DUALLY PERFECT ORTHODOX DUBREIL-JACOTIN SEMIGROUPS

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It is known that for a strong Dubreil-Jacotin semigroup the property of being perfect is equivalent to that of being naturally ordered. Here we consider the existence of a smallest idempotent and the analogous notion of being dually perfect, which we show is not equivalent to being dually naturally ordered. The main objective of the paper is to obtain a structure theorem for dually perfect orthodox Dubreil-Jacotin semigroups on which Green's relations are regular.

1. INTRODUCTION

A *strong Dubreil-Jacotin semigroup* is an ordered semigroup S for which there exists an ordered group G and an epimorphism $f : S \to G$ that is *residuated* in the sense that the pre-image under f of every principal order ideal of G is a principal order ideal of S. In particular, the pre-image of the negative cone $N(G) = \{x \in G ; x \leqslant 1\}$ of G is a principal order ideal $\xi^{\downarrow} = \{x \in S ; x \leqslant \xi\}$ of S, the so-called *bimaximum element* ξ being *equiresidual* in the sense that, for every $x \in S$, the order ideals $\{y \in S ; xy \leqslant \xi\}$ and $\{y \in S ; yx \leqslant \xi\}$ coincide and have a greatest element, denoted by $\xi : x$. It turns out (see, for example, [4, Theorem 25.7]) that the ordered group G is unique up to isomorphism and

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is given by S/A_ξ where A_ξ is the closure equivalence given by

$$(x, y) \in A_\xi \iff \xi : x = \xi : y.$$

Moreover, when S is regular the bimaximum element is the biggest idempotent.

The natural order on the set E of idempotents of S is given by

$$e \preceq f \iff e = ef = fe,$$

and S is said to be naturally ordered if the ordering $\leqslant$ of S extends this natural order, in the sense that if $e \preceq f$ then $e \leqslant f$. As shown in [2], for an orthodox strong Dubreil-Jacotin semigroup S the property of being naturally ordered is equivalent to S being perfect, in the sense that $x = x(\xi : x)x$ for every $x \in S$. The structure of such semigroups has been deeply investigated in [3]. Here we consider the existence also of a smallest idempotent, and the analogous notion of being dually perfect. This implies, but is not equivalent to, the property of being dually naturally ordered. The main objective of the paper is to obtain a structure theorem for dually perfect orthodox Dubreil-Jacotin semigroups on which Green's relations $\mathcal{L}, \mathcal{R}$ are regular.

2. PRELIMINARIES

If S is a strong Dubreil-Jacotin semigroup then a subset T of S will be called a *transversal of the A_ξ-classes* if, for every $x \in S$, $T \cap [x]_{A_\xi}$ is a singleton $\{t_x\}$; and *isotone* if $x \leqslant y$ implies $t_x \leqslant t_y$. By a *bounded* Dubreil-Jacotin semigroup we mean a strong Dubreil-Jacotin regular semigroup that has a smallest idempotent β whose subgroup $\mathcal{H}$-class H_β is an isotone transversal of the A_ξ-classes.

If S is a bounded Dubreil-Jacotin semigroup then with every $x \in S$ we can associate the element β_x given by

$$H_\beta \cap [x]_{A_\xi} = \{\beta_x\}.$$

Since $\beta_x\beta_y \in [x]_{A_\xi}[y]_{A_\xi} = [xy]_{A_\xi}$ and $\beta_x\beta_y \in H_\beta$ we have $\beta_x\beta_y = \beta_{xy}$, whence

$$(\forall e \in E) \quad \beta_e = \beta \qquad \text{and} \qquad (\forall x' \in V(x)) \quad \beta_{x'} = \beta_x^{-1}.$$

We shall say that $x \in S$ is *dually perfect* if $x = x\beta_x^{-1}x$, and that S itself is dually perfect if every element of S is dually perfect.

