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Inverse semigroups generated by linear transformations

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Abstract

Suppose X is a set with $|X| = p \geq q \geq \aleph_0$ and let $B = BL(p, q)$ denote the Baer-Levi semigroup defined on X. In 1984, Howie and Marques-Smith showed that, if $p = q$, then $BB^{-1} = I(X)$, the symmetric inverse semigroup on X, and they described the subsemigroup of $I(X)$ generated by $B^{-1}B$. In 1994, Lima extended that work to 'independence algebras', and thus also to vector spaces. In this paper, we answer the natural question: what happens when $p > q$? We also show that, in this case, the analogues BB^{-1} for sets and GG^{-1} for vector spaces are never isomorphic, despite their apparent similarities.

1. Introduction

Let X be an infinite set with cardinal p, and let q be a cardinal such that $\aleph_0 \leq q \leq p$. Let $P(X)$ denote the set of all *partial transformations* of X: that is, all transformations α whose *domain*, $\operatorname{dom} \alpha$, and *range*, $\operatorname{ran} \alpha$, are subsets of X. As usual, the *composition* $\alpha \circ \beta$ of $\alpha, \beta \in P(X)$ is the transformation with domain $Y = (\operatorname{ran} \alpha \cap \operatorname{dom} \beta)\alpha^{-1}$ such that, for all $x \in Y$,

$$x(\alpha \circ \beta) = (x\alpha)\beta,$$

and we often write $\alpha \circ \beta$ more simply as $\alpha\beta$ (compare [1] vol 1, p 29). Obviously, Y is a subset of X and $\alpha \circ \beta \in P(X)$ if $\alpha, \beta \in P(X)$. Indeed, it is well-known that $(P(X), \circ)$ is a semigroup. Let $T(X)$ denote the subsemigroup of $P(X)$ consisting of all $\alpha \in P(X)$ with domain X, and let $I(X)$ denote the *symmetric inverse semigroup* on X: that is, the set of all injective elements of $P(X)$.

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If $\alpha \in P(X)$, we let

$$c(\alpha) = |\bigcup\{y\alpha^{-1} : |y\alpha^{-1}| \geq 2\}|, \quad r(\alpha) = |\operatorname{ran}\alpha|,$$
$$g(\alpha) = |X \setminus \operatorname{dom}\alpha|, \quad d(\alpha) = |X \setminus \operatorname{ran}\alpha|,$$

and refer to these cardinal numbers as the *collapse*, *rank*, *gap* and *defect* of α, respectively. We now write

$$B = BL(p, q) = \{\alpha \in T(X) : c(\alpha) = 0,\ d(\alpha) = q\}$$

which is the *Baer-Levi semigroup* on X of type (p, q) as discussed in [1] section 8.1. In [2] Theorems 2.3 and 2.4, the authors showed that if $p = q$, then

$$B^{-1} = \{\alpha \in I(X) : g(\alpha) = p,\ d(\alpha) = 0\},$$

$$BB^{-1} = I(X) \quad \text{and} \quad B^{-1}B = \{\alpha \in I(X) : r(\alpha) = g(\alpha) = d(\alpha) = p\}.$$

In this case (namely, $p = q$), they also showed that the subsemigroup of $I(X)$ generated by $B^{-1}B$ equals $K_p = \{\alpha \in I(X) : g(\alpha) = d(\alpha) = p\}$ ([2] Theorem 2.5). In [2] Theorem 3.2, the authors proved that K_p is, in fact, the inverse subsemigroup of $I(X)$ generated by the *nilpotent elements of index 2* (that is, all $\alpha \in I(X)$ for which $\alpha \neq \emptyset$ and $\alpha^2 = \emptyset$). In [3] Proposition 4.3, Proposition 4.5 and Theorem 5.6, Lima extended these results to independence algebras, and thus also to vector spaces ('sets' and 'vector spaces' are prime examples of an independence algebra). Here, we answer the natural question: what happens when $p > q$?

