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Spherically symmetric elasticity in Relativity

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Abstract. The relativistic theory of elasticity is reviewed within the spherically symmetric context with a view towards the modelling of star interiors possessing elastic properties such as the ones expected in neutron stars. Emphasis is placed on generality in the main sections of the paper, and the results are then applied to specific examples. Along the way, a few general results for spacetimes admitting isometries are deduced, and their consequences are fully exploited in the case of spherical symmetry relating them next to the the case in which the material content of the spacetime is some elastic material. This paper extends and generalizes the pioneering work by Magli and Kijowski [1], Magli [2] and [3], and complements, in a sense, that by Karlovini and Samuelsson in their interesting series of papers [4], [5] and [6].

1. Introduction. Relativistic elasticity revisited

Let (M, g) be a spacetime, M then being a 4-dimensional Hausdorff, simply connected manifold of class C^2 at least, and g a Lorentz metric of signature $(-,+,+,+)$. The *material space* X is a 3-dimensional manifold endowed with a Riemannian metric γ, the *material metric*; points in X can then be thought of as the particles of which the material is made of. Coordinates in M will be denoted as x^a for $a = 0, 1, 2, 3$, and coordinates in X as y^A, $A = 1, 2, 3$. The material metric γ is not a dynamical quantity of the theory, but it is frozen in the material, and it roughly describes distances between neighbouring particles in the relaxed state of the material.

The spacetime configuration of the material is said to be completely specified whenever a submersion $\psi : M \to X$ is given; if one chooses coordinate charts in M and X as above, the coordinate representative of ψ is given by three fields $y^A = y^A(x^b)$, $A = 1, 2, 3$ and the physical laws describing the mechanical properties of the material can then be expressed in terms of a hyperbolic second order system of PDE. The differential map $\psi_* : T_pM \to T_{\psi(p)}X$ is then represented in the above charts by the rank 3 matrix $\left(y^A_b\right)_p$, $y^A_b = \partial_b y^A \quad A = 1, 2, 3, \quad b = 0, 1, 2, 3$ which is sometimes called *relativistic deformation gradient*. Since ψ_* has maximal rank 3, its kernel is spanned at each point by a single timelike vector which we may take as normalized to unity, the resulting vector field, say $\vec{u} = u^a\partial_a$, satisfies then $y^A_b u^b = 0, \quad u^a u_a = -1, \quad u^0 > 0$, the last condition stating that we choose it future oriented; $\vec{u}$ is called the *velocity field of the matter*, and in the above picture in which the points in X are material points, it turns out that the spacetime manifold M (or, to be precise, an open submanifold of it) is then made up by the worldlines of the material particles, whose tangent vector is precisely $\vec{u}$.

The material space is said to be in a *locally relaxed state* at $p \in M$ if, at p, it holds $k_{ab} \equiv (\psi^* \gamma)_{ab} = h_{ab}$ where $h_{ab} = g_{ab} + u_a u_b$. Otherwise, it is said to be *strained*, and a measurement of the difference between k_{ab} and h_{ab} is the *strain*, whose definition varies in the literature. We shall follow the convention in [3] and use

$$K_{ab} \equiv k_{ab} - u_a u_b \tag{1}$$

The strain tensor determines the elastic energy stored in an infinitesimal volume element of the material space (or energy per particle). That energy will be a scalar function of K_{ab} called *constitutive equation* of the material, and its specification amounts to specifying the material. We shall denote it as $v = v(I_1, I_2, I_3)$, where I_1, I_2, I_3 are any suitably chosen set of scalar invariants associated with and characterizing K_{ab} completely. Following [3] we choose

$$I_1 = \frac{1}{2}\left(\mathrm{Tr}K - 4\right),\; I_2 = \frac{1}{4}\left[\mathrm{Tr}K^2 - (\mathrm{Tr}K)^2\right] + 3,\; I_3 = \frac{1}{2}\left(\det K - 1\right) \tag{2}$$

Notice that for $K_{ab} = g_{ab}$ (equivalently $k_{ab} = h_{ab}$) the induced metric on the rest frame of an observer moving with four-velocity $\vec{u}$, h, coincides with the material metric γ (its pull-back by ψ) describing the relaxed state of the material; thus it makes sense to have zero elastic energy stored and it is immediate to see that $I_1 = I_2 = I_3 = 0$.

The energy density ρ will then be the particle number density ϵ times the constitutive equation, that is

$$\rho = \epsilon v(I_1, I_2, I_3) = \epsilon_0 \sqrt{\det K}\, v(I_1, I_2, I_3) \tag{3}$$

where ϵ_0 is the particle number density as measured in the material space, or rather, with respect to the volume form associated with $k_{ab} = (\psi^*\gamma)_{ab}$, and ϵ is that with respect to h_{ab}; see [7].

