtain necessary and sufficient conditions which ensure the exponential decay to zero of small non-linear perturbations.

1. Introduction

The asymptotic stability of spacetime cosmologies is an important problem of mathematical relativity. This problem can be addressed by considering small perturbations of a given background solution and investigating whether they asymptotically decay in time. Most of the approaches to this question have been limited to the use of linear or higher-order truncated perturbation theory, and thus, they never take fully into account the non-linearity of the Einstein-Field-Equations (EFEs). In [2], Friedrich extended his frame representation of the vacuum EFEs to the case of matter sources consisting of perfect fluids. These systems form a *first-order quasi-linear symmetric hyperbolic (FOSH) system*. In general these are of the form

$$\mathbf{A}^0(\mathbf{u})\partial_t\mathbf{u} = \mathbf{A}^j(\mathbf{u})\partial_j\mathbf{u} + \mathbf{B}(\mathbf{u})\mathbf{u}, \qquad j = 1,2,3, \tag{1}$$

where $\mathbf{u} = \mathbf{u}(\boldsymbol{x},t)$ is a smooth vector-valued function of dimension s with domain in $\Sigma \times [0,T]$ where Σ is a spacelike 3-dimensional manifold, and $\mathbf{A}^j$, $\mathbf{B}$ denote smooth $s\times s$ matrix valued-functions such that $\mathbf{A}^0$, $\mathbf{A}^j$ are symmetric and $\mathbf{A}^0$ is positive definite. The operators ∂_t and ∂_j stand for the partial derivatives with respect to the coordinate $t \in [0,T]$, and to the spatial coordinates x^j in Σ respectively. A natural way of performing a stability analysis (see also [3]) is to consider a sequence of smooth initial data $\mathbf{u}_0^\varepsilon$ for the EFEs satisfying the constraints equations on a Cauchy hypersurface Σ. The sequence is assumed to depend continously on the parameter ε in such a way that the limit $\varepsilon \to 0$ renders the data of the reference solution $\mathring{\mathbf{u}}_0$. In particular, one can write the full solution to the EFEs as the Ansatz $\mathbf{u}^\varepsilon = \mathring{\mathbf{u}} + \varepsilon\breve{\mathbf{u}}$, where $\breve{\mathbf{u}}$ is a (non-linear) perturbation whose size is controlled by the parameter ε. Using this splitting in equation (1), we are led to consider the following initial value problem for the non-linear perturbations:

$$\begin{aligned}&\left[\mathring{\mathbf{A}}^0(\mathring{\mathbf{u}}) + \varepsilon\breve{\mathbf{A}}^0(\breve{\mathbf{u}},\varepsilon)\right]\partial_t\breve{\mathbf{u}} = \left[\mathring{\mathbf{A}}^j(\mathring{\mathbf{u}}) + \varepsilon\breve{\mathbf{A}}^j(\breve{\mathbf{u}},\varepsilon)\right]\partial_j\breve{\mathbf{u}} + \left[\mathring{\mathbf{B}}(\mathring{\mathbf{u}}) + \varepsilon\breve{\mathbf{B}}(\breve{\mathbf{u}},\varepsilon)\right]\breve{\mathbf{u}},\\ &\breve{\mathbf{u}}(\boldsymbol{x},0) = \breve{\mathbf{u}}_0(\boldsymbol{x}).\end{aligned} \tag{2}$$

where $\mathring{\mathbf{A}}^\mu$ ($\mu = 0,1,2,3$) and $\mathring{\mathbf{B}}$ are defined as the non vanishing terms with $\epsilon = 0$. When the linearised system ($\epsilon = 0$) of (2) has constant coefficients, then for small enough ϵ, global existence in time t, and exponential decay to zero of a solution to (2) is well known – see [4]. Roughly this follows from the eigenvalues of the non-principal part of the linearised system ($\mathring{\mathbf{B}}$) having negative real part. To systems of the type considered here, where the matrices $\mathring{\mathbf{B}}$, $\mathring{\mathbf{A}}^\mu$ are not constant but depend smoothly on time, these methods can be easily generalized — see also [3].

2. The Einstein-Friedrich-nonlinear scalar field system

Using the Friedrich formulation of EFEs for the Einstein-Euler system [2], we obtained the following symmetric hyperbolic reduction for the Einstein-non-linear scalar field system – see [1] for details,

