Drazin-Moore-Penrose invertibility in rings*

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Abstract

Characterizations are given for elements in an arbitrary ring with involution, having a group inverse and a Moore-Penrose inverse that are equal and the difference between these elements and EP–elements is explained. The results are also generalized to elements for which a power has a Moore-Penrose inverse and a group inverse that are equal.

As an application we consider the ring of square matrices of order m over a projective free ring R with involution such that R^m is a module of finite length, providing a new characterization for range-Hermitian matrices over the complexes.

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1 Introduction

Throughout the paper and unless otherwise specified, R denotes an arbitrary ring with identity 1, $\mathsf{Mat}_{m\times n}(R)$ the set of $m\times n$ matrices and $\mathsf{Mat}_m(R)$ the ring of $m\times m$ matrices over R.

An involution $*$ in a ring is a unary operation $a\to a^*$ such that

$$(a^*)^*=a,(ab)^*=b^*a^*,(a+b)^*=a^*+b^*,$$

for all elements a,b of a ring.

Given $a\in R$, a is *(von Neumann) regular* if there exists $a^-\in R$ such that

$$aa^-a=a.$$

The set of von Neumann inverses of a will be denoted by $a\{1\}$. That is,

$$a\{1\}=\{x\in R:axa=a\}.$$

a is said to be *Moore-Penrose (MP) invertible* with respect to * , see [15] and [19], if there exists a $a^\dagger$ such that:

$$\left\{\begin{array}{rcl} aa^\dagger a & = & a\\ a^\dagger aa^\dagger & = & a^\dagger\\ \left(aa^\dagger\right)^* & = & aa^\dagger\\ \left(a^\dagger a\right)^* & = & a^\dagger a.\end{array}\right. \tag{1}$$

If the Moore-Penrose with respect to * exists then it is unique, see [1].

Necessary and sufficient conditions for the existence as well as expressions for $a^\dagger$ can be found in [16], [17], [22] and [23].

Also, the *group inverse* of a exists if there is a $a^\#$ such that

$$\left\{\begin{array}{rcl} aa^\# a & = & a\\ a^\# aa^\# & = & a^\#\\ aa^\# & = & a^\# a.\end{array}\right. \tag{2}$$

If the group inverse exists then it is unique, see [1].

Necessary and sufficient conditions for the existence as well as expressions for $a^\#$ can be found in [21].

An element $a\in R$ is said to have a *Drazin inverse* if there exists $x\in R$ such that

$$\left\{\begin{array}{rcl} a^m & = & a^{m+1}x,\ \text{for some non-negative integer } m\\ x & = & x^2a\\ ax & = & xa.\end{array}\right. \tag{3}$$

If a has a Drazin inverse, then the smallest possible non-negative integer involved in (3) is called the *Drazin index* of a. We denote by a^{D_k} the *Drazin inverse of index* k of a.

As for group and Moore-Penrose inverses, if the Drazin inverse exists then it is unique, see [1], [20].

In [1], the authors define the notion of "range -Hermitian" matrix A over the field $\mathbb{C}$ of complex numbers as a matrix satisfying $Im\, A = Im\, A^+$, in which A^+ denotes the hermitian conjugate of A. This is clearly equivalent with $A\,\mathsf{Mat}_n(\mathbb{C}) = A^+\,\mathsf{Mat}_n(\mathbb{C})$ and generalizes the notion of hermitian matrix. Then it is known, see [1, pg 164], that a complex matrix A is range-Hermitian iff $A^\# = A^\dagger$ with respect to the involution $^+$. They refer also to the concept of EP_r matrix introduced by H. Schwerdtfeger in 1950. There, however, EP_r matrices are matrices A of rank r over the complexes satisfying $Im\, A = Im\, A^T$, in which A^T denotes the transpose of A. This is clearly equivalent with $A\,\mathsf{Mat}_n(\mathbb{C}) = A^T\,\mathsf{Mat}_n(\mathbb{C})$. The matrix

$$\begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix} = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$$

over the field $\mathbb{C}$ of complex numbers is an EP_1 matrix by a theorem of H. Schwerdtfeger, see page 131 of [27], but this matrix is clearly not range-Hermitian. This shows that the concept of EP_r matrices was introduced with respect to the involution ^{T} on $\mathsf{Mat}_n(\mathbb{C})$. Therefore, we can avoid this misunderstanding about EP in $\mathsf{Mat}_n(\mathbb{C})$ by using the different notions of $^+$–EP and ^{T}–EP in $\mathsf{Mat}_n(\mathbb{C})$.

