General Relativistic Elasticity - Statics and Dynamics of Spherically Symmetric Metrics

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Abstract An introduction is provided to the theory of elasticity in general relativity. Important tensors appearing in this context are presented. In particular, attention is focussed on the elasticity difference tensor, for which an algebraic analysis is performed. Applications are given to static and non-static spherically symmetric configurations. For the latter, dynamical equations are obtained characterizing the space-time in the context of general relativistic elasticity.

1 General Relativistic Elasticity

General relativistic elasticity was formulated in the mid-twentieth century due to the necessity to study astrophysical problems such as deformations of neutron star crusts. Relevant contributions to the theory of general relativistic elasticity were given by Carter and Quintana [1], Kijowski and Magli [2], Beig and Schmidt [3], Karlovini and Samuelsson [4] and by many other authors.

The theory is based on a configuration mapping

$$\Psi : M \longrightarrow X,$$

a C^k $(k > 1)$ mapping from space-time M, endowed with a Lorentz metric g of signature $(-,+,+,+)$ and assumed to be time-orientable, to the material space X. The material space is a three-dimensional manifold, whose points represent the particles of the material. The material metric K defined on X measures the distances

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between particles in the locally relaxed state of the material. Coordinates on M are here denoted by $\{\omega^a\}$, $a=0,1,2,3$, and coordinates on X by $\{\xi^A\}$, $A=1,2,3$. Associated with Ψ are the pull-back operator Ψ^* and the push-forward operator Ψ_* which give rise to a 3×4 matrix, the relativistic deformation gradient, whose entries are $\xi^A_a=\frac{\partial\xi^A}{\partial\omega^a}$. The velocity field of the matter, u^a, satisfies the following conditions: $u^0>0$, $u^a u_a=-1$ and $u^a\xi^A_a=0$. The pulled-back material metric $k_{ab}=\xi^A_a\xi^B_b K_{AB}$ is such that $k_{ab}u^a=0$ and $\mathscr{L}_u k_{ab}=0$. It is used to construct other relativistic elastic tensors. Let n_1^2, n_2^2, n_3^2 be the eigenvalues of $k^a{}_b$, then one can write $k_{ab}=n_1^2 x_a x_b+n_2^2 y_a y_b+n_3^2 z_a z_b$, where x, y and z denote the eigenvectors of k and n_1, n_2, n_3 represent the linear particle densities (see [4]). Considering the orthonormal tetrad $\{u,x,y,z\}$, then the space-time metric takes the form $g_{ab}=-u_a u_b+x_a x_b+y_a y_b+z_a z_b$ and $h_{ab}=x_a x_b+y_a y_b+z_a z_b$ is the projection tensor.

The relativistic strain tensor $s_{ab}=\frac{1}{2}(h_{ab}-k_{ab})$ contains information about the local state of strain of the matter. The material is said to be locally relaxed at a particular point of space-time if $s_{ab}=0$.

The elasticity difference tensor $S^a{}_{bc}$ introduced by [4] can be expressed as

$$S^a{}_{bc}=\frac{1}{2}k^{-am}(D_b k_{mc}+D_c k_{mb}-D_m k_{bc}),\tag{1}$$

where k^{-1am} is such that $k^{-1am}k_{mb}=h^a{}_b$ and D_b is the spatially projected connection defined by $D_a t^{b\dots}{}_{c\dots}=h^d{}_a h^b{}_e\dots h^f{}_c\dots\nabla_d t^{e\dots}{}_{f\dots}$, where $t^{b\dots}{}_{c\dots}$ is an arbitrary tensor field, and it satisfies $D_a h_{bc}=0$. A mathematical analysis of the elasticity difference tensor is presented in [5]. It is decomposed along the eigenvectors of $k^a{}_b$ as follows

$$S^a{}_{bc}=\underset{1}{M}_{bc}x^a+\underset{2}{M}_{bc}y^a+\underset{3}{M}_{bc}z^a;\tag{2}$$

and for the three second-order symmetric tensors $\underset{1}{M}$, $\underset{2}{M}$ and $\underset{3}{M}$ the eigenvalue-eigenvector problem is studied. In particular, conditions are investigated for the three eigenvectors, x, y, z, of the pulled-back material metric to be eigenvectors for $\underset{1}{M}$, $\underset{2}{M}$ and $\underset{3}{M}$.

Here, the algebraic analysis of the elasticity difference tensor is carried out for a static and a non-static spherically symmetric space-time.

