Learning rational numbers – a study on 6th grade students' (mis)conceptions of fractions

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Abstract
This study compares students' understanding of fractions across quotient, part-whole and operator interpretations of fractions. Two questions were addressed: (1) How do students understand the equivalence and ordering of fractions in these interpretations? And (2) how do students master the fraction representation in these interpretations?

A survey was conducted using an individual questionnaire with 11 and 12-year-olds Portuguese students (N=158), who were familiar predominantly with part-whole and operator interpretations, but not with the quotient interpretation. A quantitative analysis showed that students performed better on equivalence and ordering tasks presented in quotient interpretation than in part-whole and operator interpretations; they performed better on labelling tasks in part-whole and operator interpretations than in quotient interpretations. Educational implications of these results will be discussed.

Background

Fractions are undoubtedly one of the most problematic topics in mathematics education (Behr, Wachsmuth, Post & Lesh, 1984; Kerslake, 1986; Kieren, 1993; Streefland, 1991). Several authors suggest that the knowledge of fractions demands the understanding of ordering and equivalence of fractions and the ability to use distinct modes of fractions representation, in different interpretations of this concept (Behr, Wachsmuth, Post & Lesh, 1984; Nunes, Bryant, Pretzlik, Wade, Evans & Bell, 2004; Mamede, 2008; Mamede & Nunes, 2008).

Literature presents distinct classifications of interpretations that might offer a fruitful analysis of the concept of fraction. Behr, Lesh, Post and Silver (1983) distinguished part-whole, decimal, ratio, quotient, operator, and measure as subconstructs of rational number concept; Kieren (1993) considers measure, quotient, ratio and operator as mathematical subconstructs of rational number; also Marshall (1993) based on the notion of *schemas* presents a similar classification of Behr's and colleagues distinguishing five situations: part-whole, quotient, measures, operator, and ratio. Mack (2001) proposed a different classification of interpretations using the term 'partitioning' to cover both part-whole and quotient interpretation. More recently, and following the theory of Vergnaud (1997), which emphasizes the importance of the *situations* in concept formation, Nunes, Bryant, Pretzlik, Wade, Evans and Bell (2004) presented a classification based on the notion of situation distinguishing quotient, part-whole, operator and intensive quantities situations, according to the number meanings that occur in each situation.

In spite of the differences, part-whole, quotient and operator are among the interpretations identified by all of them. However, there is not much research producing no unambiguous evidence about whether students behave differently when different situations are used. Literature provides information about students' difficulties and misunderstanding with fractions (see Behr, Wachsmuth, Post & Lesh, 1984; Hart, 1981; Kerslake, 1986; Mamede & Nunes, 2008). Nevertheless, little research has been produced on the effects of situations on students' understanding of fractions. This paper focuses on students' conception of fraction on part-whole, quotient and operator situations of fractions.

In part-whole situations, the denominator designates the number of parts into which a whole has been cut and the numerator designates the number of parts taken. So, 2/4 in part-whole situation means that a whole – for example – a chocolate was divided into four equal parts, and two were taken (Nunes *et al.*, 2004). In quotient situations, the denominator designates the number of recipients and the numerator designates the number of items being shared. In this situation, 2/4 means that 2 items – for example, two chocolates – were shared among four people. Furthermore, it should be noted that in quotient situations a fraction can have two meanings: it represents the division and also the amount that each recipient receives, regardless of how the chocolates were cut. For example, the fraction 2/4 can represent two chocolates shared among four children and also can represent the part that each child receives, even if each of the chocolates was only cut in half each (Mack, 2001; Nunes *et al.*, 2004). Finally, in an operator situation, the denominator designates the number of equal groups into which a set was divided and the numerator designates the number of groups taken. In operator situations, the connection between the numbers that describe the situation and the fraction is created by operating on these numbers. For example, if Bill has 12 sweets and eats 2/4 of them, the numbers 2 and 4 are not perceived directly in the situation; this means that one has to divide the set of sweets into 4 and take 2 groups (Nunes *et al.*, 2004).