The generalization of the notion of EP_r-matrices to an EP-morphism ϕ in a category appeared in [25] as a morphism ϕ such that ϕ and ϕ^* have images and co-images and $im\,\phi = im\,\phi^*$, $coim\,\phi = coim\,\phi^*$. Here, it is clear that EP means *–EP.

The notion of EP was also used by R.E. Hartwig, see [6], for elements in a *-regular ring, which are rings with the property that every element of it has a Moore-Penrose inverse with respect to *. Indeed, he defined an element a in a *–regular ring EP iff $aR = a^*R$ and showed that this is equivalent with the existence of $a^\#$ together with $a^\# = a^\dagger$. Here, it is also clear that EP in a *–regular ring means *–EP. It generalizes $^+$–EP, but not ^{T}–EP, in $\mathsf{Mat}_n(\mathbb{C})$ since $\mathsf{Mat}_n(\mathbb{C})$ is a $^+$–regular ring and not a ^{T}-regular ring.

But, defining *–EP in rings R with involution * as elements a for which $aR = a^*R$ and expect an equivalence with $a^\dagger = a^\#$, as for *–regular rings, is not possible. Indeed, an element a in a ring R with involution * can have the property that $aR = a^*R$ without having a MP-inverse with respect to the involution *.

As a consequence, there is the problem of characterizing the elements in a ring with involution * having a group inverse $a^\#$ and a MP-inverse $a^\dagger$ with respect to *, that are equal. These elements can be called **–group–Moore-Penrose* (*–

gMP) invertible and we show that these elements can be characterized by means of classical invertibility together with an equivalence. Moreover, there is a parallel with a result of I.J. Katz for range-Hermitian matrices over the complexes.

We also define the elements in a ring with involution * for which for some smallest natural k, $\left(a^k\right)^\# = \left(a^k\right)^\dagger$ with respect to the involution *. These elements are called *–Drazin-Moore-Penrose (*–DMP) invertible of index k. Among other characterizations, we show that a is *–DMP if and only if the core part of a is *–gMP invertible.

As an application, we characterize the $^+$–DMP invertibility in the ring of square matrices of order m over a projective free ring R with involution $^-$ such that R^m is a module of finite length, providing a new characterization for range-Hermitian matrices over the complexes.

2 Results

In a ring R with involution *, we introduce the following

Definition 1. 1. *An element a in a ring R with involution * is called* *–EP *if $aR = a^*R$.*

2. *An element a in a ring R with involution * is called* *–group-Moore-Penrose *(*–gMP) invertible, if $a^\dagger$ and $a^\#$ exist and $a^\dagger = a^\#$.*

Remarks.

1. The matrix $A = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$ over the field $\mathbb{C}$ of complex numbers is clearly ^{T}–EP but not $^+$–EP (not range Hermitian) since $A\,\mathsf{Mat}_2(\mathbb{C}) = A^T\,\mathsf{Mat}_2(\mathbb{C})$ and $A\,\mathsf{Mat}_2(\mathbb{C}) \neq A^+\,\mathsf{Mat}(\mathbb{C})$.

2. In the ring $\mathbb{Z}$ of integers with respect to the identity involution $\iota : n \to n$, all elements are ι–EP but only $0, 1, -1$ are ι–gMP.

3. In *–regular rings, such as $\mathsf{Mat}_n(\mathbb{C})$ with respect to the involution "hermitian conjugate", an element is *–EP iff it is *–gMP, see [6].

Proposition 2. *Given a in a ring R with involution *, the following conditions hold:*

1. *If $aR = a^*R$ then $a^\dagger$ exists with respect to * iff $a^\#$ exists, in which case $a^\dagger = a^\#$.*

2. *If $a^\dagger$ exists with respect to *, $a^\#$ exists and $a^\dagger = a^\#$ then $aR = a^*R$.*

Proof. (1) Suppose $aR = a^*R$ and $a^\dagger$ exists. Then also $Ra = Ra^*$ and

$$a \in aa^*R \cap Ra^*a = a^2R \cap Ra^2,$$

which implies the group invertibility of a, see [7] or [24, page 145]. Analogously, if $aR = a^*R$ and $a^\#$ exists then $a^\dagger$ exists, see [22, page 133].

In order to show $a^\# = a^\dagger$, it follows from $aR = a^*R$ and the definition of $a^\dagger$ that

$$a^\dagger R = a^*R = aR = a^{\dagger *}R$$

which imply

$$a^2R = a^\dagger R = a^{\dagger *}R = a^{*2}R.$$

So, there exist $y, z \in R$ such that $a^\dagger = a^2y, a^{\dagger *} = a^{*2}z^*$ and $a^2y = a^\dagger = za^2$. Therefore, $a^2(aya) = a = (aza)a^2$ which implies $a^\# = (aza)a(aya)$ (see [7, page 45]). This gives

$$\begin{aligned} aa^\# &= a(aza)a(aya) \\ &= a^2a^\dagger aya \\ &= a^2ya = a^\dagger a \end{aligned}$$

which is symmetric with respect to the involution $*$. Similarly,

$$\begin{aligned} a^\# a &= (aza)a(aya)a \\ &= azaa^\dagger a^2 \\ &= aza^2 = aa^\dagger \end{aligned}$$

and $a^\# a$ is also symmetric with respect to the involution $*$. This leads to $a^\dagger = a^\#$, by the uniqueness of the Moore-Penrose inverse.