2 Applications to static and dynamical configurations

Static spherically symmetric space-time

Consider a static spherically symmetric space-time with g given by the line-element

$$ds^2=-e^{2\nu(r)}dt^2+e^{2\lambda(r)}dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2\tag{3}$$

and with coordinates $\omega^a = \{t, r, \theta, \phi\}$. The space-time can be specified by the orthonormal tetrad $\{u, x, y, z\}$ using the basis vectors
$u^a = \left[\frac{1}{e^{\nu(r)}}, 0, 0, 0\right]$, $x^a = \left[0, \frac{1}{e^{\lambda(r)}}, 0, 0\right]$, $y^a = \left[0, 0, \frac{1}{r}, 0\right]$ and $z^a = \left[0, 0, 0, \frac{1}{r\sin\theta}\right]$. Due to the spherical symmetry, on X the coordinates are $\xi^A = \{\tilde{r}, \tilde{\theta}, \tilde{\phi}\}$ where $\tilde{r} = \tilde{r}(r)$, $\tilde{\theta} = \theta$ and $\tilde{\phi} = \phi$. The non-zero components of the deformation gradient are $\frac{d\xi^1}{d\omega^1} = \tilde{r}'$, $\frac{d\xi^2}{d\omega^2} = 1$, $\frac{d\xi^3}{d\omega^3} = 1$, where a prime represents a derivative with respect to r, and the line-element of the pulled-back material metric is $ds^2 = \tilde{r}'^2\, dr^2 + \tilde{r}^2\, d\theta^2 + \tilde{r}^2\, sin^2\theta\, d\phi^2$. Calculating the eigenvalues of $k^a{}_b$, one obtains $n_1^2 = \tilde{r}'^2\, e^{-2\lambda}$ and $n_2^2 = n_3^2 = \frac{\tilde{r}^2}{r^2}$. The strain tensor has three non-zero components: s_{rr}, $s_{\theta\theta}$, $s_{\phi\phi}$, and it vanishes if and only if $\tilde{r} = c\, e^{\int \frac{e^{\lambda}}{r} dr}$, $c > 0$. Solving the eigenvalue-eigenvector problem for $\underset{1}{M}$, $\underset{2}{M}$ and $\underset{3}{M}$, building up the elasticity difference tensor in (2), leads to the results listed in Table 1.

Table 1 Eigenvectors and eigenvalues for $\underset{1}{M}$, $\underset{2}{M}$ and $\underset{3}{M}$

	Eigenvectors	Eigenvalues
	x	$\mu_1 = \frac{e^{-\lambda}}{n_1} n_1'$
$\underset{1}{M}$	y	$\mu_2 = \frac{e^{-\lambda}}{r} - \frac{e^{-\lambda}}{r}\frac{n_2^2}{n_1^2} - e^{-\lambda}\frac{n_2}{n_1^2} n_2'$
	z	$\mu_3 = \mu_2$
	$x+y$	$\mu_4 = \frac{e^{-\lambda}}{n_2} n_2'$
$\underset{2}{M}$	$x-y$	$\mu_5 = -\frac{e^{-\lambda}}{n_2} n_2'$
	z	$\mu_6 = 0$
	$x+z$	$\mu_7 = \mu_4$
$\underset{3}{M}$	$x-z$	$\mu_8 = \mu_5$
	y	$\mu_9 = 0$

Non-static spherically symmetric space-time

Consider a non-static spherically symmetric space-time, whose metric g is given by the line-element $ds^2 = -e^{2\nu(t,r)}dt^2 + e^{2\lambda(t,r)}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2$. On M the coordinates are $\omega^a = \{t, r, \theta, \phi\}$. The space-time can be specified by defining the orthonormal tetrad $\{u, x, y, z\}$ with the following basis vectors:
$u^a = \left[e^{-\nu}\gamma, -e^{-\nu}\frac{\dot{\tilde{r}}}{\tilde{r}'}\gamma, 0, 0\right]$, $x^a = \left[-e^{\lambda-2\nu}\frac{\dot{\tilde{r}}}{\tilde{r}'}\gamma, e^{-\lambda}\gamma, 0, 0\right]$, $y^a = \left[0, 0, \frac{1}{r}, 0\right]$ and $z^a = \left[0, 0, 0, \frac{1}{r\sin\theta}\right]$, where $\gamma = \sqrt{\frac{e^{2\nu}\tilde{r}'^2}{e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2}}$ and a dot represents a derivative with respect to t. In this case, the coordinates on X are $\xi^A = \{\tilde{r}, \tilde{\theta}, \tilde{\phi}\}$, where $\tilde{r} = \tilde{r}(t, r)$, $\tilde{\theta} = \theta$ and $\tilde{\phi} = \phi$, so that the non-zero components of the relativistic deformation gradient take the form $\frac{\partial\xi^1}{\partial\omega^0} = \dot{\tilde{r}}$, $\frac{\partial\xi^1}{\partial\omega^1} = \tilde{r}'$, $\frac{\partial\xi^2}{\partial\omega^2} = 1$, $\frac{\partial\xi^3}{\partial\omega^3} = 1$. The line-element of the pulled-back material metric is given by $ds^2 = -\dot{\tilde{r}}^2\, dt^2 + 2\dot{\tilde{r}}\,\tilde{r}'\, dt dr + \tilde{r}'^2\, dr^2 + \tilde{r}^2\, d\theta^2 + \tilde{r}^2\, sin^2\theta\, d\phi^2$. Calculating the eigenvalues of $k^a{}_b$, one