2. Main results

Let V be a vector space over a field F and suppose $\dim V = p \geq \aleph_0$. We let $P(V)$ denote the set of all *partial linear transformations* of V: that is, all linear transformations $\alpha : A \to B$ where A, B are subspaces of V. As for partial transformations of a set, we denote the *domain* and the *range* of $\alpha \in P(V)$ by $\operatorname{dom}\alpha$ and $\operatorname{ran}\alpha$, respectively, and we define the *composition* $\alpha \circ \beta$ of $\alpha, \beta \in P(V)$ to be the linear transformation with domain $U = (\operatorname{ran}\alpha \cap \operatorname{dom}\beta)\alpha^{-1}$ such that, for all $u \in U$,

$$u(\alpha \circ \beta) = (u\alpha)\beta.$$

To simplify notation, we often write $\alpha\circ\beta$ as $\alpha\beta$. Clearly, U is a subspace of V and $\alpha\circ\beta \in P(V)$ if $\alpha, \beta \in P(V)$. Also, $(\alpha\circ\beta)\circ\gamma = \alpha\circ(\beta\circ\gamma)$ for all $\alpha, \beta, \gamma \in P(V)$, so $(P(V), \circ)$ is a semigroup. Let $T(V)$ denote the subsemigroup of $P(V)$ consisting of all linear transformations with domain V, and let $I(V)$ denote the set of all injective partial linear transformations of V. It is easy to see that $I(V)$ is an inverse subsemigroup of $P(V)$ and its idempotents are precisely the identity maps id_U on the subspaces U of V. Note that we use the 'V' in place of 'X' to denote the fact that now we are considering *linear* transformations.

As an abbreviation, we write $\{e_i\}$ to denote a subset $\{e_i : i \in I\}$ of V, taking as understood that the subscript i belongs to some (unmentioned) index set I. The subspace A of V generated by a linearly independent subset $\{e_i\}$ of V is denoted by $\langle e_i \rangle$, and we write $\dim A =$

$|I|$. Often it is necessary to define some $\alpha \in P(V)$ by first choosing a linearly independent subset $\{e_i\}$ of V and some $\{a_i\} \subseteq V$, and then letting $e_i\alpha = a_i$ for each i and extending this action by linearity to the whole of dom $\alpha = \langle e_i \rangle$. To abbreviate matters, we simply say, given $\{e_i\}$ and $\{a_i\}$ within context, that $\alpha \in P(V)$ is defined by letting

$$\alpha = \begin{pmatrix} e_i \\ a_i \end{pmatrix}.$$

Similar notation for $P(X)$ is now standard: for example, see [4].

If $\alpha \in P(V)$, we write $\ker \alpha$ for the *kernel* of α, and put

$$n(\alpha) = \dim \ker \alpha, \quad r(\alpha) = \dim \operatorname{ran} \alpha,$$
$$g(\alpha) = \operatorname{codim} \operatorname{dom} \alpha, \quad d(\alpha) = \operatorname{codim} \operatorname{ran} \alpha.$$

As usual, these are called the *nullity*, *rank*, *gap* and *defect* of α, respectively. For each cardinal q such that $\aleph_0 \leq q \leq p$, consider the *linear Baer-Levi semigroup* on V:

$$G = GS(p,q) = \{\alpha \in T(V) : n(\alpha) = 0, d(\alpha) = q\}. \tag{1}$$

As shown in [5], this is a right simple, right cancellative subsemigroup of $T(V)$ without idempotents. Thus, it is not an inverse semigroup: in fact, $\alpha \in G$ if and only if $\alpha^{-1} \in G^{-1}$ where

$$G^{-1} = \{\beta \in I(V) : g(\beta) = q, d(\beta) = 0\}. \tag{2}$$

Moreover, from the anti-isomorphism $G \to G^{-1}, \alpha \mapsto \alpha^{-1}$, we see that G^{-1} is a left simple, left cancellative subsemigroup of $I(V)$ which has no idempotents. As already remarked, Lima [3] Propositions 4.3 and 4.5, showed that if $p = q$, then

$$GG^{-1} = I(V) \quad \text{and} \quad G^{-1}G = \{\alpha \in I(V) : r(\alpha) = g(\alpha) = d(\alpha) = p\}.$$

Therefore, in this case, GG^{-1} is an inverse semigroup but $G^{-1}G$ is not even closed. We will show that if $p > q$, then both GG^{-1} and $G^{-1}G$ are inverse subsemigroups of $I(V)$.