(2) The proof is clear since $aR = aa^\dagger R = a^\dagger aR = a^*a^{\dagger *}R = a^*R$. □

Corollary 3. *The following conditions are equivalent:*

1. *a is $*$–gMP.*
2. *a is $*$–EP and $a^\#$ exists.*
3. *a is $*$–EP and $a^\dagger$ exists with respect to $*$.*

Recently, see [21], the group inverse $a^\#$ of a von Neumann regular element a in a ring has been characterized by the invertibility of the element $a^2a^- + 1 - aa^-$, or equivalently, by the invertibility of the element $a^-a^2 + 1 - a^-a$. Moreover,

$$a^\# = \left(a^2a^- + 1 - aa^-\right)^{-2} a = a\left(a^-a^2 + 1 - a^-a\right)^{-2}.$$

Also recently, see [16], [17], the Moore-Penrose inverse $a^\dagger$ of a von Neumann regular element a in a ring has been characterized by the invertibility of the element $aa^*aa^- + 1 - aa^-$, or equivalently by the invertibility of the element $a^-aa^*a + 1 - a^-a$. Moreover,

$$a^\dagger = a^* \left(aa^*aa^- + 1 - aa^-\right)^{*\,-1} = \left(a^-aa^*a + 1 - a^-a\right)^{*\,-1} a^*.$$

We now combine these two results to obtain the following characterization:

Theorem 4. *Let R be a ring with identity and with ring involution $*$. If a is von Neumann regular in R and if a^- denotes a von Neumann inverse then the following are equivalent and independent from the choice of a^-:*

1. *a is $*$–gMP.*
2. *$aa^*aa^- + 1 - aa^-$ and $a^2aa^- + 1 - aa^-$ are invertible and*
$$\left[\left(aa^*aa^- + 1 - aa^-\right)^{-1} a\right]^* = \left(a^2aa^- + 1 - aa^-\right)^{-1} a.$$
3. *$a^-aa^*a + 1 - a^-a$ and $a^-aa^2 + 1 - a^-a$ are invertible and*
$$\left[a\left(a^-aa^*a + 1 - a^-a\right)^{-1}\right]^* = a\left(a^-aa^2 + 1 - a^-a\right)^{-1}.$$

Moreover, if $u = a^2aa^- + 1 - aa^-$, $v = a^-aa^2 + 1 - a^-a$, $\tilde{u} = aa^*aa^- + 1 - aa^-$ and $\tilde{v} = a^-aa^*a + 1 - a^-a$ then

$$a^\# = a^\dagger = u^{-1}a = av^{-1} = \left(\tilde{u}^{-1}a\right)^* = \left(a\tilde{v}^{-1}\right)^*$$

and equals $a\left(a^2\right)^- a \left(a^2\right)^- a$.

Proof. Follows directly from the results in [17] and [21] if we can replace $a^2a^- + 1 - aa^-$ by $a^2aa^- + 1 - aa^-$, and analogously $a^-a^2 + 1 - a^-a$ by $a^-aa^2 + 1 - a^-a$. Indeed,

$$a^2a^- + 1 - aa^-$$

is invertible iff

$$\begin{aligned}
\left(a^2a^- + 1 - aa^-\right)^2 &= \left(a^2a^- + 1 - aa^-\right)\left(a^2a^- + 1 - aa^-\right) \\
&= a^2a^-a^2a^- + 1 - aa^- \\
&= a^3a^- + 1 - aa^-
\end{aligned}$$

is invertible. Then,

$$\begin{aligned}
\left(a^2a^- + 1 - aa^-\right)^{-2} &= \left[\left(a^2a^- + 1 - aa^-\right)^2\right]^{-1} \\
&= \left(a^3a^- + 1 - aa^-\right)^{-1}.
\end{aligned}$$