Theorem 2.1 *If S is an orthodox bounded Dubreil-Jacotin semigroup then the following statements are equivalent:*

(1) *S is dually perfect;*

(2) *S is dually naturally ordered and $(\forall x \in S)$ $\beta_x = \min [x]_{A_\xi}$.*

Proof (1) $\Rightarrow$ (2) : Suppose that (1) holds and let $e, f \in E$ be such that $e \preceq f$. Then $e = ef = fe = fef$. But as f is dually perfect we have $f = f\beta_f^{-1}f = f\beta f$.

Consequently, β being the smallest idempotent,

$$e = fef \geqslant f\beta f = f,$$

and so S is dually naturally ordered.

Now for every $x \in S$ we have $x = x\beta_x^{-1}x$ so $x\beta_x^{-1} \in E$. Consequently,

$$x = xx'x \geqslant x\beta = x\beta_x^{-1}\beta_x \geqslant \beta\beta_x = \beta_x,$$

and therefore $\beta_x = \min [x]_{A_\xi}$.

(2) $\Rightarrow$ (1) : If (2) holds then, by the dual of [5, Proposition 1.9], the smallest idempotent β is a middle unit. Since $\beta_x^{-1} = \beta_{x'} \in [x']_{A_\xi}$ we have

$$x\beta_x^{-1} \in [xx']_{A_\xi} = [\xi]_{A_\xi}$$

and so, by (2), $x\beta_x^{-1} \geqslant \beta$. Consequently,

$$x\beta_x^{-1}x = xx'x\beta_x^{-1}x \geqslant xx'\beta x = xx'x = x.$$

But on the other hand, again by (2),

$$x = xx'x \geqslant x\beta_{x'}x = x\beta_x^{-1}x.$$

Hence $x = x\beta_x^{-1}x$, whence S is dually perfect. ◇

That the condition of being dually naturally ordered is not in itself sufficient to imply that S is dually perfect is shown by the following example.

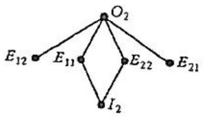
Example 2.1 Consider the set S_6 consisting of the 2×2 matrices I_2, O_2, E_{ij} where I_2 is the identity matrix, O_2 is the zero matrix, and E_{ij} has 1 in the (i,j)-th position and 0 elsewhere. From the fact that

$$E_{ij}E_{pq} = \begin{cases} O_2 & \text{if } j \neq p; \\ E_{iq} & \text{if } j = p, \end{cases}$$

it follows that S_6 is an inverse semigroup with idempotents O_2, I_2, E_{11}, E_{22}. Under the *dual* of the natural order, i.e. that given by

$$X \succeq Y \iff XX^{-1} = XY^{-1},$$

this inverse semigroup is an ordered semigroup with Hasse diagram



Clearly, S_6 is strong Dubreil-Jacotin (consisting of a single A_ξ-class, with $\xi = O_2$). The smallest idempotent is I_2 and its subgroup $\mathcal{H}$-class is the singleton $\{I_2\}$ which is trivially an isotone transversal of the A_ξ-classes. Hence S_6 is bounded. By construction, it is dually naturally ordered. But it is not dually perfect; for example, $E_{12}I_2E_{12} = O_2$ so E_{12} is not dually perfect. This also follows from Theorem 2.1 since I_2 is not the smallest element of (its A_ξ-class) S_6.

We now show that every element of a dually perfect orthodox Dubreil-Jacotin semigroup has a smallest inverse. For this purpose, for every $x \in S$ define

$$x^+ = \beta_x^{-1}x\beta_x^{-1}.$$

Theorem 2.2 *Let S be a dually perfect orthodox Dubreil-Jacotin semigroup. Then, for every $x \in S$,*

(1) $x^+ = \min V(x)$;

(2) $x^{++} = \beta x \beta$;

(3) xx^+ *is the smallest idempotent in the $\mathcal{R}$-class of x, and x^+x is that in the $\mathcal{L}$-class of x*;

(4) $(x,y) \in \mathcal{R} \iff xx^+ = yy^+, \quad (x,y) \in \mathcal{L} \iff x^+x = y^+y$.