Henceforth, we let 0 denote the linear map with domain $\{0\}$ in V: note that this mapping is a zero for the semigroup $I(V)$.

Lemma 1. Suppose $\aleph_0 \leq q < p$ and let G, G^{-1} be as defined in (1) and (2). If $\alpha \in G$ and $\beta \in G^{-1}$ then $\dim(\operatorname{ran} \alpha \cap \operatorname{dom} \beta) = p$. In particular, $\alpha\beta \neq 0$ in $I(V)$.

Proof. Suppose $\{e_i\}$ is a basis for $\operatorname{ran} \alpha \cap \operatorname{dom} \beta$ and $|I| < p$. Since $q < p$, we know $\dim(\operatorname{ran} \alpha) = p = \dim(\operatorname{dom} \beta)$, so $\{e_i\}$ can be expanded to bases $\{e_i\} \dot\cup \{a_j\}$ for $\operatorname{ran} \alpha$ and $\{e_i\} \dot\cup \{b_j\}$ for dom β where $|J| = p$. Then $\{e_i\} \dot\cup \{a_j\} \dot\cup \{b_j\}$ is linearly independent: for, if $\sum x_i e_i + \sum y_j a_j + \sum z_j b_j = 0$ for some scalars x_i, y_j and z_j, then $\sum z_j b_j \in \operatorname{ran} \alpha \cap \operatorname{dom} \beta$, and so $\sum z_j b_j = \sum r_i e_i$ for some scalars r_i. Thus, $\sum r_i e_i - \sum z_j b_j = 0$ and so $z_j = 0$ for each j. Therefore, $\sum x_i e_i + \sum y_j a_j = 0$ and, since the set $\{e_i\} \dot\cup \{a_j\}$ is linearly independent, it follows that $x_i = 0$ for each i and $y_j = 0$ for each j. Consequently, $d(\alpha) \geq |J| = p$, a contradiction. □

Recall that $\mathrm{dom}\,(\alpha\beta) \subseteq \mathrm{dom}\ \alpha$ and $\mathrm{ran}(\alpha\beta) \subseteq \mathrm{ran}\,\beta$, so $g(\alpha\beta) \geq g(\alpha)$ and $d(\alpha\beta) \geq d(\beta)$. Hence the next result is a little surprising.

Theorem 1. Suppose $\aleph_0 \leq q < p$. If G and G^{-1} are as defined in (1) and (2), then

$$GG^{-1} = \{\alpha \in I(V) : g(\alpha) \leq q, d(\alpha) \leq q\}$$

and this is an inverse subsemigroup of $I(V)$ without nilpotents.

Proof. Let $\alpha \in G$ and $\beta \in G^{-1}$. Then, from Lemma 1, $\dim(\mathrm{ran}\,\alpha \cap \mathrm{dom}\ \beta) = p$. Suppose $\{e_i\}$ is a basis for $\mathrm{ran}\,\alpha \cap \mathrm{dom}\ \beta$ and expand it to bases $\{e_i\} \dot\cup \{a_r\}$ and $\{e_i\} \dot\cup \{b_s\}$ for $\mathrm{ran}\,\alpha$ and $\mathrm{dom}\ \beta$, respectively. Since α is one-to-one, there exist unique f_i and f_r such that $e_i\alpha^{-1} = f_i$ and $a_r\alpha^{-1} = f_r$. Write $e_i\beta = g_i$ for each i and $b_s\beta = g_s$ for each s. We have $V = \langle f_i, f_r\rangle = \langle g_i, g_s\rangle$ and

$$\alpha = \begin{pmatrix} f_i & f_r \\ e_i & a_r \end{pmatrix}, \quad \beta = \begin{pmatrix} e_i & b_s \\ g_i & g_s \end{pmatrix}.$$