The remaining fact to prove is that $a^{\#} = a^{\dagger} = a\left(a^2\right)^{-} a \left(a^2\right)^{-} a$. Indeed, if $a^{\#}$ exists then a^2 is von Neumann regular and

$$\left(a^2a^- + 1 - aa^-\right)^{-1} = a\left(a^2\right)^- aa^- + 1 - aa^-$$

since

$$\begin{aligned}
\left(a^2a^- + 1 - aa^-\right)\left(a\left(a^2\right)^- aa^- + 1 - aa^-\right) &= a^2a^-a\left(a^2\right)^- aa^- + 1 - aa^- \\
&= a^2\left(a^2\right)^- aa^- + 1 - aa^- \\
&= a^2\left(a^2\right)^- a^2a^{\#}a^- + 1 - aa^- \\
&= a^2a^{\#}a^- + 1 - aa^- \\
&= 1
\end{aligned}$$

and

$$\begin{aligned}
\left(a\left(a^2\right)^- aa^- + 1 - aa^-\right)\left(a^2a^- + 1 - aa^-\right) &= a\left(a^2\right)^- aa^-a^2a^- + 1 - aa^- \\
&= a\left(a^2\right)^- a^2a^- + 1 - aa^- \\
&= a^{\#}a^2\left(a^2\right)^- a^2a^- + 1 - aa^- \\
&= a^{\#}a^2a^- + 1 - aa^- \\
&= 1.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left(a^3a^- + 1 - aa^-\right)^{-1} &= \left(a^2a^- + 1 - aa^-\right)^{-2} \\
&= \left(a\left(a^2\right)^- aa^- + 1 - aa^-\right)^2
\end{aligned}$$

and

$$a^{\#} = a^{\dagger} = \left(\left(a\left(a^2\right)^-\right)^2 aa^- + 1 - aa^-\right) a = a\left(a^2\right)^- a\left(a^2\right)^- a.$$

□

Remark.

A von Neumann regular element a in a ring R with involution $*$ has a group inverse $a^{\#}$ and a MP-inverse $a^{\dagger}$ with respect to $*$ such that $a^{\#} = a^{\dagger}$ iff

$$\left(a^3a^- + 1 - aa^-\right)^{-1} \text{ and } \left(a^-aa^*a + 1 - a^-a\right)^{-1} \text{ exist}$$

and

$$a^* = \left[\left(a^-aa^*a + 1 - a^-a\right)^* a\left(a^2\right)^- a\left(a^2\right)^-\right] a,$$

for any choice of a^-, since

$$a\left(a^-a^3+1-a^-a\right)^{-1}=\left(a^3a^-+1-aa^-\right)^{-1}a=a\left(a^2\right)^-a\left(a^2\right)^-a.$$

This property can be considered as the generalization of a result of Katz, I.J. and of its extension to Dedekind finite rings. Indeed, Katz proved, see [1, pag. 166, ex. 18], that for any square matrix A over the complexes, $A^\dagger=A^\#$ if and only if there is a matrix Y such that

$$A^*=YA.$$

His result can be lifted up to the following:

FACT 5. If a belongs to a Dedekind finite ring with a general involution * and $a^\dagger$ exists, then $a^*=ya$, for some $y\in R$, if and only if $a^\#$ exists and $a^\dagger=a^\#$.

Proof. If $a^\dagger$ exists then also $(a^\dagger)^*$ exists and equals $(a^*)^\dagger$. Since $a^*=ya$ then $a=a^*y^*$ and hence $aR\subseteq a^*R$.

Moreover, $aR\cong a^*R$ since $\phi:aR\to a^*R$, with $\phi(ax)=a^\dagger ax$, is a R-module isomorphism. Then, also $aa^\dagger R\cong a^\dagger aR$, which implies $aa^\dagger R=a^\dagger aR$, or $aR=a^*R$ by using Theorem 1 (iii) of [8]. By Proposition 2(1), $a^\#$ exists and $a^\dagger=a^\#$.

Conversely, if $a^\#$ exists and $a^\dagger=a^\#$ then

$$a^*=(aa^\dagger a)^*=a^*aa^\dagger=a^*aa^\#=a^*a^\#a.$$

It suffices to take $y=a^*a^\#$. □

To introduce the notion of *–DMP invertibility in a ring R, we first need to remark that if a is Drazin invertible with index k then a^k is *–gMP iff a^{k+1} is *–gMP. Indeed, if the Drazin index of a equals k and a^k is *–gMP, then $a^{k+1}R=a^kR=a^{k\,*}R=\left(a^*\right)^kR=\left(a^*\right)^{k+1}R$. In addition, a^{k+1} is Moore-Penrose invertible since $a^{k+1}\left(a^{k+1}\right)^*R=a^{2k+2}R=a^{k+1}R$, $R\left(a^{k+1}\right)^*a^{k+1}=Ra^{2k+2}=Ra^{k+1}$, and so $a^{k+1}\in a^{k+1}\left(a^{k+1}\right)^*R\cap R\left(a^{k+1}\right)^*a^{k+1}$. The converse is analogous.