obtains $n_1^2 = \tilde{r}'^2\, e^{-2\lambda} - \dot{\tilde{r}}^2\, e^{-2\nu}$ and $n_2^2 = n_3^2 = \frac{\tilde{r}^2}{r^2}$. The strain tensor has three more components than in the static case: s_{tt}, s_{tr}, s_{rr}, $s_{\theta\theta}$, $s_{\phi\phi}$, and it vanishes if and only if the following condition involving the functions and λ, μ and the material radius is satisfied: $\tilde{r}'^2 e^{-2\lambda} - \dot{\tilde{r}}^2 e^{-2\nu} = \frac{\tilde{r}^2}{r^2}$. Solving the eigenvalue-eigenvector problem in this case, one obtains the results listed in Table 2.

Table 2 Eigenvectors and eigenvalues for $\underset{1}{M}$, $\underset{2}{M}$ and $\underset{3}{M}$

	Eigenvectors	Eigenvalues
	x	$\mu_1 = \frac{e^{2\nu}\tilde{r}' n_1' - e^{2\lambda}\dot{\tilde{r}}\dot{n}_1}{e^{\lambda+\nu} n_1} \sqrt{\frac{1}{e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2}}$
$\underset{1}{M}$	y	$\mu_2 = \frac{r n_2 (e^{2\lambda}\dot{\tilde{r}}\dot{n}_2 - e^{2\nu}\tilde{r}' n_2') + \tilde{r}' e^{2\nu}(n_1^2 - n_2^2)}{e^{\lambda+\nu} r n_1^2} \sqrt{\frac{1}{e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2}}$
	z	$\mu_3 = \mu_2$
	$x+y$	$\mu_4 = -\frac{e^{2\lambda}\dot{\tilde{r}}\dot{n}_2 - e^{2\nu}\tilde{r}' n_2'}{e^{\lambda+\nu} n_2} \sqrt{\frac{1}{e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2}}$
$\underset{2}{M}$	$x-y$	$\mu_5 = \frac{e^{2\lambda}\dot{\tilde{r}}\dot{n}_2 - e^{2\nu}\tilde{r}' n_2'}{e^{\lambda+\nu} n_2} \sqrt{\frac{1}{e^{2\nu}\tilde{r}'^2 - e^{2\lambda}\dot{\tilde{r}}^2}}$
	z	$\mu_6 = 0$
	$x+z$	$\mu_7 = \mu_4$
$\underset{3}{M}$	$x-z$	$\mu_8 = \mu_5$
	y	$\mu_9 = 0$

Concluding remarks

Comparing the results obtained for the static case and for the non-static case, the following conclusions and remarks can be drawn.

For spherically symmetric space-times, passing from a static to a non-static configuration preserves the behaviour of the eigenvectors of the pulled-back material metric k for the tensors $\underset{1}{M}$, $\underset{2}{M}$ and $\underset{3}{M}$ building up the elasticity difference tensor: x, y, z are eigenvectors for $\underset{1}{M}$; $x+y$, $x-y$, z are eigenvectors for $\underset{2}{M}$; $x+z$, $x-z$, y are eigenvectors for $\underset{3}{M}$. In particular, the eigenvectors y and z of k remain the same for both configurations, only x changes. Furthermore, in the non-static case we can observe that the velocity field of matter u depends on the material radius; all relativistic elastic quantities (k_{ab}, n_1^2, n_2^2, s_{ab}, $S^a{}_{bc}$) are time-dependent through λ, ν and the material radius $\tilde{r}$; the condition to be satisfied for the strain tensor to vanish involve the functions ν and $\dot{\tilde{r}}$ in addition to λ and $\tilde{r}'$.

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