Therefore

$$\alpha\beta = \begin{pmatrix} f_i \\ g_i \end{pmatrix}$$

and so $g(\alpha\beta) = \dim\langle f_r\rangle = |R|$ and $d(\alpha\beta) = \dim\langle g_s\rangle = |S|$. As in the proof of Lemma 1, $\{e_i\} \dot\cup \{a_r\} \dot\cup \{b_s\}$ is linearly independent, so it can be expanded to a basis for V, say $\{e_i\} \dot\cup \{a_r\} \dot\cup \{b_s\} \dot\cup \{c_\ell\}$. Thus, $|R| \leq |R| + |L| = g(\beta) = q$ and $|S| \leq |S| + |L| = d(\alpha) = q$. Hence, $\alpha\beta$ is such that $g(\alpha\beta) \leq q$ and $d(\alpha\beta) \leq q$.

Conversely, let $\alpha : A \to B$ be an injective linear map such that $g(\alpha) \leq q$ and $d(\alpha) \leq q$. Suppose $\{a_i\}$ is a basis for A and write $a_i\alpha = b_i$ for each i. Then $\{b_i\}$ is a basis for $\mathrm{ran}\,\alpha$. Expand $\{a_i\}$ and $\{b_i\}$ to bases $\{a_i\} \dot\cup \{a_j\}$ and $\{b_i\} \dot\cup \{b_\ell\}$ for V, respectively, with $|J| = g(\alpha) \leq q$, $|L| = d(\alpha) \leq q$. Since $|L| \leq q < p$, we may write $\{a_i\}$ as $\{u_i\} \dot\cup \{u_k\}$ with $|K| = q$ and $\{u_k\} = \{v_k\} \dot\cup \{v_\ell\}$. Now define in $P(V)$

$$\beta = \begin{pmatrix} a_i & a_j \\ u_i & a_j \end{pmatrix}, \quad \gamma = \begin{pmatrix} u_i & v_\ell \\ b_i & b_\ell \end{pmatrix}.$$

It is easy to see that $\beta, \gamma \in I(V)$ and $g(\beta) = 0$, $d(\beta) = \dim\langle u_k\rangle = q$, $g(\gamma) = \dim\langle v_k, a_j\rangle = q$ and $d(\gamma) = 0$. Therefore, $\beta \in G$ and $\gamma \in G^{-1}$. Since $\alpha = \beta\gamma$, it follows that $\alpha \in GG^{-1}$.

Next, we show that GG^{-1} is an inverse semigroup. To do so, let $\alpha, \beta \in GG^{-1}$. Then, $g(\alpha) \leq q$, $d(\alpha) \leq q$, $g(\beta) \leq q$ and $d(\beta) \leq q$. Since $q < p$, $\dim(\mathrm{dom}\ \alpha) = p = \dim(\mathrm{dom}\ \beta)$. Suppose $\{a_i\}$ is a basis for $\mathrm{dom}\ \alpha$ and $\{b_i\}$ a basis for $\mathrm{dom}\ \beta$. Write $a_i\alpha = u_i$ and $b_i\beta = v_i$ for each i. Suppose $\mathrm{ran}\,\alpha \cap \mathrm{dom}\ \beta = \{0\}$. Then, $\{u_i\} \dot\cup \{b_i\}$ is linearly independent and it can be expanded to a basis $\{u_i\} \dot\cup \{b_i\} \dot\cup \{c_\ell\}$ for V. Thus, $q \geq d(\alpha) = \dim\langle b_i, c_\ell\rangle = |I| + |L| = p$, a contradiction. Therefore, $\mathrm{ran}\,\alpha \cap \mathrm{dom}\ \beta \neq \{0\}$. Let $\{e_j\}$ be a basis for $\mathrm{ran}\,\alpha \cap \mathrm{dom}\ \beta$ and expand it to bases $\{e_j\} \dot\cup \{u_r\}$ and $\{e_j\} \dot\cup \{b_s\}$ for $\mathrm{ran}\,\alpha$ and $\mathrm{dom}\ \beta$, respectively. Since α is one-to-one, there exist unique f_j and f_r such that $f_j\alpha = e_j$ and $f_r\alpha = u_r$. Write $e_j\beta = v_j$ and $b_s\beta = v_s$. Then, we have