Definition 6. *An element a in a ring R with involution * is called* *–DMP *(Drazin-Moore-Penrose) of index k if k is the smallest natural number such that $\left(a^k\right)^\#$ and $\left(a^k\right)^\dagger$ exist with respect to * and $\left(a^k\right)^\#=\left(a^k\right)^\dagger$.*

Examples.

1. The element 2_{12} in $\mathbb{Z}_{12}$, with respect to the identity involution $\iota:n\to n$ is not ι–gMP, but it is ι–DMP of index 2 since $4_{12}=\left(2_{12}^2\right)^\dagger=\left(2_{12}^2\right)^\#$. Remark that 2_{12} has no MP-inverse with respect to ι, i.e., has no group inverse.

2. Every nonzero nilpotent element with index k in the Jacobson radical of a ring with involution * is *–DMP with index k but these elements, clearly not von Neumann regular, are *not* group invertible *nor* Moore-Penrose invertible with respect to *.

Other characterizations of *–DMP of index k can be given as follows:

Theorem 7. *Let a be an element in a ring R with involution *. Then the following are equivalent:*

1. *a is *–DMP with index k.*
2. *a^{D_k} and $\left(a^k\right)^\dagger$ exist with $a^{D_k} = a^{k-1}\left(a^k\right)^\dagger$.*

Proof. Firstly, we will show that if a is *–DMP with index k then a^{D_l} exists and $l \leq k$. From a^k is group invertible with $\left(a^k\right)^\# = \left(a^k\right)^\dagger$ follows that a^{D_l} exists with $l \leq k$.

Now, suppose $l < k$. Then, since a^k is *–EP,

$$\left(a^k\right)^* R = a^k R = a^{k-1} R,$$

since $k > l$. By another hand,

$$\left(a^k\right)^* R = \left(Ra^k\right)^* = \left(Ra^{k-1}\right)^* = \left(a^{k-1}\right)^* R.$$

Therefore, $\left(a^{k-1}\right)^* R = a^{k-1} R$ and a^{k-1} is also *–EP, which is absurd since k is the smallest natural number for which a^k is *–EP.

To end this part of the proof, we remark that since k is the smallest k for which a^k is group invertible and a^k is *–EP, then $a^D = a^{k-1}\left(a^k\right)^\# = a^{k-1}\left(a^k\right)^\dagger$ (see [20]).

To show the converse, we will prove that if $a^{D_k} = a^{k-1}\left(a^k\right)^\dagger$, then $\left(a^k\right)^\# = \left(a^k\right)^\dagger$. We will simply check the group inverse equations. The first and second equations are trivially verified as they coincide with the first two Moore-Penrose equations. It suffices to show

$$a^k\left(a^k\right)^\dagger = \left(a^k\right)^\dagger a^k.$$

By one hand, $a^k\left(a^k\right)^\dagger = aa^{k-1}\left(a^k\right)^\dagger = aa^{D_k} = a^{D_k}a$, and therefore $a^k\left(a^k\right)^\dagger = \left(a^{D_k}a\right)^*$. By another hand, and since $*$ commutes with $(\cdot)^\dagger$ and $(\cdot)^D$, then $\left(a^k\right)^\dagger a^k = \left(\left(a^k\right)^\dagger a^k\right)^* = a^{*k}\left(a^{*k}\right)^\dagger = a^* a^{*k-1}\left(a^{*k}\right)^\dagger = a^* a^{*D} = a^*\left(a^{D_k}\right)^* = \left(a^{D_k}a\right)^*$. So, $a^k\left(a^k\right)^\dagger = \left(a^k\right)^\dagger a^k$. □

Let $a \in R$ be Drazin invertible with Drazin index k and consider

$$\begin{aligned} c_a &= a a^{D_k} a, \\ n_a &= \left(1 - a a^{D_k}\right) a = a - c_a. \end{aligned}$$

It should be remarked that a and $1 - aa^{D_k}$ commute, and also that n_a is nilpotent. Indeed, $n_a^k = \left(\left(1 - aa^{D_k}\right) a\right)^k = a^k \left(1 - aa^{D_k}\right) = a^k - a^{k+1} a^{D_k} = 0$. The following elementary results hold, as for matrices over the complexes (see [2]):

Lemma 8. *Let $a \in R$ be Drazin invertible with Drazin inverse a^{D_k} of index k. Let $c_a = aa^{D_k} a$ and $n_a = \left(1 - aa^{D_k}\right) a = a - c_a$. Then*

