SPECTRALLY ARBITRARY FACTORIZATION: THE NONDEROGATORY CASE

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Dedicated to E.M.Sá

Abstract. It is known that a nonsingular, nonscalar, n-by-n complex matrix A may be factored as $A = BC$, in which the spectra of B and C are arbitrary, subject to $detBdetC = detA$. We show further that B and C may be taken to be nonderogatory, even when the target spectra include repeated eigenvalues. This is a major step in a broader question of how arbitrary the Jordan forms of B and C may be, given their target spectra. In the process, a number of tools are developed, such as a special LU factorization under similarity, that may be of independent interest.

1. Introduction

In [2] it was shown that any nonsingular nonscalar matrix $A \in M_n(\mathbb{C})$ may be factored $A = BC$, so that B, $C \in M_n(\mathbb{C})$ have arbitrary spectra, subject only to the obvious determinantal condition $detA = \prod_{i=1}^{n} \beta_i \prod_{i=1}^{n} \gamma_i$, in which $\beta_1, \beta_2, ..., \beta_n$ are the eigenvalues of B and $\gamma_1, \gamma_2, ..., \gamma_n$ are the eigenvalues of C (repeats allowed). This fact has proven quite useful, and it is surprising that it was not known earlier; a slight generalization is proven in [1]. If B and C have repeated eigenvalues, no indication is given in [2] what sort of Jordan structure they may have, and unfortunately, the proof there is not easily adapted to further specify the Jordan structure.

The Jordan structure of B and C cannot generally be taken to be arbitrary. (Suppose that B and/or C have repeated eigenvalues and 1-by-1 Jordan blocks in

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case $n = 2$, for example.) Thus a natural, deeper, question is to ask for a given A and specified eigenvalues $\beta_1, \beta_2, ..., \beta_n$ for B and $\gamma_1, \gamma_2, ..., \gamma_n$ for C, $\prod_{i=1}^{n} \beta_i \gamma_i = detA$, what the Jordan form for B and C may be? One might expect that nonderogatory Jordan structure (one Jordan block for each distinct eigenvalue) for B and C is the more likely, but it is not obvious that this is always possible. Our purpose here is to show that each of B and C may be taken to be nonderogatory, for any allowed spectra specified for them. In the process, we show that a certain, more special, sort of factorization usually exists for a similarity of A, and this may be of independent interest.

We first note that the possible Jordan forms for B and C, given their eigenvalues, are a similarity invariant of A, as $S^{-1}AS = S^{-1}BCS = (S^{-1}BS)(S^{-1}CS)$. Thus for purposes of proof, A may be placed in any form allowed for it by similarity. Our general approach is to show that A is similar to a matrix with a special kind of LU factorization. Although this approach suffices to re-prove the original result (in a possibly cleaner way), it does not work in some cases for our more precise purpose. In particular, we will characterize the infrequent exceptions. However, these exceptions may be treated in another way. For this purpose (and others) we begin with a careful treatment of the 2-by-2 case, in a way different from our general approach.

Lemma 1.. *Let $A \in M_2(\mathbb{C})$ be nonsingular and nonscalar and let $\beta_1, \beta_2, \gamma_1$ and $\gamma_2 \in \mathbb{C}$ be such that $\beta_1\beta_2\gamma_1\gamma_2 = detA$. Then, there exist nonderogatory matrices B and C such that B has eigenvalues β_1, β_2, C has eigenvalues γ_1 and γ_2 and $A = BC$. In case A is nonsingular and scalar, the conclusion remains valid in case $\gamma_i = \beta_i^{-1}$, $i = 1, 2$.*

Proof. Via multiplication by scalars as needed, we may assume without loss of generality that $detA = detB = detC = 1$. Thus, A has eigenvalues $\alpha, \frac{1}{\alpha}$. We first assume that $\alpha \neq \pm 1, 0$. If $\beta_1 \neq \beta_2$ and $\gamma_1 \neq \gamma_2$, there is nothing to do, as the desired statement follows from the classical result of [2]. Thus we assume, also without loss of generality, that $\gamma_1 = \gamma_2 = 1$. Since, the eigenvalues of A are distinct, we may assume

$$\begin{bmatrix} \alpha & 0 \\ 0 & \frac{1}{\alpha} \end{bmatrix}.$$

But such an A may be factored as

$$\begin{bmatrix} \alpha\dfrac{\beta\alpha-2}{\alpha^2-1} & \alpha d \\ \dfrac{-(\alpha^2-\beta\alpha+1)^2}{\alpha d(\alpha^2-1)^2} & \dfrac{2\alpha-\beta}{\alpha^2-1} \end{bmatrix} \begin{bmatrix} \alpha\dfrac{2\alpha-\beta}{\alpha^2-1} & -d \\ \dfrac{(\alpha^2-\beta\alpha+1)^2}{d(\alpha^2-1)^2} & \dfrac{\alpha\beta-2}{\alpha^2-1} \end{bmatrix}.$$

In this case, d is a free nonzero parameter and $\beta = \beta_1 + \beta_2$. Since $d, \alpha \neq 0$, both factors are nonderogatory, completing the proof in this case.

