ASSOCIATE SUBGROUPS OF ORTHODOX SEMIGROUPS

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A *unit regular semigroup* [**1, 4**] is a regular monoid S such that $H_1 \cap A(x) \neq \varnothing$ for every $x \in S$, where H_1 is the group of units and $A(x) = \{y \in S; xyx = x\}$ is the set of associates (or pre-inverses) of x. A *uniquely unit regular semigroup* is a regular monoid S such that $|H_1 \cap A(x)| = 1$. Here we shall consider a more general situation. Specifically, we consider a regular semigroup S and a subsemigroup T with the property that $|T \cap A(x)| = 1$ for every $x \in S$. We show that T is necessarily a maximal subgroup H_α for some idempotent α. When S is orthodox, α is necessarily medial (in the sense that $x = x\alpha x$ for every $x \in \langle E \rangle$) and $\alpha S \alpha$ is uniquely unit orthodox. When S is orthodox and α is a middle unit (in the sense that $x\alpha y = xy$ for all $x, y \in S$), we obtain a structure theorem which generalises the description given in [**2**] for uniquely unit orthodox semigroups in terms of a semi-direct product of a band with an identity and a group.

Let S be a regular semigroup. Consider a subsemigroup T of S with the property that $|T \cap A(x)| = 1$ for every $x \in S$. In this case we define x^* by $T \cap A(x) = \{x^*\}$. We also define $x^{**} = (x^*)^*$ for every $x \in S$. Then $(x^*)^{**} = [(x^*)^*]^* = (x^{**})^*$ which we can write as x^{***}.

Observe that since $x^* \in A(x)$ we have $x^*xx^* \in V(x) \subseteq A(x)$. Therefore, if $x \in T$ then $x^*xx^* \in T \cap A(x) = \{x^*\}$ whence $x^*xx^* = x^*$ and consequently $x \in T \cap A(x^*) = \{x^{**}\}$, so that $x = x^{**}$. Writing $S^* = \{x^*; x \in S\}$ we therefore have $T \subseteq S^*$. Since the reverse inclusion follows from the definition of x^*, we thus have $T = S^*$. Observe also that $x^{**}x^*x^{**} \in V(x^*)$ gives $x^{**}x^*x^{**} \in T \cap A(x^*) = \{x^{**}\}$. Hence $x^{**}x^*x^{**} = x^{**}$ and so $x^* \in T \cap A(x^{**}) = \{x^{***}\}$. Thus $x^{***} = x^*$, from which it follows that $x \in T = S^*$ if and only if $x = x^{**}$.

Since $x^{**} \in V(x^*)$ we have that S^* is regular; and since $y \in S^* \cap V(x^*)$ gives $y \in S^* \cap A(x^*) = \{x^{**}\}$ we see that S^* is inverse with $(x^*)^{-1} = x^{**}$. If now $e, f \in E(S^*)$ then since e and f commute we have $ef.e.ef = ef = ef.f.ef$ whence $e, f \in S^* \cap A(ef)$ and therefore $e = (ef)^* = f$. Thus $E(S^*)$ is a singleton and so S^* is in fact a group. Denoting by α the identity element of S^* we then have the properties

$$(\forall x \in S) \quad x^*\alpha = x^* = \alpha x^*, \quad x^*x^{**} = \alpha = x^{**}x^*.$$

In what follows we shall call such a subgroup S^* an *associate subgroup* of S.

We begin by listing some basic properties arising from the existence of an associate subgroup. For every $x \in S$ we define

$$x^\circ = x^*xx^*.$$

It is clear that $x^\circ \in V(x)$ and $xx^\circ = xx^*$, $x^\circ x = x^*x$. We first investigate the relationship between x° and x^*.

THEOREM 1. $(\forall x \in S)\ x^{*\circ} = x^{**} = x^{\circ *}$.

Proof. The first equality results from the observation that

$$x^{*\circ} = x^{**}x^*x^{**} \in S^* \cap V(x^*) \subseteq S^* \cap A(x^*) = \{x^{**}\}.$$

As for the second equality, we have $x^\circ = x^\circ x^{\circ *} x^\circ$ and so

$$x = xx^\circ x = xx^\circ x^{\circ *} x^\circ x = xx^* x^{\circ *} x^* x$$

whence $x^* x^{\circ *} x^* \in S^* \cap A(x)$ and therefore $x^* x^{\circ *} x^* = x^*$. It now follows that $x^{\circ *} \in S^* \cap A(x^*) = \{x^{**}\}$. □

COROLLARY 1. $(\forall x \in S)\ x^{\circ\circ} = \alpha x \alpha$.