$$\alpha = \begin{pmatrix} f_j & f_r \\ e_j & u_r \end{pmatrix}, \quad \beta = \begin{pmatrix} e_j & b_s \\ v_j & v_s \end{pmatrix} \quad \text{and so} \quad \alpha\beta = \begin{pmatrix} f_j \\ v_j \end{pmatrix}.$$

Since $\{e_j\} \dot\cup \{u_r\} \dot\cup \{b_s\}$ is linearly independent, it can be expanded to a basis

$$\{e_j\} \dot\cup \{u_r\} \dot\cup \{b_s\} \dot\cup \{d_t\}$$

for V. Thus, $q \geq d(\alpha) = |S| + |T|$ and $q \geq g(\beta) = |R| + |T|$ and so $|S|, |R| \leq q$. Hence, $g(\alpha\beta) = |R| + g(\alpha) \leq q$ and $d(\alpha\beta) = |S| + d(\beta) \leq q$. Therefore, $\alpha\beta \in GG^{-1}$ and this shows that GG^{-1} is a semigroup. Since dom $\alpha^{-1} = \operatorname{ran}\alpha$ and $\operatorname{ran}\alpha^{-1} =$ dom α, it follows that $g(\alpha^{-1}) = d(\alpha)$ and $d(\alpha^{-1}) = g(\alpha)$ and hence GG^{-1} is an inverse semigroup. Moreover, since $d(\alpha\beta) \leq q < p$ for each $\alpha, \beta \in GG^{-1}$, we deduce that $r(\alpha\beta) = p$ and thus $\alpha\beta \neq 0$: that is, GG^{-1} has no nilpotents. □

Remark. For the above proof, it is natural to think $GG^{-1}G \subseteq G$, but this does not hold. To see this, suppose $\{e_i\}$ is a basis for V and write $\{e_i\}$ as $\{f_i\} \dot\cup \{f_j\}$ and $\{f_j\}$ as $\{a_j\} \dot\cup \{b_j\}$ where $|J| = q$. Now define $\alpha, \beta \in I(V)$ by

$$\alpha = \begin{pmatrix} f_i & f_j \\ f_i & a_j \end{pmatrix}, \quad \beta = \begin{pmatrix} f_i & b_j \\ f_i & f_j \end{pmatrix}.$$

Clearly, dom $\alpha = V = \operatorname{ran}\beta$, $d(\alpha) = \dim\langle b_j\rangle = q$ and $g(\beta) = \dim\langle a_j\rangle = q$. Thus, $\alpha \in G$ and $\beta \in G^{-1}$. Since dom $(\alpha\beta\gamma) \subseteq$ dom $(\alpha\beta) = \langle f_i\rangle \neq V$ for every $\gamma \in G$, it follows that $\alpha\beta\gamma \notin G$.

Theorem 2. Suppose $\aleph_0 \leq q < p$. If G and G^{-1} are as defined in (1) and (2), then

$$G^{-1}G = \{\alpha \in I(V) : g(\alpha) = d(\alpha) = q\}$$

and this is an inverse subsemigroup of $I(V)$ without nilpotents.

Proof. Let $\alpha \in G$ and $\beta \in G^{-1}$. Then, dom $\alpha = V = \operatorname{ran}\beta$ and so dom $(\beta\alpha) = (\operatorname{ran}\beta \cap$ dom $\alpha)\beta^{-1} = V\beta^{-1} =$ dom β and $\operatorname{ran}(\beta\alpha) = V\alpha = \operatorname{ran}\alpha$. Therefore, $g(\beta\alpha) = g(\beta) = q$ and $d(\beta\alpha) = d(\alpha) = q$.