$$
\begin{aligned}
\partial_t\phi &= \psi,\\
\partial_t\psi &= -\psi\chi - \frac{d\mathcal{V}}{d\phi},\\
2\partial_t\chi_{(bd)} - 2\,e'^{\,0}_{(b}\partial_t a_{d)} - 2\,e'^{\,j}_{(b}\partial_j a_{d)} &= \frac{2}{3}\left(\mathcal{V}(\phi)-\psi^2\right)h_{bd} - 2\chi^p{}_{(d}\chi_{b)p} + 2a_b a_d + 2E_{bd}\\
&\quad - \left(\gamma^p{}_{bd}+\gamma^p{}_{db}\right)a_p\\
\partial_t a_c - e'^{\,0}_p\partial_t\chi_c{}^p - e'^{\,j}_p\partial_j\chi_c{}^p &= \left(\frac{2}{\psi}\frac{d\mathcal{V}}{d\phi}+\chi\right)a_c - \chi_c{}^p a_p - \gamma^q{}_{cp}\chi_q{}^p + \gamma^p{}_{qp}\chi_c{}^q,\\
2\partial_t E_{bd} - 2\epsilon^{pa}{}_{(b|}e'^{\,0}_a\partial_t B_{p|d)} - 2\epsilon^{pa}{}_{(b|}e'^{\,j}_a\partial_j B_{p|d)} &= -\psi^2\left(\chi_{(bd)}-\frac{1}{3}\chi h_{bd}\right) - 4\chi E_{bd} + 10\chi^q{}_{(b}E_{d)q}\\
&\quad - 2h_{bd}\chi^{qp}E_{qp} + 4a_a B_{p(b}\epsilon_{d)}{}^{pa} - 2\gamma^q{}_{pa}B_{q(d}\epsilon_{b)}{}^{pa}\\
&\quad - 2\epsilon^{pa}{}_{(d}\gamma^q{}_{b)a}B_{pq},\\
2\partial_t B_{bd} - 2\epsilon^{ap}{}_{(d|}e'^{\,0}_a\partial_t E_{|b)p} - 2\epsilon^{ap}{}_{(d}e'^{\,j}_{|a}\partial_{j|}E_{b)p} &= -2\chi B_{bd} + 6\chi^q{}_{(b}B_{d)q} + 2\chi_{ac}B_{pq}\epsilon^{pa}{}_{(b}\epsilon_{d)}{}^{qc}\\
&\quad - 4a_a E_{p(b}\epsilon_{d)}{}^{pa} - 2\gamma^q{}_{pa}E_{q(b}\epsilon_{d)}{}^{ap}\\
&\quad - 2\epsilon^{ap}{}_{(b}\gamma^q{}_{d)a}E_{pq},\\
\partial_t e'^{\,0}_b &= -\chi_b{}^c e'^{\,0}_c + a_b,\\
\partial_t e'^{\,i}_b &= -\chi_b{}^c e'^{\,i}_c,\\
\partial_t\gamma'^a{}_{bd} &= B_{dp}\epsilon^{pa}{}_b - \chi_d{}^p\gamma'^a{}_{bp} + 2h^{ap}\chi_{d[p}a_{b]}.
\end{aligned}
\tag{3}
$$

and the following result:

Theorem 1. *The Einstein-Friedrich-nonlinear scalar field (EFsf) system consisting of the equations in* (3) *forms a quasi-linear first-order symmetric hyperbolic (FOSH) system for the scalar field (ϕ), its momentum-density (ψ), the spatial frame coefficients ($e'^{\,i}_b, e'^{\,0}_b$), the connection coefficients ($\chi_{(ab)}, a_c, \gamma'^a{}_{bd}$), and the electric and magnetic parts of the Weyl tensor ($E_{(bd)}, B_{(bd)}$), relatively to the slices of constant time t, as long as the quadratic form*

$$\sum_{a=1,2,3}\theta^a{}_i\theta^a{}_j - \frac{\partial_i\phi}{\psi}\frac{\partial_j\phi}{\psi}, \qquad i,j=1,2,3,$$

with $\theta^a{}_i$ the spatial co-frame coefficients, is positive definite.

3. Stability Analysis

In this section we use the symmetric hyperbolic system of last section to show that, for some classes of potentials, the evolution of sufficiently small nonlinear perturbations of a Friedmann-

Robertson-Walker background with a scalar field with positive potential, asymptotic exponential decay to zero in time.

3.1. The background solution

As it is well known, the metric of a Friedman-Robertson-Walker (FRW) spacetime —i.e. a spatially homogeneous and isotropic spacetime— can be written as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \left(\frac{a(t)}{\omega}\right)^2 \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j,$$