By the end of sixth grade (11- and 12-year-olds) Portuguese students are supposed to be fully acquainted with the labelling, ordering and equivalence of fractions in different situations. Nevertheless, Portuguese national tests suggest that students present some misconceptions on this domain. Do part-whole, quotient and operator situations affect students' understanding of fractions?

Methods

Participants

Seven classes of Portuguese sixth-grade students (N=158), aged 11 and 12 years, from two schools of the city of Braga, in Portugal, participated in this study. All the participants gave informed consent and permission for the study was obtained from their teachers. The two participating schools were attended by students from a range of socio-economic backgrounds.

The teachers of the participants of this study informed the researchers about the type of situations that students were familiarized with. These situations included predominantly part-whole and operator situations, whereas quotient situations were referred by them as a situation poorly explored in the classroom.

Design

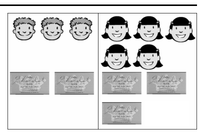
In order to have an insight of students' understanding of fractions when different situations are involved, a survey was carried out using an individual questionnaire comprising tasks related to ordering and equivalence of fractions, and labelling of fractions (with pictorial and verbal support). These types of tasks were presented in quotient, part-whole and operator situations. Tasks involving only the formal symbolic representation of fractions, without any explicit situation, were presented as well and are referred here as algebraic representation.

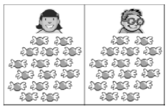
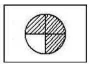
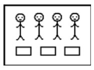
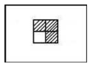
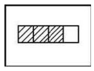
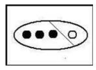
The questionnaire

The questionnaire comprises 30 tasks: 7 presented in quotient situations (QT) (2 of ordering of fractions, 2 of equivalence of fractions, 3 of representation); 7 presented in part-whole situations (PW) (2 of ordering of fractions, 2 of equivalence of fractions, 3 of representation); 7 in operator situations (OP) (2 of ordering of fractions, 2 of equivalence of fractions, 3 of representation); and 9 without any explicit situations, using only algebraic representation (AL). The tasks of the questionnaire were inspired on the studies of Kerslake (1986), Nunes *et al.* (2004) and Streefland (1991). The fractions involved in the tasks were all smaller than one.

Table 1 shows an example of a task presented in each type of situation and also an example involving the three situations. The fractions were the same across the situations, according to the type of task. Thus, for instance, an ordering task involving 2/3 and 3/5 had a correspondent task presented also in part-whole and operator situations.

Table 1: Examples of tasks presented involving different situations

Situation	Problem	Example
QT	Ordering	Three boys are going to share fairly 2 chocolate bars. Five girls are going to share fairly 3 chocolate bars. Tick the right statement:  ☐ Each boy eats more than each girl; ☐ Each girl eats more than each boy; ☐ Each boy and each girl eat the same amount of chocolate Write the number that represents the amount of chocolate eaten by each child.
PW	Representation (verbal support)	Bill ordered a pizza and divided it into 4 equal parts. He decided to eat 3 of them. What part of pizza did Bill eat? ☐ $\frac{4}{3}$ ☐ $\frac{3}{4}$ ☐ $\frac{1}{4}$ ☐ $\frac{1}{3}$ ☐ 3 ☐ Other: __

OP	Equivalence	Rita and Lewis have 16 caramels each. Rita ate $\frac{3}{4}$ of the caramels. Lewis ate $\frac{6}{8}$ of the caramels.  Tick the right statement. ☐ Rita ate more caramels than Lewis; ☐ Lewis ate more caramels than Rita; ☐ Rita and Lewis ate the same amount of caramels.
QT, PW, OP	Labeling (pictorial support)	Which figure(s) best represents the fraction $\frac{3}{4}$? ☐  ☐  ☐  ☐  ☐ 

An example of a task without any explicit situations, using only algebraic representation, is listed on Table 2.