Proof (1) Since every $x \in S$ is dually perfect we have $x\beta_x^{-1} \in E$. It follows from this that $x^+ \in V(x)$. If now $x' \in V(x)$ then, by Theorem 2.1,

$$x^+ = \beta_x^{-1}x\beta_x^{-1} = \beta_{x'}x\beta_{x'} \leqslant x'xx' = x'.$$

(2) Since $\beta_{x^+} = \beta_x^{-1}$ we have

$$x^{++} = \beta_{x^+}^{-1}x^+\beta_{x^+}^{-1} = \beta_x.\beta_x^{-1}x\beta_x^{-1}.\beta_x = \beta x \beta.$$

(3) Since $x^+ \in V(x)$ we have $xx^+ \in E$ and $(x, xx^+) \in \mathcal{R}$. If now $e \in E$ is such that $(e,x) \in \mathcal{R}$ then $(e, xx^+) \in \mathcal{R}$ and so

$$e = xx^+e \geqslant xx^+\beta = xx^+.$$

(4) This is immediate from (3). ◇

3. CONSTRUCTING DUALLY PERFECT SEMIGROUPS

We now proceed to describe a method of constructing a dually perfect orthodox Dubreil-Jacotin semigroup. By the nature of its construction, Green's relations $\mathcal{R}, \mathcal{L}$ turn out to be regular, in the sense that

$$x \leqslant y \Rightarrow xx^+ \leqslant yy^+, \quad x^+x \leqslant y^+y.$$

In Section 4 we shall prove that every dually perfect orthodox Dubreil-Jacotin semigroup on which $\mathcal{R}, \mathcal{L}$ are regular is isomorphic to a semigroup that is constructed in this way, and we shall see that a considerable simplification occurs when the semigroup is connected.

Theorem 3.1 *Let B be an ordered band with a biggest element α and a smallest element β, the latter being a middle unit. Let* End B *be the ordered semigroup of isotone endomorphisms on B and let G be an ordered group. Suppose that*

$\zeta : G \to \operatorname{End} B$, *described by* $g \mapsto \zeta_g$, *is an isotone morphism such that*

(1) $(\forall x \in B)\quad \zeta_1(x) = \beta x \beta$;

(2) $(\forall g \in G)\quad \zeta_g(\beta) = \beta$;

(3) $(\forall g, h \in G)\quad g \leqslant h \Rightarrow \zeta_g = \zeta_h$.

On the cartesian ordered set

$$[B; G]_\zeta = \{(x, g, a) \in B\beta \times G \times \beta B \; ; \; \zeta_g(a) = \zeta_1(x)\}$$

define the multiplication

$$(x, g, a)(y, h, b) = (x\zeta_g(y), gh, \zeta_{h^{-1}}(a)b).$$

Then $[B; G]_\zeta$ *is a dually perfect orthodox Dubreil-Jacotin semigroup on which* $\mathcal{R}, \mathcal{L}$ *are regular.*

Proof Observe first that the above multiplication is well defined; for

$$\zeta_{gh}[\zeta_{h^{-1}}(a)b] = \zeta_g(a)\zeta_g[\zeta_h(b)] = \zeta_1(x)\zeta_g[\zeta_1(y)] = \zeta_1[x\zeta_g(y)].$$

A purely routine calculation shows that it is also associative. Equally routine is the verification, using property (3) and the fact that ζ and all ζ_g are isotone, that under the cartesian order $[B; G]_\zeta$ is an ordered semigroup. To see that it is regular, take $(x, g, a) \in [B; G]_\zeta$ and consider the element $(\beta, g^{-1}, \beta) \in [B; G]_\zeta$. We have

$$\begin{aligned}
(x, g, a)(\beta, g^{-1}, \beta)(x, g, a) &= (x\beta, 1, \zeta_g(a)\beta)(x, g, a) && \text{by (2)}\\
&= (x\beta x\beta,\, g,\, \zeta_{g^{-1}}[\zeta_g(a)\beta]a) && \text{by (1)}\\
&= (x\beta x\beta,\, g,\, \beta a\beta a) && \text{by (1), (2)}\\
&= (x, g, a) && x \in B\beta,\ a \in \beta B.
\end{aligned}$$