Conversely, let $\alpha \in I(V)$ be such that $g(\alpha) = d(\alpha) = q$ and suppose $\{a_i\}$ is a basis for dom α. Write $a_i\alpha = b_i$ for each i and expand $\{a_i\}$ and $\{b_i\}$ to bases for V, say $\{a_i\} \dot\cup \{a_j\}$ and $\{b_i\} \dot\cup \{b_j\}$, respectively, with $|J| = q$. Since $q < p = \dim V$, it follows that $|I| = p$. Suppose $\{e_i\}$ is a basis for V and define in $I(V)$

$$\beta = \begin{pmatrix} a_i \\ e_i \end{pmatrix}, \quad \gamma = \begin{pmatrix} e_i \\ b_i \end{pmatrix}.$$

Then, $g(\beta) = \dim\langle a_j\rangle = g(\alpha) = q$, $\operatorname{ran}\beta = V =$ dom γ and $d(\gamma) = \dim\langle b_j\rangle = d(\alpha) = q$. Therefore, $\beta \in G^{-1}$ and $\gamma \in G$. Since $\alpha = \beta\gamma$, we have $\alpha \in G^{-1}G$ and the result follows.

To see that $G^{-1}G$ is a semigroup, let $\alpha, \beta \in G^{-1}G$. Then, $g(\alpha) = d(\alpha) = g(\beta) = d(\beta) = q$ and, since $q < p$, dim(dom $\alpha) = p =$ dim(dom $\beta)$. Let $\{a_i\}$ and $\{b_i\}$ be bases for dom α and dom β, respectively, and write $a_i\alpha = e_i$ and $b_i\beta = f_i$ for each i. Suppose $\operatorname{ran}\alpha \cap$dom $\beta = \{0\}$. Then $\{e_i\} \dot\cup \{b_i\}$ is linearly independent and so it can be expanded to a basis for V, say $\{e_i\} \dot\cup \{b_i\} \dot\cup \{v_j\}$. Thus, $q = d(\alpha) = \dim\langle b_i, v_j\rangle = p$, which contradicts our assumption on q and p. Therefore, $\operatorname{ran}\alpha \cap$ dom $\beta \neq \{0\}$. Let $\{c_\ell\}$ be a basis for $\operatorname{ran}\alpha \cap$ dom β and expand it to bases $\{c_\ell\} \,\dot\cup\, \{e_r\}$ and $\{c_\ell\} \,\dot\cup\, \{b_s\}$ for $\operatorname{ran}\alpha$ and $\operatorname{dom}\beta$, respectively. Since α is injective, there exist unique u_ℓ and u_r in V such that $u_\ell\alpha = c_\ell$ and $u_r\alpha = e_r$. If we write $c_\ell\beta = f_\ell$ for each ℓ and $b_s\beta = f_s$ for each s, we then have

$$\alpha = \begin{pmatrix} u_\ell & u_r \\ c_\ell & e_r \end{pmatrix}, \quad \beta = \begin{pmatrix} c_\ell & b_s \\ f_\ell & f_s \end{pmatrix}$$

and so

$$\alpha\beta = \begin{pmatrix} u_\ell \\ f_\ell \end{pmatrix}.$$

As in the proof of Lemma 1, it follows that $\{c_\ell\} \,\dot\cup\, \{e_r\} \,\dot\cup\, \{b_s\}$ is linearly independent, so it can be expanded to a basis $\{c_\ell\} \,\dot\cup\, \{e_r\} \,\dot\cup\, \{b_s\} \,\dot\cup\, \{d_k\}$ for V. Therefore, $|S| \leq |S| + |K| = \operatorname{codim}\langle c_\ell, e_r\rangle = d(\alpha) = q$ and $|R| \leq |R| + |K| = \operatorname{codim}\langle c_\ell, b_s\rangle = g(\beta) = q$. Hence, $g(\alpha\beta) = |R| + g(\alpha) = q$ and $d(\alpha\beta) = |S| + d(\beta) = q$. Thus, $\alpha\beta \in G^{-1}G$ and so $G^{-1}G$ is a semigroup. Since $g(\alpha^{-1}) = d(\alpha)$ and $d(\alpha^{-1}) = g(\alpha)$, it follows that GG^{-1} is an inverse semigroup. □