1. *$a = c_a + n_a$.*
2. *$c_a n_a = n_a c_a = 0$.*
3. *c_a is group invertible with $(c_a)^{\#} = a^{D_k}$.*
4. *$n_a^k = 0$.*
5. *$a^j = c_a^j + n_a^j$, if $j < k$.*
6. *$a^j = c_a^j$, if $j \geq k$.*

Definition 9. *For a, c_a, n_a as above, the sum*

$$a = c_a + n_a$$

is called the core nilpotent decomposition *of the element a, c_a is the* core part *of a and n_a is the* nilpotent part *of a (compare with [1], [2] for the ring of matrices over the complexes).*

We remark the fact that the core nilpotent decomposition is *unique* in the following sense: if a^{D_k} exists and x, y are such that $a = x + y$, $x^{\#}$ exists, $y^k = 0$ and $xy = yx = 0$, then $x = c_a$ and $y = n_a$ (see [1]).

Theorem 10. *Given an element a in a ring R with involution $*$, the following are equivalent:*

1. *a is $*$–DMP with index k.*
2. *a^{D_k} exists and the core part of a is $*$–gMP.*
3. *a^{D_k} exists and is $*$–gMP.*
4. *a^{D_k} exists and aa^{D_k} is symmetric.*

Proof. ($1 \Leftrightarrow 2$) Suppose a is *–DMP with index k. Then a^{D_k} exists and $a^k = c_a^k$ is *–gMP. This means that $c_a^k R = {c_a^*}^k R$, and as c_a is group invertible, also that $c_a R = c_a^* R$. So,

$$\begin{aligned} c_a c_a^* R &= c_a^2 R = c_a R, \\ R c_a^* c_a &= R c_a^2 = R c_a, \end{aligned}$$

and $c_a \in c_a c_a^* R \cap R c_a^* c_a$, which implies that c_a is Moore-Penrose invertible.

Conversely, if c_a is *–gMP, then all powers of c_a are *–gMP. In particular if k is the Drazin index of a then $c_a^k = a^k$ is *–gMP, and thus a is *–DMP of index k.

($2 \Leftrightarrow 3$) Suppose $c_a = a a^{D_k} a$ is *–gMP. Then

$$\begin{aligned} \left(a^{D_k}\right)^* R &= \left(R a^{D_k}\right)^* \\ &= \left(R a a^{D_k}\right)^* \\ &= \left(R a^{D_k} a\right)^* \\ &= \left(R a a^{D_k} a\right)^* \\ &= \left(a a^{D_k} a\right)^* R \\ &= c_a^* R \\ &= c_a R \\ &= a a^{D_k} a R \\ &= a a^{D_k} R \\ &= a^{D_k} a R \\ &= a^{D_k} R. \end{aligned}$$

Moreover, $a^{D_k} \left(a^{D_k}\right)^* R = \left(a^{D_k}\right)^2 R = a^{D_k} R$, and analogously, $R \left(a^{D_k}\right)^* a^{D_k} = R a^{D_k}$, and therefore a^{D_k} is Moore-Penrose invertible. Hence, by corollary 1, a^{D_k} is *–gMP.

Conversely, and analogously to the above, if $a^{D_k} R = \left(a^{D_k}\right)^* R$ then $c_a R = c_a^* R$. Moreover, $c_a c_a^* R = c_a^2 R = c_a R$, and also $R c_a^* c_a = R c_a$. Therefore $(c_a)^\dagger$ exists, which together $c_a R = c_a^* R$ imply c_a is *–gMP.

($2 \Leftrightarrow 4$) If c_a is *–gMP then $c_a^\dagger = c_a^\# = a^{D_k}$. Hence,

$$\begin{aligned} a a^{D_k} &= \left(a a^{D_k}\right)^2 \\ &= c_a a^{D_k} \\ &= c_a c_a^\dagger, \end{aligned}$$

which is symmetric.

Conversely, if $a a^{D_k} = a^{D_k} a$ is symmetric then we prove that a^{D_k} is the Moore-Penrose inverse of c_a. Indeed, $c_a a^{D_k}$ and $a^{D_k} c_a$ are symmetric. Obviously,

$$\begin{aligned} c_a a^{D_k} c_a &= c_a, \\ a^{D_k} c_a a^{D_k} &= a^{D_k}. \end{aligned}$$

Therefore, $c_a^\dagger = a^{D_k} = c_a^\#$ and c_a is *–gMP. $\square$

Theorem 11. *If a is *–DMP with index k and with core part c_a and nilpotent part n_a, the following hold:*

1. *If $n_a^\dagger$ exists then $a^\dagger$ exists with $a^\dagger = c_a^\dagger + n_a^\dagger = c_a^\# + n_a^\dagger$.*
2. *If $a^\dagger$ exists then $n_a^\dagger$ exists with $n_a^\dagger = \left(1 - aa^{D_k}\right) a^\dagger n_a a^\dagger \left(1 - aa^{D_k}\right)$.*

Proof. We remark that c_a belongs to the ring $aa^{D_k} R aa^{D_k}$ and n_a belongs to the ring $\left(1 - aa^{D_k}\right) R \left(1 - aa^{D_k}\right)$. Also, the previous theorem implies that $c_a^\dagger$ exists with $c_a^\dagger \in aa^{D_k} R aa^{D_k}$ (see [18]).