Proof. By the above, we have $x^{\circ\circ} = x^{\circ *} x^\circ x^{\circ *} = x^{**} x^* x x^* x^{**} = \alpha x \alpha$. □

COROLLARY 2. $(\forall x \in S)\ x^{\circ\circ\circ} = x^\circ$.

Proof. We have

$$\begin{aligned} x^{\circ\circ\circ} = x^{\circ\circ *} x^{\circ\circ} x^{\circ\circ *} = x^{***} x^{\circ\circ} x^{***} &= x^* x^{\circ\circ} x^* \\ &= x^* x x^* \text{ by Corollary 1} \\ &= x^\circ. \quad \square \end{aligned}$$

Defining $S^\circ = \{x^\circ ; x \in S\}$ we see from the above results that

$$x \in S^\circ \Leftrightarrow x = \alpha x \alpha$$

so that $S^\circ = \alpha S \alpha$. The subsemigroup S° is regular; for we have

$$\alpha x \alpha = \alpha x x^* x \alpha = \alpha x \alpha x^* \alpha x \alpha = \alpha x \alpha \,.\, \alpha x^* \alpha \,.\, \alpha x \alpha.$$

This also gives $x^* = \alpha x^* \alpha = (\alpha x \alpha)^*$. Moreover, since α is the identity element of S^* we have that $S^* \subseteq S^\circ$.

We now show that every associate subgroup of S is in fact a maximal subgroup, the uniquely unit regular situation therefore being a special case.

THEOREM 2. $S^* = H_\alpha$.

Proof. Since the maximal subgroups of S are precisely the $\mathcal{H}$-classes containing idempotents we have $S^* \subseteq H_\alpha$. To obtain the reverse inclusion, let $x \in H_\alpha$. Then $xx^* \in H_\alpha$ and $x^* x \in H_\alpha$ give $xx^* = \alpha = x^* x$ whence $x^\circ = x^* x x^* = x^* \alpha = x^*$ and $x = \alpha x \alpha = x^{\circ\circ}$. Consequently, $x = x^{\circ\circ} = x^{*\circ} = x^{**} \in S^*$. □

COROLLARY. *S° is uniquely unit regular with group of units H_α.*

Proof. Since S is regular and $H_\alpha = S^* \subseteq S^\circ$ we have that H_α is an $\mathcal{H}$-class of S°. Moreover,

$$H_\alpha \cap A(\alpha x \alpha) = S^* \cap A(\alpha x \alpha) = \{(\alpha x \alpha)^*\}$$

and $(\alpha x \alpha)^* = x^* = \alpha x^* \alpha \in S^\circ$. Since $\alpha x^\circ = x^\circ = x^\circ \alpha$ it follows that S° is uniquely unit regular with group of units H_α. □

THEOREM 3. $(\forall x, y \in S)\ (xy)^* = (x^* xy)^* x^* = y^* (xyy^*)^*$.

Proof. We have

$$xy \,.\, (x^* xy)^* x^* \,.\, xy = x \,.\, x^* xy (x^* xy)^* x^* xy = xx^* xy = xy$$

and so $(x^* xy)^* x^* \in S^* \cap A(xy)$ whence $(x^* xy)^* x^* = (xy)^*$. The other identity is established similarly. □

Observe now that $\alpha x\alpha \in E(S^\circ)$ if and only if $\alpha x\alpha x\alpha = \alpha x\alpha$. Pre-multiplying by xx^* and post-multiplying by x^*x, we see that this is equivalent to $x\alpha x = x$, i.e. to $x^* = \alpha$. Thus $\alpha x\alpha \in E(S^\circ)$ implies $x = xx^*x = xx^* . x^*x \in \langle E\rangle$. It follows from these observations that we have $E(S^\circ) \subseteq \alpha\langle E\rangle\alpha$.

THEOREM 4. *The following statements are equivalent*:
(1) α *is medial*;
(2) $(\forall x, y \in S)\ (xy)^* = y^*x^*$;
(3) $E(S^\circ) = \alpha\langle E\rangle\alpha$.