3. Isomorphism problem

Of course, there are analogues of Theorems 1 and 2 for sets. That is, if X is a set and $\aleph_0 \leq q < p = |X|$, we can show, in a manner similar to the above proofs, that

$$BB^{-1} = \{\alpha \in I(X) : g(\alpha) \leq q,\ d(\alpha) \leq q\},$$
$$B^{-1}B = \{\alpha \in I(X) : g(\alpha) = d(\alpha) = q\}$$

and that these are inverse subsemigroups of $I(X)$ without nilpotents (indeed, the above proofs hold almost verbatim, provided we omit all references to bases and their extension). Since these semigroups look so much like GG^{-1} and $G^{-1}G$, an obvious problem is to decide whether the corresponding pairs are isomorphic.

In [6] we showed that $I(X)$ and $I(V)$ are almost never isomorphic. Here we use a similar idea to show the above corresponding pairs are never isomorphic.

The idempotents of $I(X)$ have the form id_Y where $Y \subseteq X$; and they are partially ordered by

$$\mathrm{id}_A \leq \mathrm{id}_B \iff \mathrm{id}_A = \mathrm{id}_A \circ \mathrm{id}_B \iff A \subseteq B.$$

We say id_B *covers* id_A, and write $\mathrm{id}_A < \mathrm{id}_B$, if there is no idempotent in $I(X)$ strictly between id_A and id_B under this partial order: in other words, when this occurs, $B = A \cup \{x\}$ for some $x \notin A$. In addition, if $M \subseteq X$ and there are distinct $a, b \notin M$, we say

$$\mathrm{id}_M < \mathrm{id}_{M\cup\{a\}} < \mathrm{id}_{M\cup\{a,b\}} \tag{3}$$

is an *idempotent chain of length* 2; and that $\alpha \in I(X)$ *preserves* this chain if

$$\mathrm{id}_M \circ \alpha = \mathrm{id}_M \quad \text{and} \quad \mathrm{id}_{M\cup\{a,b\}} = \alpha\alpha^{-1}.$$

Clearly, when this happens, $a\alpha = x$ and $b\alpha = y$ for some $x \neq y$, and hence

$$\mathrm{id}_M < \mathrm{id}_{M\cup\{x\}} < \mathrm{id}_{M\cup\{x,y\}}$$

where $\mathrm{id}_{M\cup\{a\}} \circ \alpha = \mathrm{id}_{M\cup\{x\}}$ and $\mathrm{id}_{M\cup\{a,b\}} \circ \alpha = \mathrm{id}_{M\cup\{x,y\}}$ (this explains the terminology).

Let S denote either BB^{-1} and $B^{-1}B$, and let $D_2(S)$ equal the set of all $\alpha \in S$ of the form:

$$\alpha = \begin{pmatrix} a & b \\ x & y \end{pmatrix} \cup \text{ id}_M, \tag{4}$$

where $a \neq b, x \neq y, M \subseteq X$, and none of a, b, x, y belong to M. Clearly, $D_2(S)$ is non-empty, and we assert that its elements are characterised by the statement:

$$\alpha \in S \text{ and } \alpha \text{ preserves an idempotent chain in } S \text{ of length 2.} \tag{†}$$

Clearly, if $\alpha \in S$ has the form in (4) then $|X \setminus M| \leq q$, hence both id_M and $\text{id}_{M\cup\{a\}}$ belong to S, and so α preserves the idempotent chain of length 2 in (3). Conversely, suppose the chain in (3) belongs to S and assume $\alpha \in S$ preserves it. Then, by definition, $\text{dom } \alpha = M \cup \{a, b\}$ and $\text{id}_M \subseteq \alpha$; and moreover, since α is injective, we have $a\alpha = x$ and $b\alpha = y$ for some $x \neq y$, and so α has the form in (4).