(1) If n_a is Moore-Penrose invertible then also

$$n_a^\dagger \in \left(1 - aa^{D_k}\right) R \left(1 - aa^{D_k}\right),$$

see [18]. The equality $a^\dagger = c_a^\dagger + n_a^\dagger$ follows easily from

$$\begin{aligned} 0 &= c_a n_a \\ &= c_a n_a^\dagger \\ &= n_a^\dagger c_a \\ &= c_a^\dagger n_a \\ &= c_a^\dagger n_a^\dagger. \end{aligned}$$

(2) It is easy to show that

$$a^\dagger \left(1 - aa^{D_k}\right), \left(1 - aa^{D_k}\right) a^\dagger \in n_a \{1\}.$$

In addition,

$$n_a a^\dagger \left(1 - aa^{D_k}\right) = \left(1 - aa^{D_k}\right) aa^\dagger \left(1 - aa^{D_k}\right)$$

is symmetric, and therefore $a^\dagger \left(1 - aa^{D_k}\right)$ is a 1-3 inverse of n_a. Also,

$$\left(1 - aa^{D_k}\right) a^\dagger n_a = \left(1 - aa^{D_k}\right) a^\dagger n_a = \left(1 - aa^{D_k}\right) a^\dagger a \left(1 - aa^{D_k}\right)$$

is symmetric, which makes $\left(1 - aa^{D_k}\right) a^\dagger$ a 1-4 inverse of n_a. Hence

$$n_a^\dagger = \left(1 - aa^{D_k}\right) a^\dagger n_a a^\dagger \left(1 - aa^{D_k}\right),$$

see [28]. $\square$

It should be pointed that in the previous theorem, $a^\dagger = c_a^\dagger + n_a^\dagger$ is *not* necessarily a core nilpotent decomposition. Let

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \in \mathsf{Mat}_3(\mathbb{C})$$

with transposed conjugation as the involution. $0 + A$ is the core nilpotent decomposition of A, but since

$$A^{\dagger} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

is *not* nilpotent, $0^{\dagger} + A^{\dagger}$ is not the core nilpotent decomposition of A.

The A of this example is nilpotent of index 3. For *–DMP matrices with index 2, the following positive results hold.

Lemma 12. *If $a^2 = 0$ and $a^{\dagger}$ exists then also $\left(a^{\dagger}\right)^2 = 0$.*

Proof. The result is clear since $\left(a^{\dagger}\right)^2 = a^{\dagger}a^{\dagger} = a^{\dagger}aa^{\dagger}a^{\dagger}aa^{\dagger} = a^{\dagger}a^{\dagger *}a^{*}a^{*}a^{\dagger *}a^{\dagger}$ and $a^{*\,2} = 0$. □

Lemma 13. *If a is *–DMP with index 2 and $a^{\dagger}$ exists then $c_{a^{\dagger}} = c_a^{\dagger}$ and $n_{a^{\dagger}} = n_a^{\dagger}$.*

Proof. Since a is *–DMP then c_a is *–gMP by Theorem 9 and therefore $c_a^{\dagger} = c_a^{\#}$. So, $\left(c_a^{\dagger}\right)^{\#}$ exists and equals c_a. Also, since $c_a \in aa^{D_2}Raa^{D_2}$ then $c_a^{\dagger} \in aa^{D_2}Raa^{D_2}$. As in the previous theorem, the existence of $a^{\dagger}$ implies the Moore-Penrose invertibility of n_a, with

$$n_a^{\dagger} = \left(1 - aa^{D_2}\right) a^{\dagger} n_a a^{\dagger} \left(1 - aa^{D_2}\right) \in \left(1 - aa^{D_2}\right) R \left(1 - aa^{D_2}\right).$$

So,

$$c_a^{\dagger} n_a^{\dagger} = n_a^{\dagger} c_a^{\dagger} = 0.$$

Finally, $\left(n_a^{\dagger}\right)^2 = 0$ since $n_a^2 = 0$, and $a^{\dagger} = c_a^{\dagger} + n_a^{\dagger}$. Using the uniqueness of the core nilpotent decomposition, the result follows. □