Of course $\alpha = 0$ cannot occur, and the case $\alpha = -1$ is the same as $\alpha = 1$, again via scalar multiplication. This leaves two possibilities for A:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The former is straightforwardly treated, as claimed, leaving the latter.

If $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, and B and C both have repeated eigenvalues we may complete the proof by taking a nonderogatory square root of A, $A^{1/2}$ and passing a scalar between the two factors as necessary. Otherwise, without loss of generality, we may assume that only C has repeated eigenvalues. Then, B has distinct eigenvalues and the proof is simply completed by writing $B = AC^{-1}$ and letting B play the role of A, A the role of B and C^{-1} the role of C in the first part of this proof. □

The general approach that we take is to find a similarity of A with nonzero leading principal minors: $\beta_1\gamma_1$, $\beta_1\beta_2\gamma_1\gamma_2$, ..., $\beta_1\cdots\beta_n\gamma_1\cdots\gamma_n$ such that its LU factorization with L lower, and U upper triangular with diagonal entries $\beta_1, ..., \beta_n$ and $\gamma_1, ..., \gamma_n$ respectively have totally nonzero subdiagonal in L and superdiagonal in U. This cannot always be done. For example, if $A = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix}$ and $\beta_1\gamma_1 = \alpha_1$, then any similarity of A whose 1,1 entry is α_1 must be (lower or upper) triangular because, by the trace condition, the 2,2 entry must be $\alpha_2 (= \beta_2\gamma_2)$ and then, as the determinant is $\alpha_1\alpha_2$, the product of the two off-diagonal entries must be 0. Since the similarity of A is triangular, in no LU factorization can the 2,1 entry of L and the 1,2 entry of U both be nonzero. This problem might be solved by re-labelling either the β's or the γ's so that $\beta_1\gamma_1$ is no longer α_1. However, this cannot be done if $\beta_1 = \beta_2$ and $\gamma_1 = \gamma_2$ (in which case $\alpha_2 = \alpha_1$), but then, as in the proof of Lemma 1, there is an easy alternative.

For $A \in M_n$ with leading principal minors $\beta_1\gamma_1$, $\beta_1\beta_2\gamma_1\gamma_2$, ..., $\beta_1\cdots\beta_n\gamma_1\cdots\gamma_n \neq 0$, we say that A has a *special LU factorization* if the unique lower and upper triangular factors L, with diagonal entries $\beta_1, ..., \beta_n$, and U, with diagonal entries $\gamma_1, ..., \gamma_n$, such that $A = LU$, also have totally nonzero subdiagonal and superdiagonal, respectively. (Of course, if some other LU factorization is chosen, with diagonal entries β_i' and γ_i', $i = 1, ..., n$, $\beta_i'\gamma_i' = \beta_i\gamma_i$, the factorization will be special as well.) It is easy to see, using e.g. Cauchy-Binet, that the LU factorization of A will be special if and only if the minors

$$detA[1, ..., k, k+2; 1, ..., k+1]$$
$$detA[1, ..., k+1; 1, ..., k, k+2]$$

are all nonzero $k = 0, ..., n-2$, or equivalently by Sylvester's identity,

$$detA[1, ..., k, k+2] \neq \prod_{\substack{i=1 \\ i \neq k+1}}^{k+2} \beta_i\gamma_i$$

$k = 0, ..., n-2$. Of course, such conditions are generic, and it may happen that A (with the desired leading principal minors) fails them while a similarity of A (with the same leading principal minors) does not.

It is clear that if a similarity of A has a special LU factorization (for $\beta_1, ..., \beta_n$, $\gamma_1, ..., \gamma_n$) and if any equal β_i's (resp. equal γ_j's) are consecutively labelled, then A has a nonderogatory factorization for $\beta_1, ..., \beta_n$ and $\gamma_1, ..., \gamma_n$, as the L and U are nonderogatory (and even a bit more). Note that whether a similarity has a special LU factorization is a property both of the similarity class of A (i.e; its Jordan form) and also of the $\beta_1, ..., \beta_n$, $\gamma_1, ..., \gamma_n$ (including their ordering). We next exhibit a general constraint, on A when there is such a similarity.

Let $\lambda \in \sigma(A)$ and suppose that A has the special LU factorization $A = LU$. Then, $A - \lambda I = LU - \lambda I = L(U - \lambda L^{-1})$. Since the subdiagonal of L is totally nonzero, the same is true of L^{-1}. Since $\beta_1, ..., \beta_n$ $(\gamma_1, ..., \gamma_n)$ are the diagonal entries of L (U), the diagonal entries of $U - \lambda L^{-1}$ are

$$\gamma_1 - \frac{\lambda}{\beta_1}, \gamma_2 - \frac{\lambda}{\beta_2}, ..., \gamma_n - \frac{\lambda}{\beta_n}.$$

As L^{-1} is also lower triangular, the sub-and super-diagonal of $U - \lambda L^{-1}$ are both totally nonzero. If one of the diagonal entries $\gamma_i - \dfrac{\lambda}{\beta_i}$ of $U - \lambda L^{-1}$ were 0, then rank$(U - \lambda L^{-1})$ would be at least 2. But, $\gamma_i - \dfrac{\lambda}{\beta_i} = 0$ if and only if $\lambda = \beta_i\gamma_i$. Since rank$(A - \lambda I)$ =rank$(U - \lambda L^{-1})$, and rank$(A - \lambda I)$ is a similarity invariant, we may conclude the following.