where $a(t)$ is the Robertson-Walker scale factor, $\omega = 1 + \frac{k}{4}\delta_{ij}x^i x^j$, $\partial_i\omega = kx_i$, and $k = -1, 0, 1$ is the curvature of the spatial hypersurfaces. Since the metric is conformal flat, it follows that $\mathring{E}_{bd} = \mathring{B}_{bd} = 0$. The gauge conditions[1] for the frame are satisfied if $\mathring{e}_0{}^\mu = \delta_0{}^\mu$, $\mathring{e}_b{}^\mu = \left(\frac{\omega}{a}\right)\delta_b{}^\mu$ where $b = 1, 2, 3$. Thus, the spatial connection coefficients are given by $\mathring{\gamma}^c{}_{bd} = \frac{k}{2a^2}\left(h_{db}x^c - h_d{}^c x_b\right)$, with $x^\mu = (\omega/a)\delta^\mu{}_c x^c$. The remaining nonvanishing connection coefficients are $\mathring{\gamma}^0{}_{bd} = \mathring{\chi}_{db} = Hh_{bd}$, $\mathring{\gamma}^b{}_{0d} = \mathring{\chi}_d{}^b = Hh_d{}^b$, where $H(t) \equiv \dot{a}/a$ is the so-called *Hubble function* and $\dot{}$ denotes differentiation with respect to time t. In particular $\chi_{[bd]} = a_b = 0$ and $\chi_{(bd)} = 0$ for $b \neq d$, and for such background metrics $\mathring{\chi} = 3\dot{a}/a \equiv 3H$. Thus in the case of a FRW cosmology the Einstein-scalar field system (3) reduces to the evolution equations

$$\frac{\mathrm{d}\mathring{\phi}}{\mathrm{d}t} = \mathring{\psi}, \quad \frac{\mathrm{d}\mathring{\psi}}{\mathrm{d}t} = -3H\mathring{\psi} - \frac{\mathrm{d}\mathcal{V}}{\mathrm{d}\mathring{\phi}} \quad \text{and} \quad \frac{\mathrm{d}H}{\mathrm{d}t} = -H^2 - \frac{1}{3}\mathring{\psi}^2 + \frac{1}{3}\mathcal{V}(\mathring{\phi}), \tag{4}$$

subject to the Friedmann-scalar field constraint equation $H^2(t) - \frac{1}{6}\mathring{\psi}^2(t) - \frac{1}{3}\mathcal{V}(\mathring{\phi}) = -\frac{k}{a^2}$.

3.2. Linearised evolution equations

In order to perform the linearisation procedure we compute $\left.\frac{\mathrm{d}\mathbf{u}^\epsilon}{\mathrm{d}\epsilon}\right|_{\epsilon=0}$ and drop all (nonlinear) terms of coupled perturbations. In this way, we obtain the following system

$$\begin{aligned}
\partial_t\breve{\phi} &= \breve{\psi}, \\
\partial_t\breve{\psi} &= -\left(\frac{\mathrm{d}^2\mathring{\mathcal{V}}}{\mathrm{d}\mathring{\phi}^2}\right)\breve{\phi} - 3H\breve{\psi} - \mathring{\psi}\breve{\chi}, \\
2\partial_t\breve{\chi}_{(bd)} - 2\left(\frac{\omega}{a}\right)\delta_{(d}{}^j\partial_j\breve{a}_{b)} &= \frac{2}{3}\left(\left(\frac{\mathrm{d}\mathring{\mathcal{V}}}{\mathrm{d}\mathring{\phi}}\right)\breve{\phi} - 2\mathring{\psi}\breve{\psi}\right)h_{bd} - 4H\breve{\chi}_{(bd)} + 2\breve{E}_{bd} \\
&\quad - \frac{k}{a^2}\left(h_{bd}x^p\breve{a}_p - x_{(b}\breve{a}_{d)}\right), \\
\partial_t\breve{a}_c - \left(\frac{\omega}{a}\right)\delta_p{}^j\partial_j\breve{\chi}_c{}^p &= \left(2H + \frac{2}{\mathring{\psi}}\frac{\mathrm{d}\mathring{\mathcal{V}}}{\mathrm{d}\mathring{\phi}}\right)\breve{a}_c + \left(\frac{\mathrm{d}H}{\mathrm{d}t}\right)\breve{e}_c{}^0 + \frac{k}{2a^2}x_c\breve{\chi} - \frac{3k}{2a^2}x^q\breve{\chi}_{(qc)}, \\
2\partial_t\breve{E}_{bd} - 2\left(\frac{\omega}{a}\right)\epsilon^{pa}{}_{(b|}\delta_a{}^j\partial_j\breve{B}_{p|d)} &= -\mathring{\psi}^2\left(\breve{\chi}_{(bd)} - \frac{h_{bd}}{3}\breve{\chi}\right) - 2H\breve{E}_{bd} + \frac{k}{a^2}x_p\epsilon^{pq}{}_{(b}\breve{B}_{d)q}, \\
2\partial_t\breve{B}_{bd} - 2\left(\frac{\omega}{a}\right)\epsilon_{(d|}{}^{ap}\delta_a{}^j\partial_j\breve{E}_{p|b)} &= -2H\breve{B}_{bd} + \frac{k}{a^2}x_p\epsilon^{qp}{}_{(b}\breve{E}_{d)q}, \\
\partial_t\breve{e}_b{}^0 &= -H\breve{e}_b{}^0 + \breve{a}_b, \\
\partial_t\breve{e}_b{}^i &= -H\breve{e}_b{}^i - \left(\frac{\omega}{a}\right)\delta_c{}^i\breve{\chi}_b{}^c, \\
\partial_t\breve{\gamma}^a{}_{bd} &= -H\breve{\gamma}^a{}_{bd} - \frac{k}{2a^2}\left(x^a\breve{\chi}_{db} - x_b\breve{\chi}_d{}^a\right) \\
&\quad + H\left(\delta^a_d\breve{a}_b - h_{bd}\breve{a}^a\right) + \breve{B}_{dp}\epsilon^{pa}{}_b.
\end{aligned} \tag{5}$$