It is readily seen that the set of idempotents of $[B; G]_\zeta$ is

$$E([B; G]_\zeta) = \{(x, 1, a) \; ; \; \beta x\beta = \beta a\beta\}.$$

So $[B; G]_\zeta$ has a biggest idempotent, namely $(\alpha, 1, \alpha)$, and a smallest idempotent, namely $(\beta, 1, \beta)$. Moreover, $[B; G]_\zeta$ is orthodox; for if $(x, 1, a)$ and $(y, 1, b)$ are idempotents in $[B; G]_\zeta$ then

$$(x, 1, a)(y, 1, b) = (x\beta y\beta, 1, \beta a\beta b) = (xy, 1, ab)$$

and $\beta xy\beta = \beta x\beta y\beta = \beta a\beta b\beta = \beta ab\beta$, whence $(xy, 1, ab) \in E([B; G]_\zeta)$.

That $[B; G]_\zeta$ is dually naturally ordered results from the observation that if $(x, 1, a) \preceq (y, 1, b)$ then

$$(x, 1, a) = (xy, 1, ab) = (yx, 1, ba)$$

so that $x \preceq y$ and $a \preceq b$ in B. But since β is a middle unit it follows by the dual of [5, Proposition 1.9] that B is dually naturally ordered. Hence $y \leqslant x$ and $b \leqslant a$, and consequently $(y, 1, b) \leqslant (x, 1, a)$.

Consider now the mapping $\varphi : [B; G]_\zeta \to G$ defined by

$$\varphi(x, g, a) = g.$$

This is clearly an isotone epimorphism. Since the pre-image under φ of the negative cone of G is the principal order ideal $(\alpha, 1, \alpha)^\downarrow$, the epimorphism φ is principal and so $[B; G]_\zeta$ is Dubreil-Jacotin. Consider now the mapping $\varphi^+ : G \to [B; G]_\zeta$ given by

$$\varphi^+(g) = (\alpha, g, \alpha).$$

Clearly, φ^+ is isotone and

$$\begin{cases} \varphi^+\varphi(x, g, a) = \varphi^+(g) = (\alpha, g, \alpha) \geqslant (x, g, a);\\ \varphi\varphi^+(g) = \varphi(\alpha, g, \alpha) = g. \end{cases}$$

Consequently, φ is residuated with residual φ^+. Thus $[B; G]_\zeta$ is strong Dubreil-Jacotin. The bimaximum element is $\xi = \varphi^+(1) = (\alpha, 1, \alpha)$, and it is easy to verify that the residuals of ξ are given by

$$(\alpha, 1, \alpha) : (x, g, a) = (\alpha, g^{-1}, \alpha).$$

Thus the A_ξ-classes are given by

$$(\star)\qquad [(x, g, a)]_{A_\xi} = \{(y, h, b) \; ; \; h = g\}.$$

We now identify the $\mathcal{H}$-class of the smallest idempotent $(\beta, 1, \beta)$. For this purpose, suppose first that $(x, g, a)\,\mathcal{R}\,(\beta, 1, \beta)$. Then for some (y, h, b) and (z, k, c) we have

$$\begin{cases} (\beta, 1, \beta) = (x, g, a)(y, h, b) = (x\zeta_g(y), gh, \zeta_{h^{-1}}(a)b);\\ (x, g, a) = (\beta, 1, \beta)(z, k, c) = (\beta z, k, c). \end{cases}$$

