Nilpotents and congruences on semigroups of transformations with fixed rank

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In 1988, Howie and Marques-Smith studied P_m, a Rees quotient semigroup of transformations associated with a regular cardinal m, and described the elements which can be written as a product of nilpotents in P_m. In 1981, Marques proved that if Δ_m denotes the Malcev congruence on P_m, then P_m/Δ_m is congruence-free for any infinite m. In this paper, we describe the products of nilpotents in P_m when m is nonregular, and determine all the congruences on P_m when m is an arbitrary infinite cardinal. We also investigate when a nilpotent is a product of idempotents.

1. Introduction

Throughout this paper, X will denote an infinite set with cardinal m. All notation and terminology for semigroup theory will be from [1] and [3] unless specified otherwise, and [6] and [9] will be our authorities for advanced set theory. If $\mathscr{T}(X)$ is the full transformation semigroup on X and $\alpha \in \mathscr{T}(X)$, we put

$$Z(\alpha) = X \setminus X\alpha, \qquad d(\alpha) = |Z(\alpha)|,$$
$$S(\alpha) = \{x \in X: x\alpha \neq x\}, \qquad s(\alpha) = |S(\alpha)|,$$
$$C(\alpha) = \bigcup\{y\alpha^{-1}: |y\alpha^{-1}| \geqq 2\}, \quad c(\alpha) = |C(\alpha)|.$$

The cardinal numbers $d(\alpha)$, $s(\alpha)$ and $c(\alpha)$ are called, respectively, the *defect*, the *shift* and the *collapse* of α and were used by Howie [2] to characterise those $\alpha \in \mathscr{T}(X)$ which are the products of idempotents in $\mathscr{T}(X)$. In particular, he later showed [4] that the set

$$Q_m = \{\alpha \in \mathscr{T}(X): d(\alpha) = s(\alpha) = c(\alpha) = m\}$$

is a regular subsemigroup of $\mathscr{T}(X)$ generated by the idempotents in Q_m.

Let $r(\alpha)$ denote the *rank* of α (that is, $|X\alpha|$) and note that $I_m = \{\alpha \in Q_m: r(\alpha) < m\}$ is an ideal of Q_m. We let P_m denote the Rees quotient semigroup Q_m/I_m and identify this with $J_m \cup \{0\}$, where $J_m = \{\alpha \in Q_m: r(\alpha) = m\}$ and the product of two elements of J_m equals 0 if it lies in I_m. In [5], Howie and Marques-Smith showed that if m is a regular cardinal then the set

$$K_m = \{\alpha \in P_m: |y\alpha^{-1}| = m \text{ for some } y \in X\} \cup \{0\}$$

is a regular subsemigroup of P_m and comprises all $\alpha \in P_m$ which are products of nilpotents in P_m. In Section 2, we show that if m is *singular* (that is, nonregular) then the set

$$L_m = \{\alpha \in P_m: \text{for each } p < m, \text{ there exists } y \in X \text{ such that } |y\alpha^{-1}| > p\} \cup \{0\}$$

is a 0-bisimple regular subsemigroup of P_m and consists of all $\alpha \in P_m$ which are products of nilpotents in P_m. In Section 3, we characterise the nilpotents in L_m with index 2 that equal a product of two or of three idempotents in L_m.

In [8], Marques showed that for any infinite m, every element of P_m is a product of 4 or fewer idempotents and that 4 is best possible. In addition, she proved that if $\alpha, \beta \in J_m$ and we let

$$D(\alpha, \beta) = \{x \in X: x\alpha \neq x\beta\}, \quad dr(\alpha, \beta) = \max(|D(\alpha, \beta)\alpha|, |D(\alpha, \beta)\beta|),$$
$$\Delta_m = \{(\alpha, \beta) \in P_m \times P_m: dr(\alpha, \beta) < m\},$$

then P_m/Δ_m is congruence-free. In Section 4, we describe all the congruences on P_m for any m.

2. Nilpotents as generators

We adopt the convention introduced in [1, vol. 2, p. 241]: namely, if $\alpha \in \mathscr{T}(X)$, then we write

$$\alpha = \begin{pmatrix} A_i \\ x_i \end{pmatrix}$$

and take it as understood that the subscript i belongs to some (unmentioned) index set I, that the abbreviation $\{x_i\}$ denotes $\{x_i: i \in I\}$ and that $X\alpha = \{x_i\}$ and $A_i = x_i\alpha^{-1}$.

Our first result bears comparison with [8, Lemmas 3.1–3.3] and with [5, Proposition 2.1].

THEOREM 2.1. *If m is singular then L_m is a 0-bisimple regular semigroup.*

Proof. If $\alpha, \beta \in J_m$ and $p < m$, choose $y \in X\alpha$ such that $|y\alpha^{-1}| > p$ and let $z = y\beta$. Then

$$y\alpha^{-1} \subseteq (z\beta^{-1} \cap X\alpha)\alpha^{-1} = z(\alpha\beta)^{-1}$$

and so $|z(\alpha\beta)^{-1}| > p$: that is, L_m is a semigroup. By [8, Lemma 3.1] we know P_m is regular and so, for each $\alpha \in L_m$, there exists $\beta \in P_m$ such that $\alpha = \alpha\beta\alpha$. Let $z \in X \setminus X\alpha$ and define $\beta' \in \mathscr{T}(X)$ by putting $x\beta' = x\beta$ for all $x \in X\alpha$ and $y\beta' = z$ for all $y \in X \setminus X\alpha$. Since $d(\alpha) = m$, we have $c(\beta') = s(\beta') = m$. In addition, $X\beta' \subseteq X\beta \cup z$ implies $d(\beta') = m$, and $\alpha\beta \neq 0$ implies $r(\beta') = m$. That is, $\beta' \in L_m$ and clearly $\alpha = \alpha\beta'\alpha$.