Proof. (1)$\Rightarrow$(2): If α is medial then $\alpha = x^*$ for every $x \in \langle E\rangle$. It follows by Theorem 3 that

$$(xy)^* = y^*(x^*xyy^*)^*x^* = y^*\alpha x^* = y^*x^*.$$

(2)$\Rightarrow$(1): If (2) holds then we have $e^* \in E$ for every $e \in E$. Since S^* is a group it follows that $e^* = \alpha$ for every $e \in E$. Consequently, if $e_1, e_2 \in E$ then by (2) we have $(e_1e_2)^* = e_2^*e_1^* = \alpha\alpha = \alpha$, whence by induction we have $x^* = \alpha$ for all $x \in \langle E\rangle$. It follows that $x = x\alpha x$ for every $x \in \langle E\rangle$ whence α is medial.

(1)$\Rightarrow$(3): Suppose that α is medial and that $x \in \langle E\rangle$. Then $x^*\alpha$ and so $\alpha x\alpha \in E(S^\circ)$ whence $\alpha\langle E\rangle\alpha \subseteq E(S^\circ)$.

(3)$\Rightarrow$(1): If (3) holds and $x \in \langle E\rangle$ then $\alpha x\alpha$ is idempotent so $x^* = \alpha$ and $x = xx^*x = x\alpha x$, i.e. α is medial. □

COROLLARY. *If* α *is medial then* S° *is uniquely unit orthodox*. □

THEOREM 5. *If* S *is orthodox then* α *is medial and* $E(S^\circ) = \alpha E\alpha$.

Proof. If S is orthodox then we have $y^\circ x^\circ \in V(xy) \subseteq A(xy)$. Then

$$xy = xyy^\circ x^\circ xy = xyy^*x^*xy$$

whence $y^*x^* \in S^* \cap A(xy)$ and therefore $y^*x^* = (xy)^*$. The result therefore follows by Theorem 4. □

COROLLARY 1. *If* S *is orthodox then* $e^* = \alpha$ *for every* $e \in E$. □

COROLLARY 2. *If* S *is orthodox then any two associate subgroups of* S *are isomorphic.*

Proof. Let A, B be associate subgroups of S with respective identity elements α, β. Since S is orthodox, α and β are medial so $\beta = \beta\alpha\beta$ and $\alpha = \alpha\beta\alpha$. Thus $\beta \in V(\alpha)$ and so β belongs to the $\mathscr{D}$-class of α. Consequently we have that $B = H_\beta \simeq H_\alpha = A$. □

Observe that if we define $E^\circ = \{e^\circ; e \in E(S)\}$ then, when S is orthodox, we have $E^\circ = E(S^\circ)$. This follows immediately from Theorem 5.

THEOREM 6. *The following statements are equivalent*:
(1) α *is a middle unit*;
(2) $(\forall x, y \in S)\ (xy)^{\circ\circ} = x^{\circ\circ}y^{\circ\circ}$.

Proof. (1)$\Rightarrow$(2): If α is a middle unit then

$$(xy)^{\circ\circ} = \alpha xy\alpha = \alpha x\alpha\,.\,\alpha y\alpha = x^{\circ\circ}y^{\circ\circ}.$$

(2)$\Rightarrow$(1): If (2) holds then for all $x, y \in S$ we have

$$\begin{aligned} x^\circ xyy^\circ = x^\circ \alpha xy\alpha y^\circ &= x^\circ (xy)^{\circ\circ} y^\circ \\ &= x^\circ x^{\circ\circ} y^{\circ\circ} y^\circ \\ &= x^\circ \,.\, \alpha x\alpha \,.\, \alpha y\alpha \,.\, y^\circ \\ &= x^\circ x\alpha yy^\circ \end{aligned}$$

whence $xy = x\alpha y$ and so α is a middle unit. □

We recall now the following definitions. A medial idempotent α of a regular semigroup is said to be *normal* [**3**] if the band $\alpha\langle E\rangle\alpha$ is commutative. A regular semigroup S is said to be *locally inverse* if for every idempotent e the subsemigroup eSe is inverse. An *inverse transversal* of a regular semigroup S is an inverse subsemigroup T with the property that $|T \cap V(x)| = 1$ for every $x \in S$. If we let $T \cap V(x) = \{x^\circ\}$ then we have $T = S^\circ = \{x^\circ; x \in S\}$ and the inverse transversal S° is said to be *multiplicative* if $x^\circ xyy^\circ \in E(S^\circ)$ for all $x, y \in S$.