Table 2: Example of a task using only algebraic representation

Situation	Problem	Example
AL	Equivalence	Tick the right statement: ☐ $\frac{6}{8}$ is two times $\frac{3}{4}$; ☐ $\frac{3}{4}$ and $\frac{6}{8}$ are equivalent fractions; ☐ $\frac{3}{4}$ is smaller than $\frac{6}{8}$; ☐ $\frac{6}{8}$ is found by multiplying $\frac{3}{4}$ by 2; ☐ $\frac{3}{4}$ is two times $\frac{6}{8}$.

Results

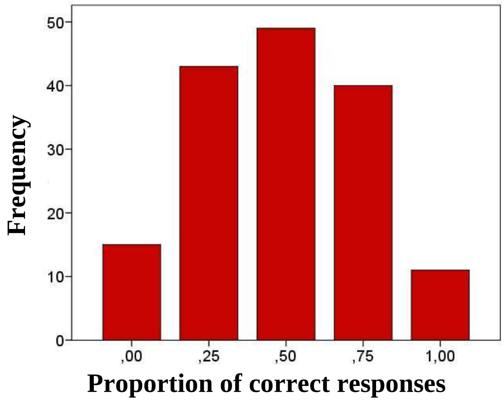
Descriptive statistics of students' performance on the tasks for each working situation are presented in Table 3, reporting the proportions of correct responses and standard deviations by task and situation. As the problems of ordering and equivalence relate to quantities represented by fractions, they demand the understanding of basic logical aspects of fractions. Thus, the ordering and equivalence problems will be referred here as logic of fractions problems.

Table 3: Proportions of correct answers and (standard deviation) by task and situation (N=158)

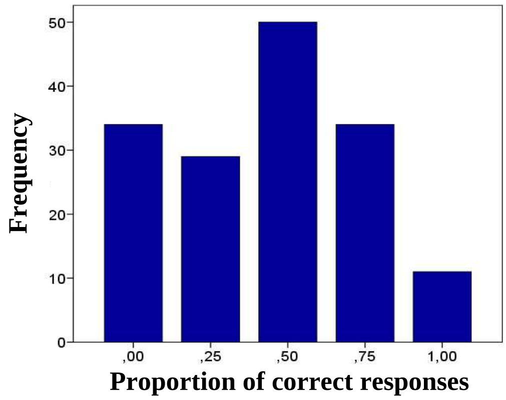
	Task	
Situation	Logic of fractions	Labelling of fractions
Quotient	.48 (.27)	.20 (.24)
Part-whole	.44 (.30)	.78 (.29)
Operator	.35 (.29)	.37 (.29)
Algebraic	.23 (.19)	.86 (.13)

The following graphs illustrate the distribution of correct responses on the logic of fractions problems (ordering and equivalence) presented in quotient situation (Graph 1), in part-whole situation (Graph 2), in Operator situation (Graph 3) and in problems presented using the formal algebraic representation (Graph 4).

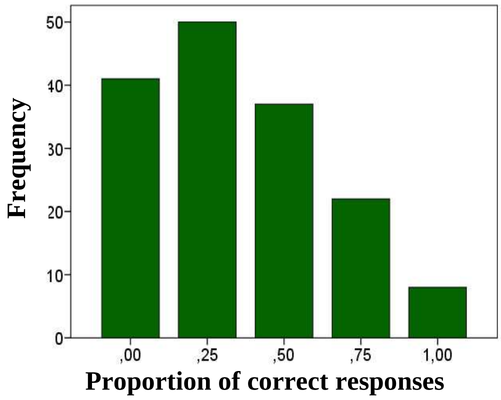
Graph 1: Proportion of correct answers on problems of ordering and equivalence of fractions presented in quotient situation