Since every element in a regular semigroup is $\mathscr{D}$-equivalent to an idempotent, to show that L_m is 0-bisimple it will suffice to show that any two idempotents in L_m are $\mathscr{D}$-equivalent. For this, let δ, ε be idempotents in L_m and let θ: $X/\ker\delta \to X\varepsilon$ be a bijection. Define $\alpha \in \mathscr{T}(X)$ by putting $x\alpha = (x\delta\delta^{-1})\theta$ for each $x \in X$. Clearly, $X\alpha =$

$X\varepsilon$ and $\ker\alpha = \ker\delta$, and so $\delta\mathcal{R}\alpha$ and $\alpha\mathcal{L}\varepsilon$. Moreover, $d(\alpha) = d(\varepsilon)$ and $C(\alpha) = C(\delta)$, and so $\alpha = L_m$. □

In [5], Howie and Marques-Smith showed that if m is regular, the subsemigroup of P_m generated by the nilpotents of P_m equals the set

$$K_m = \{\alpha \in P_m : |y\alpha^{-1}| = m \text{ for some } y \in X\} \cup \{0\}.$$

In fact, they showed that every element of K_m is a product of 3 or fewer nilpotents in P_m with *index* 2 (that is, $\lambda \in J_m$ and $\lambda^2 = 0$) and that 3 is best possible. We now solve the corresponding problem for the case when m is singular. We begin with the following generalisation of [5, Proposition 2.4].

LEMMA 2.2. *Suppose m is any infinite cardinal. Let $\alpha \in P_m$ and $X\alpha = \{x_i\}$, $A_i = x_i\alpha^{-1}$ for each $i \in I$. Then α is a product of two nilpotents with index 2 if and only if there exists $J \subseteq I$ such that $|J| < m$ and $|(\cup A_j) \cap (X\setminus X\alpha)| = m$.*

Proof. Suppose the condition holds and let $K = I\setminus J$, so that $|K| = m$ (since $r(\alpha) = m$). Choose distinct $b_j, c_k \in (\cup A_j) \cap (X\setminus X\alpha)$ and $z \in X\alpha$, and put

$$\lambda_1 = \begin{pmatrix} A_j & A_k \\ b_j & c_k \end{pmatrix} \quad\text{and}\quad \lambda_2 = \begin{pmatrix} b_j & c_k & X\setminus\{b_j, c_k\} \\ x_j & x_k & z \end{pmatrix}.$$

Then $\alpha = \lambda_1\lambda_2$, $\cup A_k \subseteq X\setminus X\lambda_1$, and $X\alpha \subseteq X\setminus\{b_j, c_k\}$. Also, $c(\lambda_1) = c(\alpha) = m$ implies $s(\lambda_1) = m$, and clearly $d(\lambda_2) = d(\alpha)$. So, $\lambda_1, \lambda_2 \in P_m$ and each is nilpotent with index 2.

Conversely, suppose $\alpha = \lambda\mu$ for nilpotents λ, μ with index 2. By [5, Lemma 2.5] and its dual we can assume $\alpha = \lambda\mu$ where $\ker\lambda = \ker\alpha$ and $X\mu = X\alpha$. In addition, for contradiction, we suppose that for all $J \subseteq I$, if $|J| < m$ then $|(\cup A_j) \cap (X\setminus X\alpha)| < m$. In this case, put $A_i\lambda = c_i$ and let $J = \{i \in I : X\lambda \cap A_i \neq \varnothing\}$ and $K = I\setminus J$. Then

$$\lambda = \begin{pmatrix} A_j & A_k \\ c_j & c_k \end{pmatrix},$$

where $(\cup A_k) \cap X\lambda = \varnothing$. If $|J| = m$ then for each j there exists z_j such that $z_j\lambda = A_j$ and $z_j\lambda^2 = A_j\lambda = c_j$: that is, $\lambda^2 \neq 0$ in P_m. Therefore, $|J| < m$ and so $|K| = m$ since $r(\lambda) = m$. But $\{c_k\} \subseteq \cup A_j$ and so $|\cup A_j| = m$. In addition, by our supposition, $|(\cup A_j) \cap (X\setminus X\alpha)| < m$. As in Figure 2.1, put

$$Z = \cup A_j,$$
$$B = (\cup A_j) \cap X\lambda \cap X\alpha,$$
$$C = [(\cup A_j) \cap X\lambda]\setminus B,$$
$$D = (X\lambda \cap X\alpha)\setminus B,$$
$$E = [(\cup A_j) \cap (X\setminus X\alpha)]\setminus C.$$

Then $|B \cup C| = m$ and $|C \cup E| < m$ imply that $|C| < m$ and hence that $|B| = m$. Thus, $|B \cup D| = m$: that is, $|X\lambda \cap X\alpha| = m$. But, since $\ker\alpha = \ker\lambda$, μ is one-to-one on $X\lambda$ and so $|(X\lambda \cap X\alpha)\mu| = m$. Consequently, $X\mu^2 = (X\alpha)\mu \supseteq (X\lambda \cap X\alpha)\mu$ implies that $\mu^2 \neq 0$ in P_m, a contradiction as required. □

The next result shows that, when m is singular, L_m is a subset of $\langle N\rangle$, the semigroup

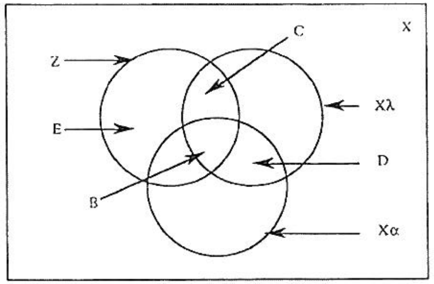

Figure 2.1.

generated by the set N of all nilpotents in P_m: part of its proof follows the basic idea of [10, p. 340].

THEOREM 2.3. *Each element of L_m is a product of 3 or fewer nilpotents with index 2.*

Proof. First suppose that there exists some z such that $|z\alpha^{-1}| = m$. By Lemma 2.2 (with $|J| = 1$) we can also suppose that $|(X\setminus X\alpha) \cap Y| < m$ where $Y = z\alpha^{-1}$, in which case $|X\alpha \cap Y| = m$. Put $X\alpha\setminus z = \{x_i\}$, $A_i = x_i\alpha^{-1}$ and choose $c_i \in (X\alpha\setminus z) \cap Y$, $d_i \in (X\setminus X\alpha)\setminus z$. Put

$$\lambda_1 = \begin{pmatrix} Y & A_i \\ z & c_i \end{pmatrix},$$
$$\lambda_2 = \begin{pmatrix} z & c_i & X\setminus\{z, c_i\} \\ z & d_i & z \end{pmatrix},$$
$$\lambda_3 = \begin{pmatrix} z & d_i & X\setminus\{z, d_i\} \\ z & x_i & z \end{pmatrix}.$$

Then $\alpha = \lambda_1\lambda_2\lambda_3$ and each $\lambda_i \in L_m$ and is nilpotent with index 2.