In the case of elastic matter, the energy-momentum tensor is obtained from the Lagrangian $\Lambda = \sqrt{-g}\rho$, which depends on y^A, y^A_a and x^a, and performing the standard decomposition w.r.t. $\vec{u}$, the velocity of the matter, it follows:

$$T_{ab} = \rho u_a u_b + p h_{ab} + P_{ab}, \tag{4}$$

where $h_{ab} = g_{ab} + u_a u_b$, $P_{ab} = h^m_a h^n_b (T_{mn} - 3p h_{mn})$, $\rho = T_{ab} u^a u^b$, $p = \frac{1}{3} h^{ab} T_{ab}$; and the heat flux vanishes: $q_a = -T_{ab} u^b + \rho u_a = 0$. The quantities ρ, p, P_{ab} are respectively the energy density, isotropic pressure and anisotropic pressure tensor that a family of observers comoving with the matter would measure at each point in the spacetime. The resulting tensor is of the diagonal Segre type $\{1, 111\}$ or any of its degeneracies, $\vec{u}$ being its (unit) timelike eigenvector (see [8]), and it then follows that an orthonormal tetrad exists, $\{u_a, x_a, y_a, z_a\}$ (with $-u_a u^a = x^a x_a = y^a y_a = z^a z_a = 1$ and the mixed products zero) with respect to which T_{ab} may be written as

$$\begin{aligned} T_{ab} = \rho u_a u_b + p_1 x_a x_b + p_2 y_a y_b + p_3 z_a z_b, \quad p = \frac{1}{3}(p_1 + p_2 + p_3), \\ h_{ab} = x_a x_b + y_a y_b + z_a z_b, \quad \text{etc.} \end{aligned} \tag{5}$$

The Dominant Energy Condition (DEC), see for instance [8], is fulfilled if and only if $\rho \geq 0,\ |p_A| \leq \rho, \mathrm{for} A = 1, 2, 3.$

2. Elasticity in spherical symmetry

Let us now consider in more detail the problem of elasticity in a spherically symmetric spacetime $(M, \bar{g})$ with associated material space (X, γ). The results given in this section generalize those in [3] in the sense that here we consider a non flat material metric γ. The reader is also referred to [9] and [10] for other related, interesting developments.

We demand that the submersion $\psi : M \longrightarrow X$ preserves the KVs, that is: $\psi_*(\vec{\xi}_A) = \vec{\eta}_A$ are also KVs on X. This implies that the metric γ is also spherically symmetric and therefore coordinates $y^A = (y, \tilde{\theta}, \tilde{\phi})$ exist with $y = y(t,r)$, $\tilde{\theta} = \theta$ and $\tilde{\phi} = \phi$, and are such that $\vec{\eta}_A = \vec{\xi}_A$ are KVs of the metric $\bar{\gamma}$. Thus, the line elements of g and γ may be written as:

$$ds^2 = -a(t,r)dt^2 + b(t,r)dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2 \tag{6}$$

$$d\Sigma^2 = f^2(y)(dy^2 + y^2 d\theta^2 + y^2 sin^2\theta d\phi^2), \tag{7}$$

Notice that this last expression is completely general, as any 3-dimensional spherically symmetric metric is necessarily conformally flat. The results in [3] correspond to $f(y) = 1$.

The velocity field of the matter takes then the form $\vec{u} = u^t(t,r)\partial_t + u^r(t,r)\partial_r$, as follows from $u^a y^A_a = 0$, and u^t and u^r can be easily derived from $g_{ab}\bar{u}^a\bar{u}^b = -1$ and $\bar{u}^0 > 0$. Now, much clarity is gained by making use of the comoving coordinates adapted to $\vec{u}$: it is easy to show that it is always possible to perform a coordinate change in the t, r plane so that, in the new coordinates one has:

$$ds^2 = -a(r,t)dt^2 + b(r,t)dr^2 + Y^2(r,t)\left(d\theta^2 + \sin^2\theta d\phi^2\right), \quad u^a = \left(a^{-1/2}, 0, 0, 0\right) \tag{8}$$

hence, for the material space (M, γ) there exist coordinates $y^A = (y, \tilde{\theta}, \tilde{\phi})$ such that $y = y(r)$, $\tilde{\theta} = \theta$ and $\tilde{\phi} = \phi$, as follows from the condition $y^A_a u^a = 0$ and the requirement that $\psi_*(\vec{\xi}_A) = \vec{\eta}_A$ are KVs of the metric $\bar{\gamma}$. Further, and since the line element of the material space is $d\sigma^2 = f^2(y)\left[dy^2 + y^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$, with $y = y(r)$, no generality is lost if we set $y = r$, as this amounts to a redefinition of the r coordinate in spacetime which leaves unchanged the form of the metric and that of $\vec{u}$ in (8). We shall do that in the sequel.

As an aside, it is interesting to notice that, in spherical symmetry (and in more general classes of spacetimes, namely type B warped ones), P_{ab} and σ_{ab} are always proportional (the latter being the shear tensor of $\vec{u}$); thus, whenever the velocity flow is non-zero, it is always possible to interpret (at least formally) elastic matter as a viscous fluid.