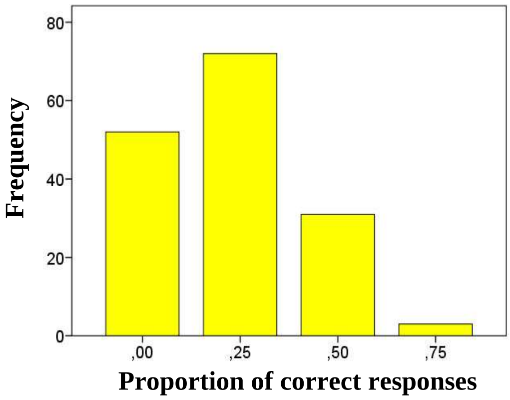
Graph 2: Proportion of correct answers on problems of ordering and equivalence of fractions presented in part-whole situation



Graph 3: Proportion of correct answers on problems of ordering and equivalence of fractions presented in operator situation



Graph 4: Proportion of correct answers on problems of ordering and equivalence of fractions presented using algebraic



Graph 1 shows that 63.3% of the students gave a correct response in at least half of the questions of logic of fractions (ordering and equivalence) presented in quotient situation; and 9.5% could not succeed in any of the logic of fraction problems presented in this type of situation. When quotient situations were used, only 7% of students gave a correct response to all of the logic of fractions problems presented; and 31% of the students gave a correct response to half of the ordering and equivalence questions.

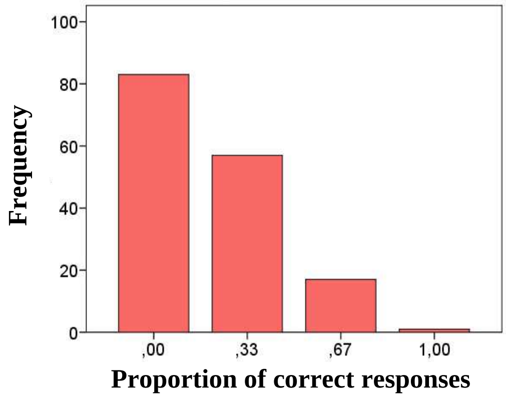
Graph 2 shows that 60.1% answered correctly at least to half of the logic of fractions problems (ordering and equivalence) presented in part-whole situation; and 21.5% of the students could not succeed in any of these problems, when presented in part-whole situation. When this type of situations was used to present the problems 31.6% of the students answered correctly to half of the ordering and equivalence problems and 7% of them could get all the problems correctly solved.

In operator situations, the students' performance on problems of ordering and equivalence of fractions is even lower, as only 42.4% of the students were able to present one correct response to at least half of the presented questions. When this type of situations was used, 5.1% of the students solved correctly all the problems of logic of fractions presented to them; and 25.9% gave no correct answer to any of these problems; only 23.4% of the students gave a correct answer to half of the questions.

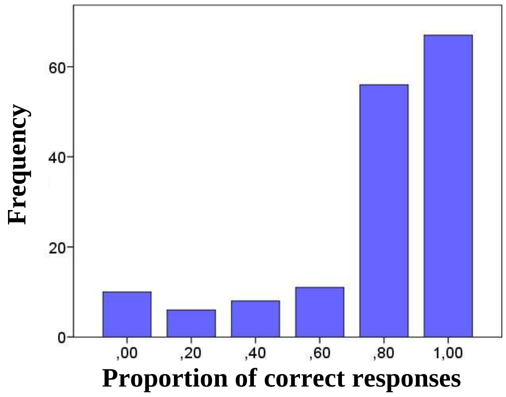
Graph 4 shows that only 47.5% of the students gave a correct response in at least half of the logic of fractions problems (ordering and equivalence) presented in problems using formal algebraic representation; and 1.3% could not succeed in any of the logic of fraction problems presented in this type of situation. When problems using algebraic representation were used, none of the students gave a correct response to all of the logic of fractions problems presented; and 8.2% of the students gave a correct response to half of the ordering and equivalence questions.

The following graphs illustrate the distribution of correct responses on the representation of fractions problems presented in quotient situation (Graph 1), in part-whole situation (Graph 2), in Operator situation (Graph 3) and in problems presented using the formal algebraic representation (Graph 4).