We now suppose that $|z\alpha^{-1}| < m$ for all $z \in X$ and write

$$\alpha = \begin{pmatrix} Y_j & u_p \\ x_j & v_p \end{pmatrix},$$

where $m > |Y_j| \geqq 2$ and $|J \cup P| = m$. If $|J| = m$, we form a disjoint union: $\{x_j\} = \{x_k\} \cup \{x_n\}$, where $|K| = m$, $|N| = cf(m)$ and $|\cup Y_n| = m$: this is possible since m is singular and so is the sum of a strictly increasing sequence of $cf(m)$ cardinals $m_n < m$ [6, pp. 26–7] and, since $\alpha \in L_m$, for each $n \in N$ there exists $Y_n \in \{Y_j\}$ such that $|Y_n| > m_n$. Now, fix $z \in X\alpha$ and write $\cup Y_n$ as a disjoint union: $\cup Y_n = \{a_k\} \cup \{a_n\} \cup \{a_p\}$ and put

$$\lambda = \begin{pmatrix} Y_k & Y_n & u_p \\ a_k & a_n & a_p \end{pmatrix} \quad\text{and}\quad \mu = \begin{pmatrix} a_k & a_n & a_p & X\setminus\{a_k, a_n, a_p\} \\ x_k & x_n & v_p & z \end{pmatrix}.$$

Then $\alpha = \lambda\mu$ and $\lambda, \mu \in L_m$ since $\cup Y_k \subseteq X\backslash X\lambda$ and $z\mu^{-1} = (\cup Y_k) \cup \{u_p\}$ where $|\cup Y_k| = m$. Also, $\lambda^2 = 0$ in P_m since $cf(m) < m$. Now, $|z\mu^{-1}| = m$ and $X\mu = X\alpha$ and so, if $|z\mu^{-1} \cap (X\backslash X\mu)| = m$, then Lemma 2.2 (with $|J| = 1$) implies that μ is a product of two nilpotents with index 2 and the result follows. On the other hand, if $|z\mu^{-1} \cap (X\backslash X\mu)| < m$ then $|(\cup Y_k) \cap (X\backslash X\alpha)| = m$ since $d(\alpha) = m$. But, in this case, $|N| = cf(m) < m$ and so Lemma 2.2 implies that α is itself a product of two nilpotents with index 2.

If $|J| < m$ then $|P| = m$ and we can write $\cup Y_j = \{b_j\} \cup \{c_p\}$. Let $z \in X\alpha$ and put

$$\lambda = \begin{pmatrix} Y_j & u_p \\ b_j & c_p \end{pmatrix} \quad \text{and} \quad \mu = \begin{pmatrix} b_j & c_p & X\backslash\{b_j, c_p\} \\ x_j & v_p & z \end{pmatrix}.$$

Then $\alpha = \lambda\mu$, $\{u_p\} \subseteq X\backslash X\lambda$, $\{u_p\} \subseteq X\backslash\{b_j, c_p\}$ and $X\mu = X\alpha$, and hence $\lambda, \mu \in L_m$, where $\lambda^2 = 0$ since $|J| < m$. Also, $|z\mu^{-1}| = m$ and so, if $|z\mu^{-1} \cap (X\backslash X\mu)| = m$, the result follows as before. If, however, $|z\mu^{-1} \cap (X\backslash X\mu)| < m$ then $|\{u_p\} \cap (X\backslash X\alpha)| < m$ and so $|(\cup Y_j) \cap (X\backslash X\alpha)| = m$ since $d(\alpha) = m$. But $|J| < m$ and another application of Lemma 2.2 completes the proof. □

The next result shows that $N \subseteq L_m$ and so, by Theorem 2.1, $\langle N \rangle \subseteq L_m$. Thus, by Theorem 2.3, L_m is precisely the subsemigroup of P_m generated by the nilpotents of P_m.

PROPOSITION 2.4. *If m is singular then every nilpotent of P_m is contained in L_m.*

Proof. Suppose there exists $\alpha \in P_m$ with $\alpha^r = 0$ and $\alpha^{r-1} \neq 0$ but $\alpha \notin L_m$. Then $|X\alpha^{r-1} \cap [X\backslash C(\alpha)]| < m$ and so $|X\alpha^{r-1} \cap C(\alpha)| = m$. Let $C(\alpha) = \{A_i\}$ and put $J = \{i \in I: X\alpha^{r-1} \cap A_i \neq \varnothing\}$. Since $\alpha \notin L_m$, there exists $p < m$ such that $|A_i| \leqq p$ for all $i \in I$. Hence, if $|J| < m$, we have

$$|\cup(X\alpha^{r-1} \cap A_j)| \leqq p \cdot |J| < m,$$

which is a contradiction. Hence, $|J| = m$ and so $[X\alpha^{r-1} \cap (\cup A_j)]\alpha$ has cardinal m, contradicting $\alpha^r = 0$. Therefore, every nilpotent of P_m lies in L_m. □

We have now shown that every element of L_m is a product of 3 or fewer nilpotents with index 2, and we have characterised when $\alpha \in P_m$ is a product of two nilpotents with index 2. It remains to show that there are elements of L_m which cannot be expressed as a product of two nilpotents with index 2: that is, that 3 is best possible. For this, consider the disjoint union $X = \{u_i\} \cup \{v_i\} \cup z$, where $|I| = m$. Put $A = \{v_i\} \cup z$ and write

$$\alpha = \begin{pmatrix} A & u_i \\ z & v_i \end{pmatrix}.$$

Then $\alpha \in L_m$ but α does not satisfy the condition of Lemma 2.2 since $X\backslash X\alpha = \{u_i\}$.

Before leaving this section we remark that both K_m and L_m can be defined for any infinite cardinal. Moreover, it can be readily checked from the proofs of [5, Proposition 2.1] and Theorem 2.1 above that in this general setting both K_m and L_m are still 0-bisimple regular semigroups such that $K_m \subseteq L_m$. To characterise when they are equal, we let n' denote the *successor* of an infinite cardinal n (that is, the least cardinal greater than n). Recall that n' is always a regular cardinal [9, Corollary 21.14].