Lemma 2.. *Suppose that the nonsingular matrix $A \in M_n$ and $\beta_1, ..., \beta_n, \gamma_1, ..., \gamma_n \in \mathbb{C}$ are given so that $\beta_1 \cdots \beta_n\gamma_1 \cdots \gamma_n = detA$. If there is an i such that $\beta_i\gamma_i = \lambda \in \sigma(A)$ and rank $(A - \lambda I) \leq 1$, then no similarity of A has a special LU factorization relative to $\beta_1, ..., \beta_n, \gamma_1, ..., \gamma_n$.*

Because of the lemma, we call a pair, consisting of a matrix $A \in M_n$ and ordered numbers $\beta_1, ..., \beta_n, \gamma_1, ..., \gamma_n$, such that rank$(A - \beta_i\gamma_i I) = 1$ for some i, *exceptional.*

For an exceptional pair, there is no similarity with a special LU factorization. However, we will show that in other relevant cases there will be a similarity with a special LU factorization, and it is relatively easy to nonderogatorially factor in another way in exceptional cases. Recall that a scalar matrix is one for which $\text{rank}(A - \lambda I) = 0$ for some λ and that these are exceptional in the case of Sourour's theorem. Our exceptional matrices are, somehow, the next closest thing to scalar matrices, as $\text{rank}(A - \lambda I) = 1$. Fortunately, there is still a nonderogatory factorization.

It is informative to revisit the 2-by-2 case with the notion of exceptional matrices in mind. If $A \in M_2$ is not scalar, it is an exceptional matrix when paired with $\beta_1, \beta_2, \gamma_1, \gamma_2$ ($\beta_1\beta_2\gamma_1\gamma_2 = detA \neq 0$) if and only if $\beta_1\gamma_1 \in \sigma(A)$ (equivalently $\beta_2\gamma_2 \in \sigma(A)$). In this event A is similar to

$$\begin{bmatrix} \beta_1\gamma_1 & t \\ 0 & \beta_2\gamma_2 \end{bmatrix}$$

with $t \neq 0$ and

$$\begin{bmatrix} \beta_1 & r \\ 0 & \beta_2 \end{bmatrix} \begin{bmatrix} \gamma_1 & s \\ 0 & \gamma_2 \end{bmatrix}$$

with $\beta_1 s + \gamma_2 r = t$ provides a nonderogatory factorization in which both factors are upper triangular. If $\beta_1\gamma_1 \notin \sigma(A)$ and A is not scalar, then a similarity of A in which $\beta_1\gamma_1$ is the 1,1 entry (which always exists, see [2] or [1]) has both off-diagonal entries nonzero and, thus, has a special LU factorization. This completes an alternative proof of the 2-by-2 case and verifies

Lemma 3.. *If $A \in M_2$ is nonsingular and nonscalar and $\beta_1, \beta_2, \gamma_1, \gamma_2 \in \mathbb{C}$ are such that $\beta_1\beta_2\gamma_1\gamma_2 = detA$, then A is similar to a matrix with special LU factorization for $\beta_1, \beta_2, \gamma_1, \gamma_2$ if and only if A, along with $\beta_1, \beta_2, \gamma_1, \gamma_2$ is not an exceptional pair.*

We will need some machinery to demonstrate a converse to lemma 2, including the case $n = 3$. We begin with a lengthy technical lemma that is more general than the case $n = 3$.

Lemma 4.. *Let $A \in M_3(\mathbb{C})$ be a nonsingular, nonscalar matrix and let $\alpha_1, \alpha_2, ..., \alpha_k \in \mathbb{C}$ be nonzero scalars such that*

$$rank(A - \alpha_i I) > 1, \quad i = 1, ..., k.$$

Then, for any nonzero $\alpha \in \mathbb{C}$ (that, in case A is not nonderogatory, is neither among $\alpha_1, \alpha_2, ..., \alpha_k$ nor $\frac{detA}{\lambda^2}$, in which λ is the repeated eigenvalue), there is a matrix B, similar to A, such that

(i) $detB[1, 2] = \alpha$;
(ii) $rank(B[1, 2] - \alpha_i I) > 1, i = 1, ..., k$;
(iii)(iiia) $detB[1, 2; 1, 3] \neq 0$

(iiib) $detB[1,3;1,2] \neq 0$.

PROOF. Our strategy is as follows. We write B as LU, in which lower (upper) triangular L (U) are chosen so as to ensure that B has desired features $(i), (ii)$ and (iii). Then, the remaining freedom in L and U is used to ensure that B lies in the similarity class of A. For this, we distinguish two cases: the one in which A is nonderogatory and the one in which A is not nonderogatory. In the latter case, we note that the repeated eigenvalue of A can not be among $\alpha_1, \alpha_2, ..., \alpha_k$ because of the rank conditions on A; in the former case, A could also have a repeated eigenvalue that might lie among $\alpha_1, \alpha_2, ..., \alpha_k$.