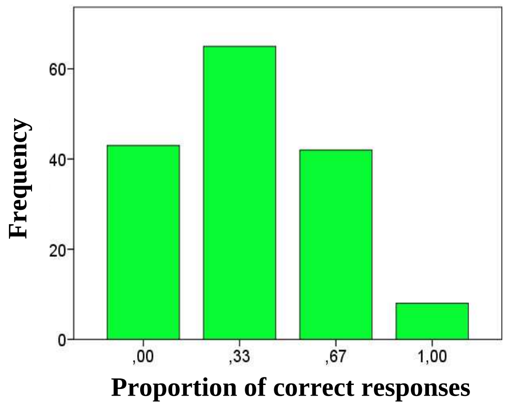
Graph 5: Proportion of correct answers on problems of representation of fractions presented in quotient situation



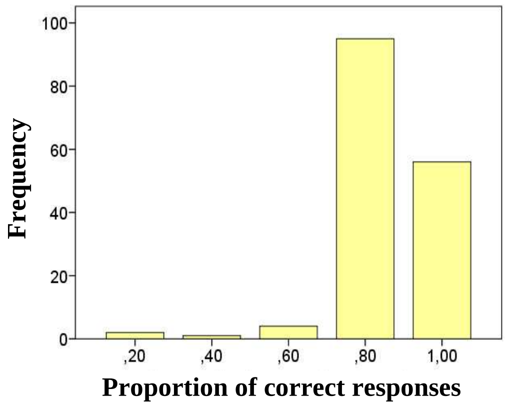
Graph 6: Proportion of correct answers on problems of representation of fractions presented in part-whole situation



Graph 7: Proportion of correct answers on problems of representation of fractions presented in operator situation



Graph 8: Proportion of correct answers on the representation of fractions presented using formal algebraic representation



Graph 5 shows that 11.4% of the students answered correctly at least to two-thirds of the problems of representation of fractions presented in quotient situation; and 52.5% of the students could not succeed in any of these problems, when presented in quotient situation. When this type of situations was used to present the problems, 10.8% of the students answered correctly to exactly two-thirds of the representation problems and 0.6% of them could get all the problems correctly solved.

Graph 6 shows that 77.8% of the students answered correctly to at least to 80% of the representation problems presented in part-whole situation; and 6.3% of the students could not succeed in any of these problems, when presented in part-whole situation. When this type of situations was used to present the problems, 35.4% of the students answered correctly exactly to 80% of the representation problems and 42.4% of them could get all the problems correctly solved.

Graph 7 shows that 31.7% answered correctly at least to two-thirds of the representation problems presented in operator situation; and 27.2% of the students could not succeed in any of these problems, when presented in operator situation. When this type of situations was used to present the problems, 26.6% of the students answered correctly to exactly 67% of the representation problems and 5.1% of them could get all the problems correctly solved.

Graph 8 shows that only 95.5% of the students gave a correct response in at least 80% of the questions of representation of fractions presented in problems using only the algebraic form; and all students answer correctly to at least one problem of this type. When problems using algebraic representation were used, 35.4% of the students gave a correct response to all of the representation of fractions problems presented; and 60.1% of the students gave a correct response to exactly 80% of the representation questions.

Students' performance on logic of fractions problems (ordering and equivalence) was better when quotient situation was involved. Their success was lower when problems of logic of fractions were presented in operator situation and when this type of problems was presented using the formal algebraic representation. Concerning representation of fractions problems, students' performance was better either when part-whole situation or formal algebraic representation was involved.

An ANOVA was conducted to analyse the effect of type of situation (quotient (QT), part-whole (PW), operator (OP), algebraic (AL)) and type of problem (logic of fractions, labelling) on students' performance. There was an interaction effect of situation × problem on students' performance, $F(3, 471)= 172.57$ ($p<.001$), indicating that the type of situation affects students' performance on the tasks. Paired contrasts showed that students performed significantly better on the logic of fractions (ordering and equivalence) tasks presented in QT situations than in AL situations; they performed significantly better on labelling tasks presented in AL than in QT situations ($p<.001$). Students performed better on logic of fractions tasks presented in OP than in AL situations ($p<.001$), but they performed better on labelling tasks in AL than in OP situations ($p<.001$). The students' performance on tasks involving the logic of fractions (ordering and equivalence) in QT situations is significantly better than their performance on labelling fractions; in PW situations the students' performance is better on labelling tasks than on logic of fractions tasks; in OP situations there is no significant differences on students' performance according to the type of task; and in AL situations the students' performance is better on labelling tasks than in logic of fractions tasks. Table 5 shows the adjusted means and standard errors of students' performance according to the type of problem they were solving and to the type of situation in which the problem was presented. These results reveal that the type of situation seems to affect students' conception of fractions.