PROPOSITION 2.5. *If m is an infinite cardinal, $K_m = L_m$ if and only if $m = n'$ for some cardinal n.*

Proof. Suppose $m = n'$ and let $\alpha \in L_m$. By the definition of L_m, some $y\alpha^{-1}$ has cardinal greater than n but at most $m = n'$: that is, $|y\alpha^{-1}| = m$ and so $\alpha \in K_m$. Conversely, suppose $m \neq n'$ for any cardinal n. If there exists $n < m$ for which there is no cardinal strictly between n and m, we have a contradiction. So, from the supposition, we deduce that for every $n < m$ there exists a cardinal k_n such that $n < k_n < m$. For each $n < m$, we let Y_n be a set with cardinal k_n: clearly, we may assume the Y_n are pairwise disjoint. We assert that $|\cup Y_n| = m$. For, if $|\cup Y_n| = p < m$ then $Y_p \subseteq \cup Y_n$ and we have

$$k_p = |Y_p| \leqq |\cup Y_n| = p < k_p,$$

a contradiction. Since $\cup Y_n$ and X therefore have the same cardinal, we may suppose the Y_n form a partition of X. We now define $\alpha \in \mathscr{T}(X)$ by writing $\cup Y_n = \{y_n\} \cup \{z_n\}$ and putting

$$\alpha = \begin{pmatrix} Y_n \\ y_n \end{pmatrix}.$$

Clearly, $r(\alpha) = c(\alpha) = s(\alpha) = m$ and, since $X\backslash X\alpha$ contains $\{z_n\}$, we also have $d(\alpha) = m$. That is, α is in L_m but not in K_m. □

3. Nilpotents as products of idempotents

In [8, Theorem 3.7] Marques proved that every element of P_m is a product of 4 or fewer idempotents in P_m: it would be interesting to know whether K_m and L_m are also generated by their idempotents. In view of Theorem 2.3, one approach to this problem would be to show that every nilpotent in L_m with index 2 is a product of idempotents in L_m. As a first step in this direction, we now characterise when a nilpotent with index 2 can be written as a product of two or of three idempotents, and we do this for each of the semigroups P_m, L_m and K_m: in what follows, S denotes any one of these latter semigroups.

PROPOSITION 3.1. *Let $\alpha \in S$ be any nilpotent with index 2. Then α is a product of three idempotents in S if and only if $|C(\alpha)\backslash X\alpha| = m$.*

Proof. Suppose α is a product of three idempotents in S and for convenience write $U = C(\alpha)$, $V = X\backslash U$. Then, by [8, Lemma 3.6], $|U\backslash X\alpha| = m$ or $|V \cap V\alpha| = m$ or $|U\alpha \cap V| < m$. Note that the second possibility cannot occur since α is nilpotent with index 2. If the third possibility occurs then $|U\alpha \cap U| = m$ and hence $|U\alpha| = m$. In this case, let $C(\alpha) = \{A_i\}$, $J = \{i \in I: X\alpha \cap A_i \neq \varnothing\}$ and $K = I\backslash J$. Now, $|J| < m$ since α is nilpotent with index 2, and so $|K| = m$. Hence, $|\cup A_k| = m$ where $(\cup A_k) \cap X\alpha = \varnothing$, and so $|U\backslash X\alpha| = m$. This proves necessity.

For the converse, suppose $|U\backslash X\alpha| = m$ and write

$$\alpha = \begin{pmatrix} A_i & u_p \\ x_i & v_p \end{pmatrix}.$$

If $|I| < m$, we choose $a_i \in A_i$ and distinct $b_i, c_p \in (U\backslash X\alpha)\backslash\{a_i\}$, a set that has cardinal

m, and put

$$\delta_1 = \begin{pmatrix} u_p & A_i \\ u_p & a_i \end{pmatrix},$$

$$\delta_2 = \begin{pmatrix} \{u_p, c_p\} & \{a_i, b_i\} & Y \\ c_p & b_i & y \end{pmatrix},$$

$$\delta_3 = \begin{pmatrix} \{c_p, v_p\} & \{b_i, x_i\} & Z \\ v_p & x_i & z \end{pmatrix},$$

where $Y = X\backslash\{u_p, c_p, a_i, b_i\}$, $Z = X\backslash\{c_p, v_p, b_i, x_i\}$ and $y \in Y$, $z \in Z$. Note that $\ker \delta_1 = \ker \alpha$, and both $c(\delta_2)$ and $c(\delta_3)$ are at least $|P \cup I|$ and this equals m since $r(\alpha) = m$. It is therefore clear that each δ_i is an idempotent in S and that $\alpha = \delta_1\delta_2\delta_3$.

If instead $|I| = m$, we let $J = \{i \in I: X\alpha \cap A_i \neq \varnothing\}$ and $K = I\backslash J$. Then, as before, $|J| < m$ and $|K| = m$. Hence, $|\bigcup A_k| = m$ where $(\bigcup A_k) \cap X\alpha = \varnothing$. In this case, choose $a_j \in A_j$ and distinct $a_k, b_k \in A_k$ (possible since each $|A_k| \geq 2$) and then choose disjoint sets $\{b_j\}$ and $\{c_p\}$ in $\{b_k\}$. An argument similar to that in the last paragraph shows that the following transformations are the required idempotents in this case:

$$\delta_1 = \begin{pmatrix} u_p & A_j & A_k \\ u_p & a_j & a_k \end{pmatrix},$$

$$\delta_2 = \begin{pmatrix} \{u_p, c_p\} & \{a_j, b_j\} & a_k & X\backslash\{u_p, c_p, a_j, b_j, a_k\} \\ c_p & b_j & a_k & y \end{pmatrix},$$

$$\delta_3 = \begin{pmatrix} \{c_p, v_p\} & \{b_j, x_j\} & \{a_k, x_k\} & X\backslash\{c_p, v_p, b_j, x_j, a_k, x_k\} \\ v_p & x_j & x_k & z \end{pmatrix}. \quad \square$$

The next result should be compared with [5, Lemma 2.5]: we use it to characterise when a nilpotent in S with index 2 is a product of two idempotents in S.