We note also that the conditions (iii) are precisely equivalent to the statements that the $3,2$ entry of L and the $2,3$ entry of U be nonzero. Suppose

$$B = \begin{bmatrix} 1 & 0 & 0 \\ x_1 & 1 & 0 \\ x_3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & y_3 \\ 0 & \alpha & 1 \\ 0 & 0 & \gamma \end{bmatrix} = \begin{bmatrix} 1 & 1 & y_3 \\ x_1 & \alpha + x_1 & 1 + x_1 y_3 \\ x_3 & \alpha + x_3 & 1 + x_3 y_3 + \gamma \end{bmatrix}.$$

Then, condition (i) and (ii) are automatically satisfied and we choose $\gamma = \dfrac{detA}{\alpha}$, so that $detB = \alpha\gamma = detA$, one of the conditions for similarity. Now

$$B[1,2] = \begin{bmatrix} 1 & 1 \\ x_1 & \alpha + x_1 \end{bmatrix},$$

so that $det(B[1,2] - \alpha_i I) = -\alpha_i x_1 + \alpha - \alpha_i(1 + \alpha - \alpha_i)$.

Since $\alpha_i \neq 0$ and this expression is linear in x_1, we choose $x_1 \neq 0$ to be any one of the infinitely many values that makes it nonzero. There will be another (finite) restriction placed upon x_1 later, but, otherwise, we henceforth imagine it to be data, as are α and γ. At this point all required conditions on B are met, except that it must be a similarity of A.

To ensure that B be a similarity of A, we first choose x_3, y_3 so as to achieve the characteristic polynomial of A. This may require another restriction upon x_1, which may be made without loss of generality. Then we consider the nonderogatory case, which entails one final, and feasible, restriction upon x_1. First suppose that $\Sigma_1 = traceA$ and Σ_2 are the first and second elementary symmetric functions of the eigenvalues of A. Since $detB = detA$, if we arrange Σ_1 as trace B and Σ_2 as the sum of the 2-by-2 minors of B, then B and A will have the same spectra. Thus, by calculation from B we have

$$\Sigma_1 = 2 + \alpha + \gamma + x_1 + x_3 y_3$$

and

$$\Sigma_2 = \alpha + \gamma + 1 + x_1 + x_1\gamma + \alpha x_3 y_3 + \alpha\gamma - x_3 - \alpha x_1 y_3$$

From these we first obtain

$$x_3y_3 = \Sigma_1 - (2 + \alpha + \gamma + x_1),$$

which is nonzero by one more restriction upon x_1. We may then write

$$y_3 = \frac{\Sigma_1 - (2 + \alpha + \gamma + x_1)}{x_3},$$

assuming $x_3 \neq 0$, which will be justified later. Now, upon substitution into the desired expression for Σ_2, we have

$$\Sigma_2 = (1+\alpha)(1+\gamma)+x_1(1+\gamma)+\alpha(\Sigma_1-(2+\alpha+\gamma+x_1))-x_3-\alpha x_1(\frac{\Sigma_1 - (2 + \alpha + \gamma + x_1)}{x_3})$$

or

$$x_3+\frac{\alpha x_1}{x_3}(\Sigma_1-(2+\alpha+\gamma+x_1)) = (1+\alpha)(1+\gamma)+x_1(1+\gamma)-\Sigma_2-\alpha(\Sigma_1-(2+\alpha+\gamma+x_1)).$$

This is a monic quadratic in x_3, which with one more restriction on x_1 (since $\alpha \neq 0$) may be made to have nonzero linear and constant terms, ensuring the existence of a nonzero x_3 and the cospectrality of B with A.

For the nonderogatory case, we need that whenever $\lambda \in \sigma(A)$, $rank(B-\lambda I) = 2$. This is ensured with a few final linear restrictions upon x_1, by choosing x_1 so that $det(B[1,2] - \lambda I) \neq 0$ when $\lambda \in \sigma(A)$. In the end, only finitely many values of x_1, are excluded, ensuring existence of the desired B in the nonderogatory case.

In case A is not nonderogatory, there are two possible Jordan structures for A, as A is not scalar:

$$(1) \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix}, \; b \neq a \;\text{ and }\; (2) \begin{bmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}.$$

We may assume, without loss of generality, that $a = 1$, and then $detA = b$ in case (1) and $det(A) = 1$ in case (2). In either event, because of the hypotheses, no $\alpha_i = 1$, and α cannot be b in case (1) or 1 in case (2).