As a consequence, the linearised system is of the following form

$$\mathring{\mathbf{A}}^0\partial_t\breve{\mathbf{u}} - \mathring{\mathbf{A}}^j(t,\boldsymbol{x})\partial_j\breve{\mathbf{u}} = \mathring{\mathbf{B}}(t,\boldsymbol{x})\breve{\mathbf{u}}.$$

Since we are considering perturbations over a Friedmann-Robertson-Walker background with flat spatial sections, then, the linearized matrices $\mathring{\mathbf{A}}^\mu$ and $\mathring{\mathbf{B}}$ are functions of cosmic time t, only. In this case the characteristic polynomial of $\mathring{\mathbf{B}}$ has the following form

$$\begin{aligned}&(\lambda+H)^{21}\times(\lambda+2H)^3\times\left(\lambda^2+6H\lambda+2\mathring{\psi}^2+8H^2\right)^3\times\\&\times\left(\lambda^2-\left(H+2\frac{\mathring{\mathcal{V}}'}{\mathring{\psi}}\right)\lambda-\left(\frac{\mathrm{d}H}{\mathrm{d}t}+2H^2+2H\frac{\mathring{\mathcal{V}}'}{\mathring{\psi}}\right)\right)^3\times f(\lambda)\end{aligned}\tag{6}$$

where f is a polynomial of degree 8 in λ with coefficients depending on the background quantities $(\mathring{\mathcal{V}}'', \mathring{\mathcal{V}}', \mathring{\psi}, H)$. Here $\mathring{\mathcal{V}}'$ and $\mathring{\mathcal{V}}''$ denote, respectively, the first and second derivatives of the potential with respect to $\mathring{\phi}$. Then, in order to obtain conditions which ensure that the characteristic polynomial have eigenvalues with negative real part, we can make use of the *Liénard-Chipart theorem*—see e.g. [5]. This theorem states that a polynomial $f(z) = a_0z^n + a_1z^{n-1} + a_2z^{n-2} + \cdots + a_n, \quad a_0 > 0$, with real coefficients, has roots with negative real part if and only if *all coefficients of f are positive* and the *Hurwitz determinants* defined by

$$\delta_0 \equiv 1, \quad \delta_l \equiv det\begin{pmatrix} a_1 & a_3 & a_5 & \cdots & a_{2l-1}\\ a_0 & a_2 & a_4 & \cdots & a_{2l-2}\\ 0 & a_1 & a_3 & \cdots & a_{2l-3}\\ \vdots & \vdots & \vdots & & \vdots\\ 0 & 0 & 0 & \cdots & a_l\end{pmatrix} \qquad l = 1,\ldots,n.$$

are positive. It is easy to see that the first two term in (6) requires $H(t) > 0$, which is simply the condition for an (ever) expanding background. Then if there is a constant $H_0 > 0$, such that for all t, $H(t) > H_0$, the conditions in the potential, for its first and second derivatives ($\mathcal{V}'$ and $\mathcal{V}''$) can be infered – see [1] for details.

Acknowledgments

JAVK wass supported by an EPSRC Advanced Research Fellowship and by a project research grant from the Leverhulme Trust (F/07 476/AI). AA and FM are supported by projects FCT/PTDC/MAT/108921/2008 and CERN/FP/116377/2010 and by CMAT, Univ. Minho, through FCT plurianual funding. AA thanks FCT grant SFRH/BD/48658/2008. FM thanks FCT grant SFRH/BSAB/967/2010.

References

[1] Alho A, Mena F C, Valiente-Kroon J, *The Einstein-Friedrich-non-linear scalar field system and the stability of scalar field cosmologies*, arxiv.org/abs/1006.3778
[2] Friedrich H 1998 *Physical Review D* **57** 2317-2322
[3] Reula O A 1999 *Physical Review D* **60** 083507
[4] Kreiss H O and Lorenz J 1998 *Acta Numerica* **7** 203-285
[5] Rahman Q I and Schmeisser G 2002 *Analytic Theory of Polynomials: Critical Points, Zeros and Extremal Properties* London Mathematical Society Monographs (Clarendon Press)