Lemma 3.2. *Let T be a regular semigroup with a zero 0. If $a \in T$, $a^2 = 0$ and $a = xy$ for some idempotents x, y in T, then $a = x_1y_1$ for some idempotents x_1, y_1 in T such that $x_1 \mathscr{R} a$ and $y_1 \mathscr{L} a$.*

Proof. Put $x_1 = aa'x$ and $y_1 = ya'a$ where a' is any inverse of a in T. It is then easy to check that x_1 and y_1 are idempotents in T with the desired properties. $\square$

The characterisations of $\mathscr{L}$ and $\mathscr{R}$ on $\mathscr{T}(X)$ given in [1, Lemmas 2.5 and 2.6 and their proofs] hold almost verbatim for the Rees quotient semigroup:

$$D_m = \mathscr{T}(X)/\{\alpha \in \mathscr{T}(X): r(\alpha) < m\}.$$

That is, for nonzero $\alpha, \beta \in D_m$ we have: $\alpha \mathscr{L} \beta$ if and only if $X\alpha = X\beta$, and $\alpha \mathscr{R} \beta$ if and only if $\ker \alpha = \ker \beta$. Since each of P_m, L_m and K_m are regular subsemigroups of D_m, we may apply [3, Proposition II.4.5] to conclude that the same characterisations hold for each of P_m, L_m and K_m.

Proposition 3.3. *Suppose α is a nilpotent in S with index 2 and write $\ker \alpha = \{A_i\}$. Then α is a product of two idempotents in S if and only if:*

(a) $|C(\alpha)\backslash X\alpha| = m$, and

(b) *for each $i \in I$, $A_i\alpha \in A_i$ or $A_i \cap (X\backslash X\alpha) \neq \varnothing$.*

Proof. Suppose (a) and (b) hold and write

$$\alpha = \begin{pmatrix} B_r & B_t & u_p & u_q \\ b_r & x_t & u_p & v_q \end{pmatrix},$$

where B_r, B_t contain at least two elements, $b_r \in B_r$, $x_t \notin B_t$, and $u_q \notin X\alpha$. Note that both R and P have cardinal less than m since α is nilpotent with index 2. Also, either $|T| = m$ or $|Q| = m$ (or both) since $r(\alpha) = m$. By (b) we can choose $b_t \in B_t \cap (X\backslash X\alpha)$ and put

$$\varepsilon_1 = \begin{pmatrix} B_r & B_t & u_p & u_q \\ b_r & b_t & u_p & u_q \end{pmatrix},$$

$$\varepsilon_2 = \begin{pmatrix} b_r & \{b_t, x_t\} & u_p & \{u_q, v_q\} & Y \\ b_r & x_t & u_p & v_q & y \end{pmatrix},$$

where $Y = X\backslash[X\alpha \cup \{b_t\} \cup \{u_q\}]$, $y \in Y$, $\{b_t\} \cap \{x_t\} = \varnothing$, $\{u_q\} \cap \{v_q\} = \varnothing$, and $X\backslash X\varepsilon_2 = X\backslash X\alpha$. Also, note that $X\backslash X\varepsilon_1$ contains $\{x_t\} \cup \bigcup\{B_t\backslash b_t\}$. If $|T| = m$, put $J = R \cup T$ and $K = \{j \in J: X\alpha \cap B_j \neq \varnothing\}$. Then, since α is a nilpotent with index 2, $|K| < m$ and this implies $|J\backslash K| = m$ and $|\bigcup\{B_j: j \in J\backslash K\}| = m$. But $(\bigcup\{B_j: j \in J\backslash K\}) \cap X\alpha = \varnothing$ and $(\bigcup\{B_j: j \in J\backslash K\}) \cap \{b_t\} = \varnothing$. Therefore, $\bigcup\{B_j: j \in J\backslash K\} \subseteq Y$ and so $|Y| = m$. If, on the other hand, $|T| < m$, we use (a) and $Y \supseteq [C(\alpha)\backslash X\alpha]\backslash\{b_t\}$ to conclude that $|Y| = m$. Hence, $\varepsilon_1, \varepsilon_2 \in S$, and sufficiency follows.

Conversely, suppose $\alpha = \varepsilon_1\varepsilon_2$ and $X\alpha = \{x_i\}$, $A_i = x_i\alpha^{-1}$. By Lemma 3.2, we can also suppose $\ker \varepsilon_1 = \ker \alpha$ and $X\varepsilon_2 = X\alpha$, in which case we have, for each $i \in I$, $A_i\varepsilon_1 = a_i$, some element of A_i. Suppose $A_i\alpha = x_i \notin A_i$ and $A_i \cap (X\backslash X\alpha) = \varnothing$. Then $A_i \subseteq X\alpha$ and so each a_i equals some x_j: that is, $a_i = a_j\alpha$ where $j \neq i$ by the last supposition. But

$$A_j\alpha = A_j\varepsilon_1\varepsilon_2 = a_j\varepsilon_2 = x_j = a_i$$

then gives a contradiction. Hence, (b) holds and, since $\alpha = \varepsilon_1\varepsilon_2\varepsilon_2$ is a product of three idempotents, (a) follows from Proposition 3.1. $\square$

It remains to note that the transformation α defined immediately after the proof of Proposition 2.2 is a nilpotent in S with index 2 and has the property that $|C(\alpha)\backslash X\alpha| < m$: consequently, by Propositions 3.1 and 3.2, it cannot be written as the product of two or of three idempotents in S.

4. Congruences

The congruences on $\mathscr{T}(X)$ were determined by Malcev [7], with an alternative account being given in [1, section 10.8]. An important type of congruence on $\mathscr{T}(X)$, the so-called *Malcev congruence*, induces a congruence on P_m as follows. For each $\alpha, \beta \in P_m$, put

$$D(\alpha, \beta) = \{x \in X: x\alpha \neq x\beta\}, \quad dr(\alpha, \beta) = \max\{|D(\alpha, \beta)\alpha|, |D(\alpha, \beta)\beta|\}$$

and for each n such that $\aleph_0 \leqq n \leqq m$, let

$$\Delta_n = \{(\alpha, \beta) \in P_m \times P_m \colon dr(\alpha, \beta) < n\}.$$

Then each Δ_n is a congruence on P_m (compare [1, Lemma 10.6]) and in [8] Marques showed that P_m/Δ_m is congruence-free for any infinite cardinal m. We now show that the *proper* (that is, nonuniversal) congruences on P_m are precisely the Δ_n together with the identity congruence; in fact, by making an obvious adjustment to the definition of Δ_n, we shall prove the same thing for K_m and L_m: in what follows, S will denote any one of P_m, L_m and K_m, and Δ_n will denote the corresponding congruence on S.

We begin with a simple but useful result: it was first observed by Professor G. B. Preston in lectures at Monash University in 1966. Note that since P_m, L_m and K_m are 0-bisimple regular semigroups, they are also 0-simple.