THEOREM 7. *If S is orthodox then the following statements are equivalent*:

(1) *α is a normal medial idempotent*;
(2) *$S^\circ = \alpha S\alpha$ is inverse*;
(3) $(\forall x, y \in S)\ (xy)^\circ = y^\circ x^\circ$;
(4) *S is locally inverse*;
(5) *S° is a multiplicative inverse transversal of S.*

Proof. (1)$\Rightarrow$(2): By Theorem 4, $E(S^\circ) = \alpha E\alpha$ which by (1) is a semilattice.

(2)$\Rightarrow$(1): By Theorem 5, α is medial; and by (2) it is normal.

(1)$\Rightarrow$(3): If (1) holds then α is a middle unit by [**3**, Theorem 2.2]. It follows that for all $x, y \in S$ we have

$$\begin{aligned} (xy)^\circ &= (xy)^*xy(xy)^* \\ &= y^*x^*xyy^*x^* \quad \text{by Theorem 4} \\ &= y^* \,.\, \alpha x^*x\alpha \,.\, \alpha yy^*\alpha \,.\, x^* \\ &= y^* \,.\, \alpha yy^*\alpha \,.\, \alpha x^*x\alpha \,.\, x^* \quad \text{by (1)} \\ &= y^*yy^*x^*xx^* \\ &= y^\circ x^\circ. \end{aligned}$$

(3)$\Rightarrow$(2): By Corollary 1 of Theorem 5 we have $e^* = \alpha$ and hence $e^\circ = e^*ee^* = \alpha e\alpha = e$. Suppose then that (3) holds. Then for $e, f \in E(S^\circ)$ we have

$$(ef)^\circ = f^\circ e^\circ = fe.$$

It follows that $ef = ef \,.\, fe \,.\, ef = efef$ and so S° is orthodox. Moreover, we have $ef \in E(S^\circ)$ and so $(ef)^\circ = ef$. Hence $ef = fe$ for all $e, f \in E(S^\circ)$ and so S° is inverse.

(1)$\Rightarrow$(4): For $e \in E$ and $x \in S$ we have, by Theorem 5 and its Corollary 1,

$$(exe)^* = e^*x^*e^* = \alpha x^*\alpha = x^*.$$

Hence $exe = exe(exe)^*exe = exe \,.\, ex^*e \,.\, exe$ and so eSe is regular. That the idempotents in eSe commute is shown precisely as in [**3**, Theorem 4.3].

(4)$\Rightarrow$(2): This is clear.

(2)$\Rightarrow$(5): If (2) holds then by the above so does (1) whence α is middle unit; and so does (3). Suppose then that $x \in S^\circ$ and that $y \in S^\circ \cap V(x)$. We have $y = y^{\circ\circ}$ and $yxy = y$, $xyx = x$. By (2) and (3) it follows that

3 $$y = y^{\circ\circ} = \alpha y \alpha = (\alpha x \alpha)^{-1} = (\alpha x \alpha)^\circ = \alpha x^\circ \alpha = x^\circ.$$

Hence S° is an inverse transversal of S. Since, for all $x, y \in S$,

$$\begin{aligned}(x^\circ xyy^\circ)^\circ &= (x^\circ xyy^\circ)^* x^\circ xyy^\circ (x^\circ xyy^\circ)^* \\ &= \alpha x^\circ xyy^\circ \alpha \quad \text{by Corollary 1 of Theorem 5} \\ &= x^\circ xyy^\circ,\end{aligned}$$

we have that $x^\circ xyy^\circ \in E(S^\circ)$ and so S° is multiplicative.