Table 5: Adjusted Means and Standard Errors (in brackets) of students performance by type of problem and type of situation (N=158)

	Type of Problem	
Type of situation	Ordering and equivalence	Labelling
Quotient	.48 (.02)	.20 (.02)
Part-whole	.44 (.02)	.78(.02)
Operator	.35 (.02)	.37 (.02)
Algebraic	.23 (.02)	.86 (.01)

The type of situation in which fractions are used seems to affect differently the students understanding of the concept of fraction. These students were not familiar with quotient situations but they were able to success in solving ordering and equivalence problems presented in this situation, in spite of failing in labelling tasks in this type of situations. Their success in these situations suggests that quotient situations easily match with students' informal knowledge of fractions.
The results also indicate that students can succeed in solving fractions representations problems even without mastering the logic issues of fractions, such as ordering and equivalence of fractions. This was the case of their performance on solving representation problems in part-whole situations and formal algebraic representation.

Our findings also suggest that operator situations are more difficult for students than part-whole and quotient situations. Their levels of success were even lower than those achieved when quotients situations were used concerning the ordering and equivalence problems. Nevertheless, this type of situation was part of their formal instruction of fractions.

Discussion and conclusions

The findings of this research suggest that the type of situation in which fractions are used affects students' understanding of fractions, and that students' success on algebraic problems does not guarantee the understanding of the concept of fractions.

These results converge with the results of previous research carried out by Nunes *et al.* (2004), who conducted a survey involving 9 and 11-year-olds students (N=130), to analyse their performance in solving equivalence problems in quotient and part-whole situations. Their results showed that students succeeded on 35% of problems presented in part-whole, contrasting with 66% of success achieved in quotient situation. Also Mack (1990) conducted a research with students of 11-12-years-old (N=8) to analyse their understanding of fractions building on informal knowledge. Mack's results showed that four out of five students could not compare 1/6 and 1/8 given symbolically; however, these students were able to solve this problem easily when quotient situations were involved.

The idea that the type of situations in which fractions are used affects students understanding of the concept of fraction has becoming more consistent. More recently, Mamede (2008) conducted an intervention to analyse the effects of different situations on children's understanding of fractions when building on their informal knowledge of fractions. The study involved 6-7-year-olds children (N=37) who were introduced to fractions using quotient, part-whole and operator situations. These children had no previous formal instruction about fractions. The results showed that children performed better on solving equivalence and ordering problems involving quantities represented by fractions in quotient situations, and presented poor performances when part-whole and operator situations were involved. These results suggest that children easily build on their informal knowledge when quotient situations are involved, but not so easily when part-whole and operator situations are involved. In agreement with Streefland (1991), who referred that fractions evolve from everyday experience of fair sharing, quotient situations seems to be relevant situations to build on students' informal knowledge of fractions.

Another relevant issue that emerged from our study concerns the wrong idea that students who succeed on tasks of algebraic representation of fractions may transmit to their teachers. Students are able to succeed easily on these tasks even if they only possess a poor understanding of ordering and equivalence of fractions. Perhaps this is due to the fact of learning labels is easier than learning logical aspects of fractions. It is possible for students to succeed on labelling problems, in spite of misunderstand fractions in different situations.

The effect of situations in which fractions are used on students understanding of fractions is a relevant issue for the students' acquisition of number. More research is needed to address these issues in order to help students to overcome their difficulties.

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