The first of these gives $h^{-1} = g$, whence

$$\beta = \zeta_g(a)b = \zeta_1(x)b = \beta xb;$$

and the second gives $x = \beta z$, whence $\beta x = x$. Consequently, since $x \in L$, we have

$$\beta = xb \geqslant x\beta = x$$

and therefore $x = \beta$. Arguing similarly with $\mathcal{L}$, we deduce that

$$H_{(\beta,1,\beta)} \subseteq \{(\beta, g, \beta) \; ; \; g \in G\}.$$

But the equations

$$(\beta, 1, \beta)(\beta, g, \beta) = (\beta, g, \beta), \qquad (\beta, g, \beta)(\beta, g^{-1}, \beta) = (\beta, 1, \beta)$$

show that $(\beta, g, \beta)\,\mathcal{R}\,(\beta, 1, \beta)$; and similarly $(\beta, g, \beta)\,\mathcal{L}\,(\beta, 1, \beta)$. Hence

$$(\star\star)\qquad H_{(\beta,1,\beta)} = \{(\beta, g, \beta) \; ; \; g \in G\}.$$

It now follows from $(\star)$ and $(\star\star)$ that

$$H_{(\beta,1,\beta)} \cap [(x, g, a)]_{A_\xi} = \{(\beta, g, \beta)\}.$$

Consequently, $[B;G]_\zeta$ is bounded. Since $(\beta,g,\beta) = \min[(x,g,a)]_{A_\zeta}$, it now follows by Theorem 2.1 that $[B;G]_\zeta$ is dually perfect.

To prove that $\mathcal{R}$ is regular on $[B;G]_\zeta$ we use Theorems 2.1, 2.4 and the fact that $xx^+ = x\beta_x^{-1}$; we have

$$(x,g,a)\beta^{-1}_{(x,g,a)} = (x,g,a)(\beta,g,\beta)^{-1} = (x,g,a)(\beta,g^{-1},\beta) = (x,1,\zeta_g(a)\beta).$$

That $\mathcal{R}$ is regular now follows from the fact that ζ and ζ_g are isotone. Similarly, $\mathcal{L}$ is regular on $[B;G]_\zeta$. ◇

Note that in the above construction we have $H_{(\beta,1,\beta)} \simeq G$ and $E([B;G]_\zeta) \simeq B$. The first isomorphism is given by the assignment $(\beta,g,\beta) \mapsto g$. As for the second, consider the mapping $\lambda : B \to E([B;G]_\zeta)$ given by $\lambda(x) = (x\beta,1,\beta x)$. This is a morphism since, β being a middle unit,

$$\lambda(x)\lambda(y) = (x\beta.\beta y\beta,1,\beta x\beta.\beta y) = (xy\beta,1,\beta xy) = \lambda(xy).$$

It is injective; for if $(x\beta,1,\beta x) = (y\beta,1,\beta y)$ then $x\beta = y\beta$ gives $x = x\beta x = y\beta x = yx$, and $\beta x = \beta y$ gives $y = y\beta y = y\beta x = yx$, so $x = y$. Finally, it is surjective since given $(e\beta,1,\beta f) \in E([B;G]_\zeta)$ we have $\beta e\beta = \beta f\beta$ so

$$\lambda(ef) = (ef\beta,1,\beta ef) = (e\beta f\beta,1,\beta e\beta f) = (e\beta e\beta,1,\beta f\beta f) = (e\beta,1,\beta f).$$

4. THE ISOMORPHISM

We now show that every dually perfect orthodox Dubreil-Jacotin semigroup on which $\mathcal{R},\mathcal{L}$ are regular is isomorphic to a semigroup as constructed in Theorem 3.1.

Theorem 4.1 *Let S be a dually perfect orthodox Dubreil-Jacotin semigroup on which $\mathcal{R},\mathcal{L}$ are regular. Let E be the band of idempotents of S and let β be the smallest element of E. Then β is a middle unit and the mapping $\vartheta : H_\beta \to \operatorname{End} E$ described by $\beta_x \mapsto \vartheta_{\beta_x}$, where*

$$(\forall e \in E) \qquad \vartheta_{\beta_x}(e) = \beta_x e\beta_x^{-1},$$

is an isotone morphism that satisfies conditions (1), (2), (3) of Theorem 3.1 and, as ordered semigroups,

$$S \simeq [E;H_\beta]_\vartheta.$$

Proof Since S is dually perfect, it is dually naturally ordered and so β is a middle unit. It follows that $\beta_x e\beta_x^{-1} \in E$. Moreover, each ϑ_{β_x} is a morphism; for, given $e,f \in E$, we have

$$\vartheta_{\beta_x}(e)\vartheta_{\beta_x}(f) = \beta_x e\beta_x^{-1}\beta_x f\beta_x^{-1} = \beta_x e\beta f\beta_x^{-1} = \beta_x ef\beta_x^{-1} = \vartheta_{\beta_x}(ef).$$