LEMMA 4.1. *Every proper congruence ρ on a 0-simple semigroup T is 0-restricted (that is, $\{0\}$ is a ρ-class).*

To describe the congruences on S, we need a result which is analogous to [1, Theorem 10.69(ii)].

LEMMA 4.2. *If ρ is a congruence on S and there exist $(\alpha, \beta) \in \rho$ such that $1 \leqq dr(\alpha, \beta) = \xi < \aleph_0$, then $\Delta_{\aleph_0} \subseteq \rho$.*

Proof. We begin by closely following the ideas of [1, vol. 2, p. 244]. Let $D = D(\alpha, \beta)$ and, without loss of generality, suppose $|D\alpha| = \xi$, $C = D\alpha \cup D\beta = \{c_i\}$, $X\alpha \backslash C = X\beta \backslash C = \{e_j\}$, $M_i = c_i\alpha^{-1}$, $N_i = c_i\beta^{-1}$, and $R_j = e_j\alpha^{-1} = e_j\beta^{-1}$. Note that possibly one (but not both) of M_i, N_i is empty but nonetheless $\cup M_i = \cup N_i$ and this set contains D. We therefore have:

$$\alpha = \begin{pmatrix} M_i & R_j \\ c_i & e_j \end{pmatrix} \sim \beta = \begin{pmatrix} N_i & R_j \\ c_i & e_j \end{pmatrix}, \tag{4.1}$$

where $\alpha \sim \beta$ denotes that α, β are ρ-equivalent. Again without loss of generality, suppose some $c_0 = a\alpha \neq a\beta$ where $a \in M_0$. Then, since $\cup M_i = \cup N_i$, $a \in N_1$ for some index $1 \in I$ different from 0. Note that I is finite and so $|J| = m$ since $r(\alpha) = m$. We can therefore write $\{R_j\} = \{R_p\} \cup \{R_q\}$ where $|P| = |Q| = m$ and, for some fixed index $2 \in P$, choose $b \in R_2$ as well as $r_q \in R_q$. Put $A = [(\cup M_i)\backslash a] \cup \cup R_p$ and let

$$\varphi_1 = \begin{pmatrix} a & A & R_q \\ a & b & r_q \end{pmatrix}.$$

Note, that $|A| = m$ and $X \backslash X\varphi_1 \supseteq \{r_p\}\backslash b$. Thus, φ_1 is an element of S and we have:

$$\varphi_1\alpha = \begin{pmatrix} a & A & R_q \\ c_0 & e_2 & e_q \end{pmatrix} \sim \varphi_1\beta = \begin{pmatrix} a & A & R_q \\ c_1 & e_2 & e_q \end{pmatrix}.$$

Now put $Y = X \backslash (\{c_0, c_1, e_2\} \cup \{e_q\})$ and let

$$\varphi_2 = \begin{pmatrix} c_0 & \{c_1, e_2\} \cup Y & e_q \\ a & b & r_q \end{pmatrix}.$$

Note that Y has cardinal m since it contains $X \backslash X\alpha$, and $X \backslash X\varphi_2$ contains $\{r_p\}\backslash b$.

Thus, φ_2 is an element of S and we have:

$$\varphi_1\alpha\varphi_2 = \begin{pmatrix} a & A & R_q \\ a & b & r_q \end{pmatrix} \sim \varphi_1\beta\varphi_2 = \begin{pmatrix} a \cup A & R_q \\ b & r_q \end{pmatrix}.$$

Distinguish $R_1, \ldots, R_n$ in $\{R_q\}$ and choose $r_i \in R_i$ for $i = 1, \ldots, n$. Write $T = Q \backslash \{1, \ldots, n\}$ and put

$$\psi = \begin{pmatrix} a & R_1 & \ldots & R_n & A & R_t \\ r_1 & r_2 & \ldots & a & b & r_t \end{pmatrix}.$$

Once again, note that $|A| = m$ and $X \backslash X\psi$ contains $\{r_p\}\backslash b$. Thus, ψ is an element of S and we have:

$$\varphi_1\alpha\varphi_2\psi = \begin{pmatrix} a & R_1 & \ldots & R_n & A & R_t \\ r_1 & r_2 & \ldots & a & b & r_t \end{pmatrix} \sim \varphi_1\beta\varphi_2\psi = \begin{pmatrix} a \cup A & R_1 & \ldots & R_n & R_t \\ b & r_2 & \ldots & a & r_t \end{pmatrix}.$$

Now observe that if $\lambda = \varphi_1\alpha\varphi_2\psi$ and $\mu = \varphi_1\beta\varphi_2\psi$ then we have:

$$\lambda^{n+1} = \begin{pmatrix} a & R_1 & \ldots & R_n & A & R_t \\ a & r_1 & \ldots & r_n & b & r_t \end{pmatrix} \sim \mu^{n+1} = \begin{pmatrix} a \cup A \cup R_1 \cup \ldots \cup R_n & R_t \\ b & r_t \end{pmatrix}, \tag{4.2}$$

where n is any positive integer and $|T| = m$. Finally, let σ, τ be any two distinct elements of S such that $dr(\sigma, \tau) = n < \aleph_0$ and write

$$\sigma = \begin{pmatrix} G_k & W_t \\ u_k & v_t \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} H_k & W_t \\ u_k & v_t \end{pmatrix}$$

in the same way as we did for α, β in (4.1): that is, possibly one (but not both) of G_k, H_k is empty but in any case $\cup G_k = \cup H_k$; and $|T| = m$ since $r(\sigma) = m$ and we may suppose, without loss of generality, that $|K| = n$. Then, using the notation in (4.2), we define

$$\omega_1 = \begin{pmatrix} G_1 & \ldots & G_n & W_t \\ r_1 & \ldots & r_n & r_t \end{pmatrix}$$

and note that $\ker \omega_1 = \ker \sigma$ and $X \backslash X\omega_1$ contains $\{r_p\}\backslash b$. Hence, $\omega_1 \in S$ and, pre-multiplying (4.2) by ω_1, we obtain

$$\begin{pmatrix} G_1 & \ldots & G_n & W_t \\ r_1 & \ldots & r_n & r_t \end{pmatrix} \sim \begin{pmatrix} \cup G_k & W_t \\ b & r_t \end{pmatrix}. \tag{4.3}$$