Now suppose

$$B = LU = \begin{bmatrix} 1 & 0 & 0 \\ x_1 & 1 & 0 \\ x_3 & 1 & 1 \end{bmatrix} \begin{bmatrix} \gamma_1 & 1 & y_3 \\ 0 & \gamma_2 & 1 \\ 0 & 0 & \gamma_3 \end{bmatrix} = \begin{bmatrix} \gamma_1 & 1 & y_3 \\ x_1\gamma_1 & x_1 + \gamma_2 & x_1y_3 + 1 \\ x_3\gamma_1 & x_3 + \gamma_2 & 1 + x_3y_3 + \gamma_3 \end{bmatrix}.$$

Our requirements on B are now that $rank(B - I) = 1$, $\gamma_1\gamma_2 = \alpha$ and $\gamma_1\gamma_2\gamma_3 = b$ in case (1) and $= 1$ in case (2). Then $rank(B - I) = 1$ (assuming B is nonscalar) if

$$det(B[1,2]-I) = det(B-I)[1,2;1,3] = det(B-I)[2,3;1,2] = det(B[2,3]-I) = 0.$$

From these, we obtain that x_1, x_3, y_3 and γ_1 must be chosen so that

$$x_1 = (\gamma_1 - 1)(\gamma_2 - 1)$$

$$1 = (\gamma_2 - 1)(\gamma_3 - 1)$$
$$x_3 = \gamma_2(1 - \gamma_3)x_1$$

and

$$y_3 = \frac{1}{\gamma_2 - 1}, \quad \gamma_2 \neq 1.$$

This yields the solution for B,

$$\begin{bmatrix} -\frac{1}{b}\left(-b\alpha+\alpha^2\right) & 1 & \frac{1}{\alpha}\left(b-\alpha\right) \\ -\frac{1}{b}\frac{-b\alpha+\alpha^2}{b\alpha-b^2}\left(b\alpha+\alpha^3-b\alpha^2\right) & -b\frac{\alpha}{-b\alpha+\alpha^2}+\frac{1}{b\alpha-b^2}\left(b\alpha+\alpha^3-b\alpha^2\right) & \frac{1}{\alpha}\frac{b-\alpha}{b\alpha-b^2}\left(b\alpha+\alpha^3-b\alpha^2\right)+1 \\ -\frac{1}{b(-b+\alpha)}\left(-b\alpha+\alpha^2\right)\left(b-b\alpha+\alpha^2\right) & -b\frac{\alpha}{-b\alpha+\alpha^2}+\frac{1}{-b+\alpha}\left(b-b\alpha+\alpha^2\right) & \frac{b}{\alpha}+\frac{1}{\alpha}\frac{b-\alpha}{-b+\alpha}\left(b-b\alpha+\alpha^2\right)+1 \end{bmatrix}$$

in case (1) and the same, with $b = 1$, in case (2). Of course, $B[1,2]$ necessarily has an eigenvalue 1, so that its other eigenvalue is α (accounting for one of the restrictions). These matrices B complete the proof. □

For general n, we wish to show the following

Theorem 5.. *Let $A \in M_n$ be nonsingular and nonscalar and suppose that $\beta_1, ..., \beta_n, \gamma_1, ...\gamma_n \in \mathbb{C}$ are given so that $\prod_{i=1}^{n} \beta_i\gamma_i = detA$. Then A is similar to a matrix with special LU factorization for $\beta_1, ..., \beta_n, \gamma_1, ...\gamma_n$ if and only if there is no i, $1 \leq i \leq n$, such that rank$(A - \beta_i\gamma_i I) = 1$.*

Proof. Of course, the case $n = 2$ is in lemma 2; the case $n = 3$ is an easy corollary to lemma 4 by taking $k = 3$, $\alpha_1 = \beta_1\gamma_1$, $\alpha_2 = \beta_2\gamma_2$, $\alpha_3 = \beta_3\gamma_3$ and $\alpha = \beta_1\gamma_1\beta_2\gamma_2$. Nota that the restrictions in the lemma, in the not nonderogatory case are consistent with these requirements. Then, the 2-by-2 case may be applied to the upper left block, without disturbing the nonzero almost principal minor conditions that ensure that the LU factorization will be special. Using the determinant condition, the product of the $3,3$ entries of L and U will be $\beta_3\gamma_3$ and these entries can be chosen separately as β_3 and γ_3 without loss of the generalities. This gives

Corollary 6.. *Let $A \in M_3(\mathbb{C})$ be nonsingular and nonscalar and suppose that $\beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3 \in \mathbb{C}$ are given so that $\prod_{i=1}^{3} \beta_i\gamma_i = detA$. Then, A is similar to a matrix with special LU factorization for $\beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3$ if and only if there is no $i, 1 \leq i \leq n$, such that rank$(A - \beta_i\gamma_i I) = 1$.*

We now turn to a proof of theorem 5. The overall strategy is an induction on n. Lemma 1 and corollary 6 supply the initial cases $n = 2, 3$. At the induction step, we need only show that the requirements on A have been transferred to the upper left $(n-1)$-by-$(n-1)$ principal submatrix of some similarity of A, in which the last two almost principal minors are nonzero. It is easy to observe that when induction is applied by a block similarity acting upon the upper left submatrix, the nonzero, almost principal minor condition will not be altered. This ensures that the special LU factorization will extend.

In the event that the rank conditions, $rank(A-\beta_i\gamma_i I) > 1$, are strongly satisfied: $\min_i rank(A-\beta_i\gamma_i I) > 3$, they are obviously conveyed by standard rank inequalities to the upper left submatrix. Then, by observing the relationship between the minors of B and its inverse, the proof is easily completed in this case, using the following two lemmas.