3 Application

Let R be a projective free ring with identity and involution $r \mapsto \overline{r}$ such that R^m be a module of finite length, which means that R^m has ACC and DCC for submodules, see [3], [13]. Let $+ : (a_{ij}) \rightarrow \left(\overline{a_{ij}}\right)^T$ be the involution on $\mathsf{Mat}_m(R)$. It follows from Fitting's Decomposition Theorem, see [3], [5], [10] and [13], that every matrix A is similar to a matrix of the form $G \oplus N$, with G invertible and N nilpotent with an index k, since R is also supposed to be projective free. So,

$$A = \begin{pmatrix} Q_1 & Q_2 \end{pmatrix} \begin{pmatrix} G & 0 \\ 0 & N \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$$

with $\begin{pmatrix} Q_1 & Q_2 \end{pmatrix} = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}^{-1}$.

By Theorem 9, A is $^+$–DMP of index k if and only if AA^{D_k} is symmetric with respect to $^+$. But,

$$\begin{aligned}
AA^{D_k} &= A^k\left(A^k\right)^{\#} \\
&= \begin{pmatrix} Q_1 & Q_2 \end{pmatrix}\begin{pmatrix} G^k & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}\begin{pmatrix} Q_1 & Q_2 \end{pmatrix}\begin{pmatrix} G^{-k} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \\
&= \begin{pmatrix} Q_1 & Q_2 \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \\
&= Q_1P_1
\end{aligned}$$

and, the symmetry of Q_1P_1 together with $P_1Q_1 = I$ implies that

$$Q_1 = P_1^{\dagger}.$$

But also $P_2P_1^{\dagger} = 0$, i.e., $P_2P_1^{+}\left(P_1P_1^{+}\right)^{-1} = 0$ or $P_2P_1^{+} = 0$ and $P_1P_2^{+} = 0$. This means that P_2^{+} is a cokernel of P_1 in the sense of [26], and Theorem 3.1 (page 77) implies

$$\begin{bmatrix} Q_1 & Q_2 \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}^{-1} = \begin{bmatrix} P_1^{\dagger} & P_2^{\dagger} \end{bmatrix}.$$

Therefore,

1.

$$\begin{aligned}
A \text{ is } {}^+\text{–gMP} \quad & \text{iff } A = \begin{bmatrix} P_1^{\dagger} & P_2^{\dagger} \end{bmatrix}\begin{bmatrix} G & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \\
& \text{iff } A = P_1^{\dagger}GP_1 \\
& (P_1 \text{ retraction, } G \text{ invertible})
\end{aligned}$$

It is easy to verify $A^{\#} = A^{\dagger}$ by means of the product formulas $(paq)^{\#}$ and $(paq)^{\dagger}$, see [21], [17]. Indeed,

$$\begin{aligned}
A^{\#} &= \left(P_1^{\dagger}GP_1\right)^{\#} \\
&= \left(P_1^{+}\left[\left(P_1P_1^{+}\right)^{-1}G\right]P_1\right)^{\#} \\
&= P_1^{+}\left(P_1P_1^{+}\right)^{-1}G^{-1}P_1 \\
&= P_1^{\dagger}G^{-1}P_1 \\
&= A^{\dagger} \text{ with respect to } {}^+.
\end{aligned}$$

2. A is $^+$–DMP of index k iff

$$\begin{aligned} A &= \begin{bmatrix} P_1^\dagger & P_2^\dagger \end{bmatrix} \begin{bmatrix} G & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \\ &= P_1^\dagger G P_1 + P_2^\dagger N P_2 \end{aligned}$$

(G invertible, N nilpotent of index k and $\begin{bmatrix} P_1 \\ P_2 \end{bmatrix}^{-1} = \begin{bmatrix} P_1^\dagger & P_2^\dagger \end{bmatrix}$). Clearly,

$$\left(A^k\right)^{\#} = \left(A^k\right)^{\dagger} = P_1^\dagger G^{-1} P_1.$$

Remark

In [2], we can find the following characterization for range-Hermitian matrices over $\mathbb{C}$:

- there exists a unitary matrix $U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$ and an invertible $r \times r$ matrix G, $r = rank\, A$, such that

$$\begin{aligned} A &= \begin{bmatrix} U_1^+ & U_2^+ \end{bmatrix} \begin{bmatrix} G & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \\ &= U_1^+ G\, U_1. \end{aligned}$$

Since $\mathbb{C}$ is projective free and $\mathbb{C}^n$ has finite length, the following is now a unitary free characterization for range-Hermitian matrices over $\mathbb{C}$:

- there exists an $r \times n$ matrix P_1 of full rank and an invertible $r \times r$ matrix G, $r = rank\, A$, such that

$$A = P_1^\dagger G P_1.$$

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