The pulled-back material metric k is

$$k^a_b = \text{diag}\ \left(0, f^2(r)b^{-1}, r^2 f^2(r) Y^{-2}, r^2 f^2(r) Y^{-2}\right). \tag{9}$$

The operator $K^a_b = g^{ac}k_{cb} - u^a u_b$, used to measure the state of strain of the material has one eigenvalue equal to 1 (corresponding to the eigenvector $\vec{u}$), while the other eigenvalues η and s (algebraic multiplicity two) are $f^2(r)b^{-1}$, $s = r^2 f^2(r) Y^{-2}$, and one can then calculate the three invariants I_1, I_2, I_3 of K introduced in (2). In [3], the energy-momentum tensor was calculated from these invariants for a flat material metric. A similar calculation shows that

$$T^a_b = \rho\,\delta^a_b - \frac{\partial\rho}{\partial I_3}\det K\, h^a_b + \left(\text{Tr}\, K\,\frac{\partial\rho}{\partial I_2} - \frac{\partial\rho}{\partial I_1}\right) k^a_b - \frac{\partial\rho}{\partial I_2}\, k^a_c\, k^c_b. \tag{10}$$

Thus, Einstein's Field Equations read

$$-\frac{\dot{Y}}{Y^2 a} - \frac{\dot{Y}}{Y}\frac{\dot{b}}{ab} + \frac{2Y''}{Yb} + \frac{Y'^2}{Y^2 b} - \frac{Y'}{Y}\frac{b'}{b^2} - \frac{1}{Y^2} = \epsilon v\, 8\pi, \qquad 2\dot{Y}' - \frac{a'}{a}\dot{Y} - \frac{\dot{b}}{b}Y' = 0, \tag{11}$$

$$-\frac{\dot{Y}^2}{Y^2 a} + \frac{\dot{Y}}{Y}\frac{\dot{a}}{a^2} + \frac{Y'}{Y}\frac{a'}{ab} + \frac{Y'^2}{Y^2 b} - \frac{2\ddot{Y}}{Ya} - \frac{1}{Y^2} = -\epsilon\, 2\,\eta\,\frac{\partial v}{\partial \eta}\, 8\pi, \tag{12}$$

$$\frac{1}{2}\frac{\dot{Y}}{aY}\left(\frac{\dot{a}}{a}-\frac{\dot{b}}{b}\right)-\frac{1}{4}\frac{a'}{ab}\left(\frac{a'}{a}+\frac{b'}{b}\right)+\frac{1}{2}\frac{Y'}{bY}\left(\frac{a'}{a}-\frac{b'}{b}\right)$$
$$+\frac{Y''}{Yb}-\frac{\ddot{Y}}{Ya}+\frac{1}{2ab}\left(a''-\ddot{b}+\frac{\dot{b}}{2}\left(\frac{\dot{a}}{a}+\frac{\dot{b}}{b}\right)\right)=-\epsilon\, s\,\frac{\partial v}{\partial s}\,8\pi. \tag{13}$$

3. Shearfree solutions. Examples

We next consider the case of spacetimes with an elastic material content such that the velocity of the matter is shearfree, in which case, the interpretation as a viscous fluid with kinematical viscosity is not possible, and therefore the anisotropy in the pressures must be a consequence of the elastic properties of the material. In the shear-free case, coordinates exist such that

$$ds^2=-a(r,t)dt^2+Y^2(r,t)\left(dr^2+d\theta^2+\sin^2\theta d\phi^2\right) \tag{14}$$

and two cases can be distinguished: static or non-static.

In the static case, it is easy to see that solutions with a well posed constitutive equation and satisfying the DEC do indeed exist, as the following example shows:

$$ds^2=-e^{-5r^2}dt^2+e^{5r^2}\left(dr^2+d\theta^2+\sin^2\theta d\phi^2\right). \tag{15}$$

One has

$$\rho=\epsilon v=\frac{e^{-5r^2}}{8\pi}(25r^2+9),\quad p_1=-2\epsilon\eta\frac{\partial v}{\partial\eta}=-\frac{e^{-5r^2}}{8\pi}(25r^2+1),\quad p_2=-\epsilon s\frac{\partial v}{\partial s}=\frac{e^{-5r^2}}{8\pi}25r^2$$

which is obviously well behaved: satisfies the dominant energy condition and is non-singular at the origin. After some tedious algebra, we get: $f(r)=\exp(\frac{5}{2}r^2)(75r^2+1)^{-\frac{1}{3}}$, whence expressions for η, s and $\epsilon=\epsilon_0 s\sqrt{\eta}$ can be easily derived, and also: $v=F(\Sigma)\left[(75r^2+1)(75r^2+27)^{12}\right]^{-\frac{1}{39}}$, where $F(\Sigma(r)) = \left(25r^2+9\right)^{-\frac{12}{39}}\left(75r^2+1\right)^{-\frac{1}{39}}$, and r must be the only real solution of $r^3-75e^{\frac{3}{2}\Sigma}r^2-e^{\frac{3}{2}\Sigma}=0$.

Thus we have proven that a solution exists, which is regular at the origin $r=0$, satisfies the dominant energy condition and possesses a constitutive equation that can be given in closed form.

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