Now put $Z = X \backslash (\{r_k\} \cup \{r_t\})$, choose $c \in Z$, and let

$$\omega_2 = \begin{pmatrix} r_1 & \ldots & r_n & r_t & Z \\ u_1 & \ldots & u_n & v_t & c \end{pmatrix}.$$

Note that this is well-defined even if $c \in X\sigma$. In addition, $Z \supseteq \{r_p\}$ and $X \backslash X\omega_2$ contains $(X \backslash X\sigma)\backslash c$, so $\omega_2 \in S$. Hence, since $b \in \{r_p\}$, after postmultiplying (4.3) by ω_2, we obtain:

$$\sigma \sim \begin{pmatrix} \cup G_k & W_t \\ c & v_t \end{pmatrix}.$$

In a similar way, we can show that

$$\tau \sim \begin{pmatrix} \cup H_k & W_t \\ c & v_t \end{pmatrix}$$

and so, by the transitivity of ρ, we conclude that $(\sigma, \tau) \in \rho$ as required. □

We now aim to prove a result corresponding to [1, Theorem 10.69(i)] and need a version of [1, Lemma 10.73]. For the latter, we slightly modify [8, Lemma 3.10].

LEMMA 4.3. *If $\alpha, \beta \in S$ and $dr(\alpha, \beta) = \xi \geq \aleph_0$, then there exists $Y \subseteq D(\alpha, \beta)$ such that $Y\alpha \cap Y\beta = \varnothing$ and $\max(|Y\alpha|, |Y\beta|) = \xi$.*

Proof. Suppose, without loss of generality, that $|D\alpha| = \xi$ where $D = D(\alpha, \beta)$. Put $Y = (D\alpha \setminus D\beta)\alpha^{-1} \cap D$ and note that $Y\alpha = D\alpha \setminus D\beta$ and $Y\beta \subseteq D\beta$, so $Y\alpha \cap Y\beta = \varnothing$. If $|D\beta| < \xi$ then $|Y\alpha| = \xi$ and $\max(|Y\alpha|, |Y\beta|) = \xi$, and the result follows. If, on the other hand, $|D\beta| = \xi$ we let

$$\mathscr{F} = \{Z \subseteq D : Z\alpha \cap Z\beta = \varnothing\}$$

and note that $\mathscr{F}$ is nonempty since it contains ξ singletons. Moreover, Zorn's Lemma cn be applied to choose a maximal Z in $\mathscr{F}$, and then $Z\alpha \cap Z\beta = \varnothing$ where $|Z\beta| \leq \xi$ since $Z \subseteq D$ and $|D\beta| = \xi$. If now $|Z\beta| = \xi$, the result follows (and the same conclusion holds if $|Z\alpha| = \xi$). So, we suppose both $Z\alpha$ and $Z\beta$ have cardinal less than ξ. Then, $D \setminus Z \neq \varnothing$ (otherwise, $D = Z$ and so $|D\alpha| < \xi$, contradicting our initial supposition). Since Z is maximal in $\mathscr{F}$, we have:

$$(Z \cup d)\alpha \cap (Z \cup d)\beta \neq \varnothing$$

for each $d \in D \setminus Z$, and hence $d\alpha \in Z\beta$ or $d\beta \in Z\alpha$. Put

$$D_1 = \{d \in D \setminus Z : d\beta \in Z\alpha\} \quad \text{and} \quad D_2 = \{d \in D \setminus Z : d\alpha \in Z\beta\}$$

and note that $D \setminus Z = D_1 \cup D_2$, $D = D_1 \cup D_2 \cup Z$, $D_1\beta \subseteq Z\alpha$ and $D_2\alpha \subseteq Z\beta$. Thus, $D\alpha = D_1\alpha \cup D_2\alpha \cup Z\alpha$ where the last two sets on the right have cardinal less than ξ. Hence, $|D_1\alpha| = \xi$ and $|D_1\beta| < \xi$: this means we can apply the very first case (with D_1 replacing D), and the proof is complete. □

We can now prove the following result.

LEMMA 4.4. *If ρ is a proper congruence on S and there exists $(\alpha, \beta) \in \rho$ such that $dr(\alpha, \beta) = \xi \geq \aleph_0$ then $\Delta_{\xi'} \subseteq \rho$.*

Proof. We adopt the same notation as introduced at and before (4.1), with the proviso that now ξ is infinite. By Lemma 4.2, there exists $Y \subseteq D$ such that $Y\alpha \cap Y\beta = \varnothing$ and $\max(|Y\alpha|, Y\beta|) = \xi$. If $|Y\beta| = \xi$ then $Y \subseteq D$ and $|D\beta| \leq \xi$ together imply that $|D\beta| = \xi$. Hence, we may assume that $|Y\alpha| = \xi$ and let $Y\alpha = \{c_k\} \subseteq \{c_i\}$ where $|K| = \xi$. Let $L_k = c_k\alpha^{-1}$: note that each L_k equals some M_i and $\cup L_k \subseteq \cup M_i$. We now consider two cases.

Case (1). $\xi = m$. Write $\{L_k\} = \{L_r\} \cup \{L_t\}$ where $|R| = |T| = m$, choose $y_r \in L_r$, $y_t \in L_t$ and $r_j \in R_j$, and let

$$\varphi_1 = \begin{pmatrix} y_r & (\cup M_i)\setminus\{y_r\} & R_j \\ y_r & a & r_j \end{pmatrix},$$

where $a \in \{y_t\} \subseteq (\cup M_i)\setminus\{y_r\} = A$, say, is chosen so that $a\beta \notin \{c_r\}$ (if necessary, when $a\beta = c_0$ for some index $0 \in R$, we re-index $\{y_r\}\setminus y_0$ and add y_0 to A). Note that $|A| = m$ and $X\setminus X\varphi_1$ contains $\{y_t\}$, so $\varphi_1 \in S$. In addition, $a\alpha \notin \{c_r\}$. Therefore, after premultiplying (4.1) by φ_1, we obtain:

$$\varphi_1\alpha = \begin{pmatrix} y_r & A & R_j \\ c_r & a\alpha & e_j \end{pmatrix} \sim \varphi_1\beta = \begin{pmatrix} y_r & A & R_j \\ y_r\beta & a\beta & e_j \end{pmatrix},$$

where $\{c_r\} \cap \{y_r\beta\} = \varnothing$ by the choice of Y, and the set

$$Q = X\setminus(\{c_r\} \cup \{e_j\}) \supseteq \{c_t\} \cup \{y_r\beta\} \cup \{a\alpha, a\beta\}$$

has cardinal m since $|T| = m$. Put

$$\varphi_2 = \begin{pmatrix} c_r & Q & e_j \\ y_r & a & r_j \end{pmatrix}$$

and note that $X\setminus X\varphi_2 \supseteq \{y_t\}$. Hence, $\varphi_2 \in S$ and we have:

$$\varphi_1\alpha\varphi_2 = \begin{pmatrix} y_r & A & R_j \\ y_r & a & r_j \end{pmatrix} \sim \varphi_1\beta\varphi_2 = \begin{pmatrix} \{y_r\} \cup A & R_j \\ a & r_j \end{pmatrix}. \tag{4.4}$$

Since $r(\varphi_1\alpha\varphi_2) = m = |R|$ and ρ is nonuniversal, it follows from (4.4) that $|J| = m$.