(5)$\Rightarrow$(2): This is clear. □

Example. Let B be a rectangular band and let B^1 be obtained from B by adjoining an identity element 1. Let $S = \mathbb{Z} \times B^1 \times \mathbb{Z}$ and define on S the multiplication

$$(m, x, p)(n, y, q) = (m_k + n, xy, p + q_k)$$

where, for a fixed integer $k > 1$, m_k is the greatest multiple of k that is less than or equal to m. It is readily seen that S is a semigroup. Simple calculations reveal that the set of associates of $(m, x, p) \in S$ is

$$A(m, x, p) = \begin{cases} \{(n, y, q); n_k = -m_k, q_k = -p_k\} & \text{if } x \neq 1; \\ \{(n, 1, q); n_k = -m_k, q_k = -p_k\} & \text{if } x = 1, \end{cases}$$

and that the set of inverses of $(m, x, p) \in S$ is

$$V(m, x, p) = \begin{cases} \{(n, y, q); n_k = -m_k, y \neq 1, q_k = -p_k\} & \text{if } x \neq 1; \\ \{(n, 1, q); n_k = -m_k, q_k = -p_k\} & \text{if } x = 1. \end{cases}$$

The set of idempotents of S is

$$E = \{(m, x, p); m_k = 0 = p_k\}$$

and so S is orthodox. For every $(m, x, p) \in S$ define

$$(m, x, p)^* = (-m_k, 1, -p_k).$$

Then S^* is an associate subgroup of S. The identity element of S^* is $\alpha = (0, 1, 0)$.

It is readily seen that α is a middle unit. Now

$$(m, x, p)^\circ = (m, x, p)^*(m, x, p)(m, x, p)^* = (-m_k, x, -p_k),$$

whence simple calculations give

$$\begin{aligned}[(m, x, p)(n, y, q)]^\circ &= (-m_k - n_k, xy, -p_k - q_k), \\ (n, y, q)^\circ (m, x, p)^\circ &= (-m_k - n_k, yx, -p_k - q_k).\end{aligned}$$

Now $xy \neq yx$ for distinct $x, y \in B$ so, by Theorem 7, α is not medial normal.

We now proceed to describe the structure of orthodox semigroups with an associate subgroup of which the identity element is a middle unit. For this purpose, let B be a band with a middle unit α and let End B be the monoid of endomorphisms on B. Define

$$\text{End}_\alpha B = \{f \in \text{End } B; f \text{ preserves } \alpha \text{ and } \text{Im} f = \alpha B \alpha\}.$$

Then $\text{End}_\alpha B$ is a subsemigroup of End B.

Consider the mapping $\varphi : B \to B$ given by $\varphi(x) = \alpha x \alpha$ for every $x \in B$. Since α is a middle unit, we have $\varphi \in \operatorname{End} B$. Moreover, φ clearly preserves α and $\operatorname{Im} \varphi = \alpha B \alpha$. Hence $\varphi \in \operatorname{End}_\alpha B$. In fact, φ is the identity element of $\operatorname{End}_\alpha B$; for if $f \in \operatorname{End}_\alpha B$ then

$$(\forall x \in B) \quad f\varphi(x) = f(\alpha x \alpha) = \alpha f(x)\alpha = \varphi f(x) = f(x),$$

the last equality following from the fact that $\varphi|_{\alpha B \alpha} = \mathrm{id}_{\alpha B \alpha}$. Hence $f\varphi = \varphi f = f$ and so $\operatorname{End}_\alpha B$ is a monoid.

THEOREM 8. *Let B be a band with a middle unit α and let G be a group. Let $\zeta : G \to \operatorname{End}_\alpha B$, described by $g \mapsto \zeta_g$, be a 1-preserving morphism. On the set*

$$[B; G]_\zeta = \{(x, g, a) \in B\alpha \times G \times \alpha B;\ \zeta_g(a) = \zeta_1(x)\}$$

define the multiplication

$$(x, g, a)(y, h, b) = (x\zeta_g(y), gh, \zeta_{h^{-1}}(a)b).$$

Then $[B; G]_\zeta$ is an orthodox semigroup with an associate subgroup of which the identity element $(\alpha, 1, \alpha)$ is a middle unit. Moreover, we have $E([B; G]_\zeta) \simeq B$ and $H_{(\alpha,1,\alpha)} \simeq G$.