Clearly, each ϑ_{β_x} is isotone, so $\vartheta_{\beta_x} \in \operatorname{End} E$.

Since, for all $e \in E$,

$$\vartheta_{\beta_x}[\vartheta_{\beta_y}(e)] = \beta_x\beta_y e\beta_y^{-1}\beta_x^{-1} = \beta_x\beta_y e(\beta_x\beta_y)^{-1} = \vartheta_{\beta_x\beta_y}(e),$$

we also have

$$\vartheta_{\beta_x}\vartheta_{\beta_y} = \vartheta_{\beta_x\beta_y},$$

so $\vartheta : H_\beta \to \operatorname{End} E$ is also a morphism.

That ϑ is isotone is a consequence of the regularity of $\mathcal{R}$ on S. To see this, observe first that

$$(\forall x \in S)(\forall e \in E) \qquad \beta^{-1}_{\beta_x e} = \beta_x^{-1}.$$

In fact, on passing to quotients modulo A_ξ and using Theorem 2.1, we have that

$$[\beta_x e]_{A_\xi} = [xe]_{A_\xi} = [x]_{A_\xi},$$

whence the result follows by Theorem 2.1 again. Using this fact and the regularity of $\mathcal{R}$, we then have

$$\begin{aligned} \beta_x \leqslant \beta_y &\Rightarrow (\forall e \in E) \quad \beta_x e \leqslant \beta_y e \\ &\Rightarrow (\forall e \in E) \quad \beta_x e\beta^{-1}_{\beta_x e} \leqslant \beta_y e\beta^{-1}_{\beta_y e} \\ &\Rightarrow (\forall e \in E) \quad \beta_x e\beta_x^{-1} \leqslant \beta_y e\beta_y^{-1} \\ &\Rightarrow \vartheta_{\beta_x} \leqslant \vartheta_{\beta_y}, \end{aligned}$$

so that ϑ is isotone.

We now show that ϑ satisfies conditions (1), (2), (3) of Theorem 3.1. As for (1), we have

$$\vartheta_\beta(e) = \beta e\beta^{-1} = \beta e\beta;$$

and as for (2),

$$\vartheta_{\beta_x}(\beta) = \beta_x\beta\beta_x^{-1} = \beta.$$

To establish (3), we use the regularity of $\mathcal{L}$ on S. In fact, we have

$$\begin{aligned} \beta_x \leqslant \beta_y &\Rightarrow (\forall e \in E) \quad e\beta_y^{-1} \leqslant e\beta_x^{-1} \\ &\Rightarrow (\forall e \in E) \quad \beta^{-1}_{e\beta_y^{-1}} e\beta_y^{-1} \leqslant \beta^{-1}_{e\beta_x^{-1}} e\beta_x^{-1} \\ &\Rightarrow (\forall e \in E) \quad \beta_y e\beta_y^{-1} \leqslant \beta_x e\beta_x^{-1} \\ &\Rightarrow \vartheta_{\beta_y} \leqslant \vartheta_{\beta_x}. \end{aligned}$$

But as ϑ is isotone $\beta_x \leqslant \beta_y$ gives $\vartheta_{\beta_x} \leqslant \vartheta_{\beta_y}$. Consequently, we have

$$\beta_x \leqslant \beta_y \Rightarrow \vartheta_{\beta_x} = \vartheta_{\beta_y},$$

which is (3).