Lemma 7.. *Suppose that $A \in M_2(\mathbb{C})$ is a nonscalar matrix with eigenvalues λ_1, λ_2 and that $\alpha \in \mathbb{C}$ is given. Then, there is a matrix similar to A with α in the $2,2$ position and nonzero entries in the $1,2$ and $2,1$ position if and only if $\alpha \neq \lambda_1, \lambda_2$.*

Proof. The matrix A is similar to

$$\begin{bmatrix} \lambda_1 + \lambda_2 - \alpha & u \\ v & \alpha \end{bmatrix}$$

in which for any $u \neq 0$,

$$v = \frac{\alpha(\lambda_1 + \lambda_2 - \alpha) - \lambda_1\lambda_2}{u}.$$

The scalar v is nonzero, unless $\alpha = \lambda_1$ or $\alpha = \lambda_2$. □

Lemma 8.. *Suppose that $A \in M_n(\mathbb{C})$, $n \geq 3$, is a nonscalar, nonsingular matrix and that $\alpha \in \mathbb{C}$ is given. Then, there is a matrix similar to A with α in the n, n position and nonzero entries in the $n-1, n$ and $n, n-1$ positions.*

Proof. According to the prior lemma, we need only note that there is a similarity of A so that the 2-by-2 principal submatrix lying in the last two rows and columns does not have the eigenvalue α. □

To complete the induction when the rank conditions are not strongly satisfied ($rank(A-\beta_i\gamma_i I) = 2$ *or* 3 for some i) we consider the possible Jordan structures for A in this event. They are either

(1) nondiagonal: $aI \bigoplus J_4(a)$, $aI \bigoplus J_3(a) \bigoplus [b]$, $aI \bigoplus J_2(a) \bigoplus J_2(b)$, $aI \bigoplus J_2(a) \bigoplus [b] \bigoplus [c]$, $aI \bigoplus J_3(a)$, $aI \bigoplus J_2(a) \bigoplus [b]$, $aI \bigoplus J_2(b)$ or
(2) diagonal: $aI \bigoplus [b] \bigoplus [c] \bigoplus [d]$, $aI \bigoplus [b] \bigoplus [c]$.

Here, it is assumed a is among the list $\beta_1\gamma_1, ..., \beta_n\gamma_n$ and is different from b, c or d, but b, c, d or any subset may be equal. Of course, the block aI may be absent in dimension 4 in some cases. In each of the nondiagonal cases, a 3-by-3 principal submatrix may be found, to which lemma 4 is applied. Via permutation similarity, this submatrix may be be placed in the lower right. Then an α may be chosen along with a similarity acting upon this lower right block, so that the induction hypothesis applies to the upper left $(n-1)$-by-$(n-1)$ principal submatrix of the result. Because they held in the 3-by-3 submatrix, by identifying them as a direct summand, the nonvanishing almost principal minor conditions hold in the n-by-n matrix and they are undisturbed by the application of induction via a block similarity on the upper left $(n-1)$-by-$(n-1)$ submatrix. Because of these, the partial special LU factorization delivered by induction extends to the entire matrix.

It remains to consider the diagonal cases (2). In this case it is more convenient to place a 3-by-3 matrix, that is not exceptional, in the upper left corner. These is no loss of generality in illustrating how to do this in case $n = 4$ and

$$A = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \end{bmatrix}.$$

This matrix is clearly similar to one of the form

$$\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & 0 & b \end{bmatrix},$$

in which all the $*$'s are nonzero and none are equal to a or b. This matrix is in turn similar to one of the form

$$\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix},$$

with the same restrictions. Now, the nonvanishing almost principal minor conditions are met and they are unaltered when lemma 4 and corollary 6 are applied to the upper left 3-by-3 block. In this way the induction step may be carried out in the diagonal cases, completing the proof. □

We now turn to nonderogatory factorization in the exceptional cases. Here we take advantage of the fact that, although the β's and γ's are given, we may control their respective ordering (which we have not previously exploited except that equal values be consecutively labelled, which we may continue to assume). It can happen that A, together with $\beta_1, ..., \beta_n, \gamma_1, ...\gamma_n$ is exceptional, but it is not if the β's and γ's are re-ordered. For example,

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

is exceptional with $\beta_1 = 1, \beta_2 = \dfrac{3}{2}, \gamma_1 = 1, \gamma_2 = 2$ (because the eigenvalues of A are 1, and 3) but not with $\beta_1' = \dfrac{3}{2}, \beta_2' = 1, \gamma_1 = 1, \gamma_2 = 2$.