Furthermore, every such semigroup is obtained in this way. More precisely, let S be an orthodox semigroup with an associate subgroup of which the identity element α is a middle unit. For every $y \in S$ let y^* be given by $H_\alpha \cap A(y) = \{y^*\}$, and for every $x \in H_\alpha$ let $\vartheta_x : E(S) \to E(S)$ be given by $\vartheta_x(e) = xex^*$. Then $\vartheta_x \in \operatorname{End}_\alpha E(S)$, the mapping $\vartheta : H_\alpha \to \operatorname{End}_\alpha E(S)$ described by $x \mapsto \vartheta_x$ is a 1-preserving morphism and

$$S \simeq [E(S); H_\alpha]_\vartheta.$$

Proof. Observe first that the multiplication on $[B; G]_\zeta$ is well defined, for we have $x\zeta_g(y) \in B\alpha \,.\, \alpha B \alpha \subseteq B\alpha$ and $\zeta_{h^{-1}}(a)b \in \alpha B \alpha \,.\, \alpha B \subseteq \alpha B$, with

$$\zeta_{gh}[\zeta_{h^{-1}}(a)b] = \zeta_g(a)\zeta_g[\zeta_h(b)] = \zeta_1(x)\zeta_g[\zeta_1(y)] = \zeta_1[x\zeta_g(y)].$$

A purely routine calculation shows that it is also associative. That the semigroup $[B; G]_\zeta$ is regular follows from the fact that

$$\begin{aligned}
(x, g, a)(\alpha, g^{-1}, \alpha)(x, g, a) &= (x\zeta_g(\alpha), gg^{-1}, \zeta_g(a)\alpha)(x, g, a) \\
&= (x\alpha, 1, \zeta_g(a)\alpha)(x, g, a) \\
&= (x\alpha\zeta_1(x), g, \zeta_{g^{-1}}[\zeta_g(a)\alpha]a) \\
&= (x\alpha x\alpha, g, \zeta_1(a)\alpha a) \\
&= (x, g, \alpha a \alpha a) \\
&= (x, g, a).
\end{aligned}$$

It is readily seen that the set of idempotents of $[B; G]_\zeta$ is

$$E([B; G]_\zeta) = \{(x, 1, a);\ \alpha x = a\alpha\},$$

and that the idempotent $(\alpha, 1, \alpha)$ is a middle unit of $[B; G]_\zeta$. If now $(x, 1, a)$ and $(y, 1, b)$ are idempotents then

$$\begin{aligned}
(x, 1, a)(y, 1, b) &= (x\zeta_1(y), 1, \zeta_1(a)b) \\
&= (x\alpha y\alpha, 1, \alpha a\alpha b) \\
&= (xy, 1, ab),
\end{aligned}$$

with $\alpha xy = a\alpha y = ab\alpha$. Hence we see that $[B; G]_\zeta$ is orthodox. Moreover, we have $E([B; G]_\zeta) \simeq B$. To see this, consider the mapping $f: E([B; G]_\zeta) \to B$ given by

$$f(x, 1, a) = xa.$$

Now f is surjective since for every $e \in B$ we have $(e\alpha, 1, \alpha e) \in E([B; G]_\zeta)$ with $f(e\alpha, 1, \alpha e) = e\alpha . \alpha e = e$. To see that f is also injective, suppose that $(x, 1, a)$ and $(y, 1, b)$ are idempotents with $f(x, 1, a) = f(y, 1, b)$. Then $xa = yb$ with $\alpha x = a\alpha$ and $\alpha y = b\alpha$. It follows that $x = x\alpha x = xa\alpha = yb\alpha = y\alpha y = y$ and similarly $a = b$. Finally, f is a morphism; for

$$f[(x, 1, a)(y, 1, b)] = f(xy, 1, ab) = xyab$$

and

$$\begin{aligned} xyab = x\alpha ya\alpha b &= xb\alpha xb \\ &= xb = x\alpha xb\alpha b = xa\alpha yb \\ &= xayb = f(x, 1, a)f(y, 1, b). \end{aligned}$$

It is also readily seen that

$$(y, h, b) \in A(x, g, a) \Leftrightarrow \zeta_g(y) \in A(x), h = g^{-1}, \zeta_{g^{-1}}(b) \in A(a).$$

Since α is a middle unit of B, it follows that $(\alpha, g^{-1}, \alpha) \in A(x, g, a)$. Defining

$$(x, g, a)^* = (\alpha, g^{-1}, \alpha),$$

we see that $[B; G]_\zeta^*$ is an associate subgroup, with identity element $(\alpha, 1, \alpha)$, that is isomorphic to G. It follows from Theorem 2 that $G \simeq H_{(\alpha,1,\alpha)}$.