Using Theorem 3.1, we can now construct the semigroup $[E;H_\beta]_\vartheta$ which is dually perfect orthodox Dubreil-Jacotin with $\mathcal{R},\mathcal{L}$ regular. Now since S is dually perfect we have

$$(\forall x \in S) \qquad x = x\beta_x^{-1}x = x\beta_x^{-1}.\beta_x.\beta_x^{-1}x,$$

where $x\beta_x^{-1} = xx^+ \in E\beta$ and $\beta_x^{-1}x = x^+x \in \beta E$. Moreover,

$$(\forall x \in S) \qquad \vartheta_{\beta_x}(\beta_x^{-1}x) = \beta x\beta_x^{-1} = \vartheta_\beta(x\beta_x^{-1}).$$

We can therefore define a mapping $\psi : S \to [E; H_\beta]_\vartheta$ by the assignment

$$\psi(x) = (x\beta_x^{-1},\ \beta_x,\ \beta_x^{-1}x).$$

Using the fact that β is a middle unit, we see that ψ is a morphism; in fact, for all $x, y \in S$,

$$\begin{aligned}\psi(x)\psi(y) &= (x\beta_x^{-1},\ \beta_x,\ \beta_x^{-1}x)(y\beta_y^{-1},\ \beta_y,\ \beta_y^{-1}y)\\ &= (x\beta_x^{-1}.\vartheta_{\beta_x}(y\beta_y^{-1}),\ \beta_{xy},\ \vartheta_{\beta_y^{-1}}(\beta_x^{-1}x).\beta_y^{-1}y)\\ &= (x\beta_x^{-1}\beta_x y\beta_y^{-1}\beta_x^{-1},\ \beta_{xy},\ \beta_y^{-1}\beta_x^{-1}x\beta_y\beta_y^{-1}y)\\ &= (xy\beta_{xy}^{-1},\ \beta_{xy},\ \beta_{xy}^{-1}xy)\\ &= \psi(xy).\end{aligned}$$

Since $\mathcal{R}$ and $\mathcal{L}$ are regular, it is clear that ψ is isotone.

That ψ is injective is clear from the fact that $x\beta_x^{-1}.\beta_x.\beta_x^{-1}x = x$. To see that ψ is also surjective, let $(x, \beta_y, a) \in [E; H_\beta]_\vartheta$. Then we have

$$\vartheta_{\beta_y}(a) = \vartheta_\beta(x) = \beta x\beta = \beta x,$$

and

$$\vartheta_{\beta_y^{-1}}(x) = \vartheta_\beta(a) = \beta a\beta = a\beta.$$

Also, x and a are idempotents so, as before,

$$\beta_{x\beta_y a}^{-1} = \beta_y^{-1}.$$

Consequently,

$$\begin{aligned}\psi(x\beta_y a) &= (x\beta_y a\beta_y^{-1},\ \beta_y,\ \beta_y^{-1}x\beta_y a)\\ &= (x\vartheta_{\beta_y}(a),\ \beta_y,\ \vartheta_{\beta_y^{-1}}(x)a)\\ &= (x\beta x,\ \beta_y,\ \vartheta_\beta(a)a)\\ &= (x, \beta_y, a).\end{aligned}$$

Hence ϑ is an order isomorphism. ◇

5. PARTICULAR CASES

A remarkable feature of Theorem 3.1 is the condition (3) which can be stated in the form : if $g, h \in G$ are comparable then $\zeta_g = \zeta_h$; or, roughly speaking, that there is a single morphism ζ_g for every Hasse diagram component of G. Suppose then that G is *connected* in the sense that it consists of a single component, so

that for all $g, h \in G$ there is a finite zig-zag chain joining g to h. Then clearly (3) is satisfied; and since $\zeta_g = \zeta_1$ for all $g \in G$, condition (1) implies condition (2). In this case we then have that

$$[B; G]_\zeta = \{(x, g, a) \in B\beta \times G \times \beta B \ ;\ \beta x = a\beta\}$$

and since β is a middle unit the multiplication becomes

$$(x, g, a)(y, h, b) = (xy, gh, ab).$$

So if G is connected $[B; G]_\zeta$ is a subalgebra of the cartesian ordered cartesian product semigroup $B\beta \times G \times \beta B$. Focussing more closely, we see by the principal theorem of [1] that $[B; G]_\zeta$ reduces to the cartesian ordered cartesian product of G with the spined product $B\beta \,|\!\times\!|\, \beta B$.