Next we assert that S satisfies:

$$\begin{aligned} &\text{if } \alpha \in S \text{ preserves a chain } \text{id}_M < \text{id}_{M\cup\{a\}} < \text{id}_{M\cup\{a,b\}} \\ &\text{and } \alpha\gamma, \alpha^2\gamma \neq \text{id}_M \text{ for some idempotent } \gamma \in S \text{ that covers } \text{id}_M, \\ &\text{and if } \alpha^2 \text{ is not idempotent, then } \gamma\alpha = \text{id}_M\,. \end{aligned} \tag{$*$}$$

To see this, suppose $\alpha \in S$ satisfies the initial condition, hence it has the form in (4), and assume an idempotent $\gamma \in S$ covers id_M and $\alpha\gamma \neq \text{id}_M$. Then $\text{ran}\,\alpha \cap \text{dom}\,\gamma \neq M$ and so, without loss of generality,

$$\gamma = \begin{pmatrix} x \\ x \end{pmatrix} \cup \text{ id}_M\,.$$

Suppose $x = a$: in this case, if $b = y$ then $\alpha = \text{id}_{\{a,b\}} \cup \text{id}_M$ is idempotent; and if $b \neq y$ then $\alpha^2 = a_a \cup \text{ id}_M$ is idempotent, contradicting the supposition in both cases. On the other hand, if $x = b$ then $y \neq b$ (since α is injective) and $y \neq a$ (since α^2 is not idempotent). Hence, in this case, $\alpha^2 = a_y \cup \text{id}_M$, so $\alpha^2\gamma \neq \text{id}_M$ implies $y \in \text{dom }\gamma$, which is impossible since $y \notin M \cup \{x\}$. Therefore, $x \notin \{a, b\}$ and it follows that $\gamma\alpha = \text{id}_M$.

Clearly, property $(*)$ will be preserved under an isomorphism φ from BB^{-1} onto GG^{-1} (and likewise for the other pair of analogues). This is because the notions of 'cover' and 'idempotent chain of length 2' can be described algebraically, and also because BB^{-1} is an inverse semigroup (hence, both α^{-1} and $\alpha\alpha^{-1}$ belong to BB^{-1} if $\alpha \in BB^{-1}$). However, we assert that the image of S under φ does not satisfy $(*)$.

To see this, choose linearly independent $a, b \in V$, then $\{b, a + b\}$ is also linearly independent. Suppose N is a subspace of V such that $N \cap \langle a, b\rangle = \{0\}$ and assume

$$\beta = \begin{pmatrix} a & b \\ b & a+b \end{pmatrix} \cup \text{ id}_N \in S\varphi$$

(this containment simply restricts the codimension of N as required). Now β preserves the idempotent chain of length 2 in $S\varphi$ given by

$$\text{id}_N < \text{id}_{N+\langle a\rangle} < \text{id}_{N+\langle a,b\rangle}$$

since $\mathrm{id}_N \circ \beta = \mathrm{id}_N$ and $\mathrm{id}_{N+\langle a,b\rangle} = \beta\beta^{-1}$. Hence β is the image of some element of $D_2(S)$. However, β does not satisfy $(*)$. For, clearly β^2 is not idempotent. Also, if $c = a + b$ then $\gamma = \mathrm{id}_{N+\langle c\rangle}$ is an idempotent in $S\varphi$ which covers id_N. In addition, $\beta\gamma \neq \mathrm{id}_N$ and $\beta^2\gamma \neq \mathrm{id}_N$. However, also $\gamma\beta \neq \mathrm{id}_N$, and the last assertion follows.

The above argument shows that, despite their similarities, these corresponding pairs of semigroups are essentially different, something which is not apparent when they are regarded as semigroups of endomorphisms of an independence algebra.

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