Conversely, suppose that S is an orthodox semigroup with an associate subgroup G the identity element α of which is a middle unit of S. Then by Theorem 2 we have $G = H_\alpha$. Let x^* be given by $H_\alpha \cap A(x) = \{x^*\}$ for every $x \in S$. Observe first that for every $x \in H_\alpha$ we have $xex^* \in E(S)$ for every $e \in E(S)$. In fact, $xex^* . xex^* = xe\alpha ex^* = xex^*$. For $x \in H_\alpha$ the mapping $\vartheta_x: E(S) \to E(S)$ given by $\vartheta_x(e) = xex^*$ is then a morphism; for

$$\vartheta_x(ef) = xefx^* = xe\alpha fx^* = xex^* . xfx^* = \vartheta_x(e)\vartheta_x(f).$$

Moreover, ϑ_x preserves α. Since α is the identity of H_α it is clear that $\operatorname{Im} \vartheta_x \subseteq \alpha E(S)\alpha$. Since for every $e \in \alpha E(S)\alpha$ it is clear that $\vartheta_x(x^*ex) = e$, it follows that $\operatorname{Im} \vartheta_x = \alpha E(S)\alpha$ for every $x \in H_\alpha$, and therefore $\vartheta_x \in \operatorname{End}_\alpha E(S)$. The mapping $\vartheta: H_\alpha \to \operatorname{End}_\alpha E(S)$ given by $x \mapsto \vartheta_x$ is then a morphism; for

$$\vartheta_x[\vartheta_y(e)] = xyey^*x^* = xye(xy)^* = \vartheta_{xy}(e).$$

Furthermore, ϑ is 1-preserving since $\vartheta_\alpha(e) = \alpha e\alpha = \varphi(e)$ where φ is the identity of $\operatorname{End}_\alpha E(S)$. We can therefore construct the semigroup $[E(S); H_\alpha]_\vartheta$.

Since for every $x \in S$ we have $xx^* = xx^*\alpha \in E(S)\alpha$ and $x^*x = \alpha x^*x \in \alpha E(S)$ with

$$\vartheta_{x^{**}}(x^*x) = x^{**}x^*xx^* = \alpha xx^*\alpha = \vartheta_\alpha(xx^*),$$

we can define a mapping $\psi: S \to [E(S); H_\alpha]_\vartheta$ by

$$\psi(x) = (xx^*, x^{**}, x^*x).$$

We show as follows that ψ is an isomorphism.

That ψ is injective follows from the fact that if $\psi(x)=\psi(y)$ then $xx^*=yy^*$, $x^{**}=y^{**}$ and $x^*x=y^*y$ give

$$x=xx^*x^{**}x^*x=yy^*y^{**}y^*y=y.$$

To see that ψ is surjective, let $(e,x,f)\in[E(S);H_\alpha]_\vartheta$. Then $xfx^*=\vartheta_x(f)=\vartheta_\alpha(e)=\alpha e\alpha$. Consider the element $s=exf$. Using Theorem 4 and Corollary 1 of Theorem 5, we have

$$s^{**}=e^{**}x^{**}f^{**}=\alpha x\alpha=x.$$

It follows that $s^*=x^*$ and so

$$ss^*=exfx^*=e\alpha e\alpha=e\alpha=e.$$

Since α is a middle unit, we also have

$$s^*s=x^*exf=x^*\alpha e\alpha xf=x^*xfx^*xf=\alpha f\alpha f=\alpha f=f.$$

Consequently, $\psi(s)=(ss^*,s^{**},s^*s)=(e,x,f)$ and so ψ is surjective.