Thus we first show how it can happen that A is exceptional, together with the β's and γ's, no matter how they are ordered. Since for $n > 2$, rank$(A - \lambda I) = 1$ for at most one value of λ, the answer is relatively simple. (For $n = 2$, we have nothing left to do because of lemma 1.) Suppose that $\beta_1, ..., \beta_n, \gamma_1, ...\gamma_n$ and λ are given. If there is a β_i such that for some j, $\gamma_j = \dfrac{\lambda}{\beta_i}$, call β_i, γ_j a $\lambda-$pair. By the *multiplicity* of β_i (γ_j), we mean the number of distinct indices k such that $\beta_k = \beta_i$ ($\gamma_k = \gamma_j$) and write $m(\beta_i)$ and $m(\gamma_j)$. (Note that only the number of equal values among β's, or among the γ's, are counted.) By the *total multiplicity*, $TM(\beta_i, \gamma_j)$, of a $\lambda-$pair β_i, γ_j we mean $m(\beta_i) + m(\gamma_j)$. For at most one $\lambda-$pair could $TM(\beta_i, \gamma_j)$ exceed n. We observe the following

Lemma 9.. *Let $\beta_1, ..., \beta_n; \gamma_1, ...\gamma_n$ and $\lambda \in \mathbb{C}$ be given. It is possible to re-order the β's and γ's: $\beta_{j_1}, ..., \beta_{j_n}; \gamma_{k_1}, ...\gamma_{k_n}$ so that*

(i) *equal β's (resp. γ's) occur consecutively; and*
(ii) $\beta_{j_i}\gamma_{k_i} \neq \lambda$, $i = 1, ..., n$

if and only if there is no pair β_p, γ_q such that $TM(\beta_p, \gamma_q) > n$, with respect to λ.

Proof. The necessity of the condition is obvious. So, we turn to the proof of sufficiency.

We assume, without loss of generality, that there are k distinct β's $\beta_1, ..., \beta_k$ with respective multiplicities $m_1, ..., m_k$ and k distinct γ's $\gamma_1, ..., \gamma_k$ with respective multiplicities $n_1, ..., n_k$ such that $\beta_i\gamma_i = \lambda$, $i = 1, ..., k$. In this event $\sum_{i=1}^{k} TM(\beta_i, \gamma_i) = 2n$, the maximum possible. (Otherwise, unmatched β's or γ's may be identified or grouped as matched $\lambda-$pairs.) The case $k = 1$ cannot occur, by hypothesis, and the case $k = 2$ is easily checked. An ordering is easily constructed, given the assumption.

We next show that the required implication follows inductively, given the case $k = 3$. Suppose $k \geq 4$, and assume without loss of generality that the β's and γ's are ordered by descending total multiplicity. In this event $TM(\beta_{k-1}, \gamma_{k-1}) + TM(\beta_k, \gamma_k) \leq n$, and if we identify β_k (γ_k) with β_{k-1} (γ_{k-1}), our assumption on total multiplicity remains satisfied. When the renamed β's and γ's are properly ordered according to the inductive case $k - 1$, we are at liberty to order within the β_{k-1}, β_k (γ_{k-1}, γ_k) block so that requirement (i) is met. Requirement (ii) is obviously met.

Thus, the proof of sufficiency rests upon the case $k = 3$. In this event, note that since $TM(\beta_i, \gamma_i) \leq n$ and $\sum_{i=1}^{3} TM(\beta_i, \gamma_i) \leq 2n$ we have $TM(\beta_i, \gamma_i) + TM(\beta_j, \gamma_j) \geq n$ for each pair $i \neq j$. Also, for each distinct pair i, j, $1 \leq i, j \leq 3$, and $k \neq i, j$, $1 \leq$

$k \leq 3$, either

$$m_i + m_j + n_i \leq n \text{ or } m_k + n_j + n_k \leq n \qquad (*)$$

Now, suppose there is a distinct pair i, j such that both $m_i + m_j + n_i \leq n$ and $n_i + n_j + m_j \leq n$. Then, we may place first in the β list all the β_j's, then β_i's and in the γ list last all the γ_i's and next to last all the γ_j's. This determines the placement of the remaining β's and γ's in their respective lists and, given the stated inequalities, it is easily checked that for no l, $1 \leq l \leq 3$, is a β_l in the same position in the β list as γ_l in the γ list. Requirement (i) is also met by design.

Why must there be such a pair i, j? Suppose not. Then, of the 6 possible such pairings, each must have one of the triple sums greater than n. A careful inspection of the possibilities, in view of $(*)$ yields that if this occurs, then either: $m_1+m_2+n_1$, $m_2+m_3+n_2$ and $m_1+m_3+n_3 > n$ or $n_1+n_2+m_2$, $n_2+n_3+m_3$ and $n_1 + n_3 + m_1 > n$. However, either leads to a contradiction, as, in either case, the sum of the 3 sums is identically $3n$, so that not all 3 sums can exceed n, completing the proof □

We note that both parts, then, of being exceptional for all orderings of the β's are rather rare, and increasingly so for larger n. We must first have a λ for which $\text{rank}(A - \lambda I) = 1$ and we must have a pair for which $TM > n$; and the pair must be a λ-pair.

Now we suppose that $A \in M_n$, $\text{rank}(A - \lambda I) = 1$, and there is a β_i such that $TM(\beta_i, \frac{\lambda}{\beta_i}) > n$. In this ("unavoidably exceptional") event, we wish to nonderogatorily factor A (giving the correct eigenvalues for the factors) without the benefit of a special LU factorization.