Another situation where a simplification occurs is when the dually naturally ordered band B is a $\vee$-semilattice. In this case $E([B; G]_\zeta)$ is a semilattice and so $[B; G]_\zeta$ is an inverse semigroup. Moreover, β is the identity of B so (1) holds; and $\zeta_1 = \mathrm{id}_B$ and each ζ_g is an isotone automorphism, whence (2) is satisfied. So we can obtain the structure of a dually perfect Dubreil-Jacotin inverse semigroup on which $\mathcal{R}, \mathcal{L}$ are regular by taking B to be a $\vee$-semilattice, $\zeta : G \to \mathrm{Aut}\, B$ an isotone morphism and retaining only property (3).

Finally, combining the above two observations we can deduce that a connected dually perfect Dubreil-Jacotin inverse semigroup on which $\mathcal{R}, \mathcal{L}$ are regular is the cartesian ordered cartesian product of an ordered group and a $\vee$-semilattice that has a biggest element and a smallest element.

Example 5.1 Let $I = \{x \in \mathbb{R}\ ;\ 0 \leqslant x \leqslant 1\}$ and consider the cartesian ordered set $S = I \times I \times \mathbb{Z}$ made into an ordered semigroup by the multiplication

$$(a, b, x)(c, d, y) = (a \vee c, b, x + y).$$

The idempotents are the elements of the form $(a, b, 0)$, so S is orthodox. The mapping $\pi : S \to \mathbb{Z}$ given by $\pi(a, b, x) = x$ is an isotone epimorphism which is residuated, with residual $\pi^+ : \mathbb{Z} \to S$ given by $\pi^+(x) = (1, 1, x)$. So S is strong Dubreil-Jacotin. The bimaximum element is $\xi = \pi^+(0) = (1, 1, 0)$, and residuals are given by $\xi : (a, b, x) = (1, 1, -x)$. The A_ξ-classes are given by

$$[(a, b, x)]_{A_\xi} = \{(c, d, x)\ ;\ c, d \in I\}.$$

The $\mathcal{R}$-classes and $\mathcal{L}$-classes are given by

$$R_{(a,b,x)} = \{(a, b, y)\ ;\ y \in \mathbb{Z}\}, \quad L_{(a,b,x)} = \{(a, c, y)\ ;\ c \in I,\ y \in \mathbb{Z}\}.$$

So the $\mathcal{H}$-class of the smallest idempotent $(0, 0, 0)$ is

$$H_{(0,0,0)} = \{(0, 0, y)\ ;\ y \in \mathbb{Z}\}.$$

It follows that

$$H_{(0,0,0)} \cap [(a, b, x)]_{A_\xi} = \{(0, 0, x)\}$$

and so S is bounded. Here $\beta_{(a,b,x)} = (0,0,x) = \min[(a,b,x)]_{\Lambda_\xi}$. Since S is dually naturally ordered, it follows by Theorem 2.1 that S is dually perfect. Since

$$(a,b,x)\beta^{-1}_{(a,b,x)} = (a,b,x)(0,0,x) = (a,b,0);$$
$$\beta^{-1}_{(a,b,x)}(a,b,x) = (0,0,x)(a,b,x) = (a,0,0),$$

we see that $\mathcal{R}, \mathcal{L}$ are regular. In this case, S is connected. The coordinatisation of the isomorphism theorem is

$$(a,b,x) \sim ((a,b,0),\ (0,0,x),\ (a,0,0)).$$

The subset $S_* = I \times \{0\} \times \mathbb{Z}$ of S is a connected inverse subsemigroup. Setting the middle components to 0 in the above or, equivalently, ignoring them, we have in this case the coordinatisation

$$(a,x) \sim ((a,0),\ (0,x)).$$

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