Finally, ψ is a morphism since

$$\begin{aligned}\psi(x)\psi(y)&=(xx^*,x^{**},x^*x)(yy^*,y^{**},y^*y)\\&=(xx^*\vartheta_{x^{**}}(yy^*),x^{**}y^{**},\vartheta_{y^*}(x^*x)y^*y)\\&=(xx^*x^{**}yy^*x^*,(xy)^{**},y^*x^*xy^{**}y^*y)\\&=(xyy^*x^*,(xy)^{**},y^*x^*xy)\\&=(xy(xy)^*,(xy)^{**},(xy)^*xy)\\&=\psi(xy).\end{aligned}$$

Hence we have that $S\simeq[E(S);H_\alpha]_\vartheta$. □

That the structure theorem in **[2]** for uniquely unit orthodox semigroups is a particular case of Theorem 8 can be seen as follows. Suppose that S is uniquely unit orthodox. Then, taking $\alpha=1$ in Theorem 8, the mappings ϑ_x become automorphisms on $E(S)$. For, $xex^*=xfx^*$ gives $e=1e1=x^*xex^*x=x^*xfx^*x=1f1=f$ so that ϑ_x is injective; and $\vartheta_x(x^*ex)=xx^*exx^*=1e1=e$ so that ϑ_x is surjective. Therefore, in the construction of the first part of Theorem 8 we can take ζ to be a group morphism from G to Aut B. In this case the elements of $[B;G]_\zeta$ are the triples (x,g,a) with $a=\zeta_{g^{-1}}(x)$. Since the third component of the triple is therefore completely determined by the first two components we can effectively ignore third components. Then it is clear that $[B;G]_\zeta$ reduces to the semi-direct product described in **[2]**.

Theorem 8 can of course be illustrated using the example that precedes it. Here we have $\alpha=(0,1,0)$ and the "building bricks" in the construction are the bands $E(S)\alpha$ consisting of the elements of the form $(0,x,p)$, $\alpha E(S)$ consisting of the elements of the form $(m,x,0)$, and the subgroup H_α consisting of the elements of the form $(m_k,1,p_k)$. Simple calculations give $(m,x,p)(m,x,p)^*=(0,x,p-p_k)$, $(m,x,p)^*(m,x,p)=(m-m_k,x,0)$, and $(m,x,p)^{**}=(m_k,1,p_k)$. The isomorphism $S\simeq[E(S);H_\alpha]_\vartheta$ is then given via the coordinatisation

$$(m,x,p)\sim((0,x,p-p_k),(m_k,1,p_k),(m-m_k,x,0)).$$

DEFINITION. If S is an orthodox semigroup with an associate subgroup of which the identity element is a middle unit then we shall say that S is *compact* if $x^\circ=x^*$ for every $x\in S$.

THEOREM 9. *Let S be an orthodox semigroup with an associate subgroup of which the identity element is a middle unit. Then the following statements are equivalent*:

(1) *S is compact*;

(2) *$E(S)$ is a rectangular band.*

Proof. (1)$\Rightarrow$(2): If (1) holds then $\alpha S\alpha = S^\circ = S^*$ and is a subgroup of S whence $\alpha E(S)\alpha = \{\alpha\}$. Thus $\alpha f\alpha = \alpha$ for every $f \in E(S)$. If now $e, f, g \in E(S)$ then, since α is a middle unit,

$$efg = e\alpha f\alpha g = e\alpha g = eg.$$

Thus every $f \in E(S)$ is a middle unit of $E(S)$, so $E(S)$ is a rectangular band.

(2)$\Rightarrow$(1): If $E(S)$ is a rectangular band then $\alpha E(S)\alpha = \{\alpha\}$. It follows that, for every $x \in S$,

$$x^{\circ\circ}x^\circ = \alpha x\alpha x^\circ = \alpha xx^\circ = \alpha xx^* = \alpha xx^*\alpha = \alpha.$$

Hence, by Theorem 1,

$$\begin{aligned} x^\circ = \alpha x^\circ &= x^*x^{**}x^\circ = x^*x^{\circ *}x^\circ \\ &= x^*x^{\circ\circ}x^\circ = x^*\alpha = x^*, \end{aligned}$$

whence S is compact. $\square$

In the compact situation, Theorem 8 simplifies considerably. To see this, observe that for every $x \in H_\alpha$ we have $\vartheta_x(e) = xex^* = x\alpha e\alpha x^* = x\alpha x^* = xx^* = \alpha$. The structure maps ϑ_x therefore "evaporate" and S is isomorphic to the cartesian product semigroup $E(S)\alpha \times H_\alpha \times \alpha E(S)$.

ACKNOWLEDGEMENT. We gratefully acknowledge the NATO Collaborative Reseach Grant 910765.

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