Lemma 10.. *Suppose that $A \in M_n$, $n > 1$, is nonsingular and that $\beta_1, ..., \beta_n, \gamma_1, ..., \gamma_n$ such that $\beta_1 \cdots \beta_n \gamma_1 \cdots \gamma_n = detA$ are given. If $rank(A - aI) = 1$ and for every permutation τ there is an i, $1 \leq i \leq n$, such that $\beta_{\tau(i)}\gamma_i = a$, then there exist $B \in M_n$ with eigenvalues $\beta_1, ..., \beta_n$ and $C \in M_n$ with eigenvalues $\gamma_1, ..., \gamma_n$ such that $A = BC$, and B and C are nonderogatory.*

Proof. Without loss of generality, we may suppose that $a = 1$ and that the $1-$pair β_i, γ_j whose total multiplicity exceeds n is $\beta_i = 1$, $\gamma_j = 1$. The former is by multiplication of A by a scalar, as necessary, which does not change factorizability, and the latter, then, by passing a scalar factor between B and C, as necessary.

By similarity, we may then assume that

$$A = \begin{bmatrix} I & 0 \\ 0 & \begin{bmatrix} 1 & 0 \\ t & \lambda \end{bmatrix} \end{bmatrix},$$

in which $\lambda = detA$, and that $\beta_1 \cdots \beta_r, \gamma_1 \cdots \gamma_s = 1$, with $r+s$ the total multiplicity of the 1−pair 1,1. Then any remaining β's and γ's are not 1, and we suppose that they are ordered so that any equal β (γ) values occur consecutively. Let $m = min\{r,s\} \geq 1$, and we suppose without loss of generality that $m = r$. For convenience, we distinguish three possibilities for consideration: $m < n-1$, $m = n-1$ and $m = n$.

If $m < n-1$, re-partition A as

$$A = \begin{bmatrix} I_m & 0 \\ 0 & A' \end{bmatrix}.$$

Either A' is 2-by-2, in which case it may be factored by lemma 1; A' is non-exceptional for some re-labelling of $\beta_{m+1}, ..., \beta_n, \gamma_{m+1}, ..., \gamma_n$ and its factorizability is reduced to that case; or A' may be further reduced in the manner that A has been so far. In any event, we may assume (inductively, if necessary) that $A' = B'C'$, with B' and C' nonderogatory and having the desired eigenvalues. Let

$$J = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & 1 & \\ & & & 1 & 1 \end{bmatrix}$$

be the m-by-m basic Jordan block associated with 1, and write

$$\begin{aligned} \begin{bmatrix} I & 0 \\ 0 & A' \end{bmatrix} &= \begin{bmatrix} J^{-1} & 0 \\ X & B' \end{bmatrix} \begin{bmatrix} J & 0 \\ Y & C' \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ XJ + B'Y & A' \end{bmatrix}. \end{aligned}$$

Choosing $Y = -B'^{-1}XJ$ and X so that its last column is not in the column space of $B' - I$ (if that matrix is singular, and arbitrarily otherwise) ensures that the first factor is nonderogatory and has the desired eigenvalues. As no eigenvalue of C' is 1, and C' is nonderogatory, the second factor is nonderogatory, regardless of Y. But, because of the choice of Y, the two matrices factor the desired one.

The case $m = n-1$ is similar by re-partitioning A as

$$A = \begin{bmatrix} I_m & 0 \\ 0...0\ t & \lambda \end{bmatrix},$$

writing

$$A = \begin{bmatrix} J^{-1} & 0 \\ x & \beta_n \end{bmatrix} \begin{bmatrix} J & 0 \\ y & \gamma_n \end{bmatrix},$$

choosing $y = \frac{1}{\beta_n}([0\,0\,...\,t] - xJ)$ and $x = [0\,0\,...\,0\,1]$.

If $m = n$, λ must be 1 and we write (for $t \neq 1$, which we may assume)

$$A = \left(J + \begin{bmatrix} 0 & & \cdots & & 0 \\ \vdots & & \cdots & & \vdots \\ 0 & & \cdots & & 0 \\ 0 & \cdots & 0 & t & t & 0 \end{bmatrix} \right) J^{-1}$$

to give the desired factorization, which complete the proof. □

Theorem 11.. *Let A be an n-by-n nonsingular nonscalar matrix and suppose that $\beta_1, ..., \beta_n, \gamma_1, ...\gamma_n \in \mathbb{C}$ are given complex numbers, repetitions allowed, so that*

$$\prod_{i=1}^{n} \beta_i \gamma_i = detA.$$

Then, there exist nonderogatory matrices B with eigenvalues $\beta_1, ..., \beta_n$ and C with eigenvalues $\gamma_1, ...\gamma_n$ such that

$$A = BC.$$

Proof. In the event that neither the β's nor the γ's include repetitions, there is nothing to prove. If repetitions do occur and the β's and γ's may be ordered so that repeats among the β's occur consecutively and among the γ's occur consecutively, and A is not exceptional for $\beta_1\gamma_1, ..., \beta_n\gamma_n$, then the factorization is guaranteed by theorem 5. If A is exceptional for any pair of consecutive orderings, then the result follows from lemma 10.

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