Case (2). $\xi < m$. In this case, $|J| = m$ and we can write $\{R_j\} = \{R_p\} \cup \{R_q\}$ where $|P| = |Q| = m$. As before, choose $y_k \in L_k$, $r_j \in R_j$ and let

$$\varphi_1 = \begin{pmatrix} y_k & ((\cup M_i)\setminus\{y_k\}) \cup \cup R_p & R_q \\ y_k & b & r_q \end{pmatrix},$$

where b is chosen in R_0 for some index $0 \in P$ such that $b\alpha = e_0 = b\beta \notin \{c_k\}$. Note that $((\cup M_i)\setminus\{y_k\}) \cup \cup R_p = B$, say, has cardinal m and $X\setminus X\varphi_1$ contains $\{r_p\}$. Thus, $\varphi_1 \in S$ and we have:

$$\varphi_1\alpha = \begin{pmatrix} y_k & B & R_q \\ c_k & b\alpha & e_q \end{pmatrix} \sim \varphi_1\beta = \begin{pmatrix} y_k & B & R_q \\ y_k\beta & b\beta & e_q \end{pmatrix},$$

where $\{c_k\} \cap \{y_k\beta\} = \varnothing$ by choice of Y, and the set

$$E = X\setminus(\{c_k\} \cup \{e_q\}) \supseteq \{e_p\} \cup \{y_k\beta\} \cup \{e_0\}$$

has cardinal m since $|P| = m$. Put

$$\varphi_2 = \begin{pmatrix} c_k & E & e_q \\ y_k & b & r_q \end{pmatrix}$$

and note that $X\setminus X\varphi_2 \supseteq \{r_p\}$. Hence, $\varphi_2 \in S$ and we have:

$$\varphi_1\alpha\varphi_2 = \begin{pmatrix} y_k & B & R_q \\ y_k & b & r_q \end{pmatrix} \sim \varphi_1\beta\varphi_2 = \begin{pmatrix} \{y_k\} \cup B & R_q \\ b & r_q \end{pmatrix}. \tag{4.5}$$

We have shown that in both cases there are ρ-equivalent elements of S with the form of (4.5): that is, where $|K| = \xi$ and $|Q| = m$. Now, let σ, τ be any two distinct elements of S such that $dr(\sigma, \tau) \leqq \xi$ and write

$$\sigma = \begin{pmatrix} G_n & W_t \\ u_n & v_t \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} H_n & W_t \\ u_n & v_t \end{pmatrix}$$

in the same way as we did for α, β in (4.1): that is, possibly one (but not both) of G_n, H_n is empty but in any case $\cup G_n = \cup H_n$; and $|N| \leqq \xi$, $|T| \leqq m$ (at least one of N, T has cardinal m since $r(\sigma) = m$). Then, using the notation in (4.5), we choose $\{y_n\} \subseteq \{y_k\}$, $\{r_t\} \subseteq \{r_q\}$ and define

$$\omega_1 = \begin{pmatrix} G_n & W_t \\ y_n & r_t \end{pmatrix}.$$

Note that $\ker \omega_1 = \ker \sigma$, and $X \setminus X\omega_1$ contains B which has cardinal m. Hence, $\omega_1 \in S$ and, premultiplying (4.5) by ω_1, we obtain

$$\begin{pmatrix} G_n & W_t \\ y_n & r_t \end{pmatrix} \sim \begin{pmatrix} \cup G_n \cup B & W_t \\ b & r_t \end{pmatrix}. \tag{4.6}$$

Now put $Z = B \cup [\{y_k\} \setminus \{y_n\}] \cup [(\cup R_q) \setminus \{r_t\}]$, a set with cardinal m, and let

$$\omega_2 = \begin{pmatrix} y_n & r_t & Z \\ u_n & v_t & b \end{pmatrix}.$$

Note that $d(\omega_2) = m$ since $d(\sigma) = m$, so $\omega_2 \in S$. Hence, after postmultiplying (4.6) by ω_2, we obtain:

$$\sigma \sim \begin{pmatrix} \cup G_n \cup B & W_t \\ b & v_t \end{pmatrix}.$$

In a similar way, we can show that

$$\tau \sim \begin{pmatrix} \cup H_n \cup B & W_t \\ b & v_t \end{pmatrix}.$$

where $\cup G_n = \cup H_n$, and the result follows by the transitivity of ρ. □

We now use Lemmas 4.2 and 4.4 to obtain our main result.

THEOREM 4.5. *If ρ is a proper congruence on S different from the identity, then $\rho = \Delta_n$ for some $n \geqq \aleph_0$.*

Proof. Let n equal the least cardinal greater than $dr(\alpha, \beta)$ where $(\alpha, \beta) \in \rho$. By Lemma 4.2, n is infinite and $\rho \subseteq \Delta_n$. Let $(\alpha, \beta) \in \Delta_n$ and suppose $dr(\alpha, \beta) = \xi$. If $dr(\lambda, \mu) < \xi$ for all $(\lambda, \mu) \in \rho$, we contradict the definition of n. Hence, there exists $(\lambda, \mu) \in \rho$ with $dr(\lambda, \mu) \geqq \xi$ and then Lemmas 4.2 and 4.4 imply that $\Delta_{\xi'} \subseteq \rho$ in which case $(\alpha, \beta) \in \rho$: that is, $\rho = \Delta_n$ as required. □

Note that Marques' work [8] showing that P_m/Δ_m is congruence-free follows as a consequence of the above theorem. In addition, we have shown that K_m/Δ_m and L_m/Δ_m are congruence-free whenever m is, respectively, regular and singular.

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