The Jordan Form Problem for $C = AB$: the Balanced, Diagonalizable Case

by

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Abstract We consider a key case in the fundamental and substantial problem of the possible Jordan canonical forms of $A, B, C \in M_n(F)$ when $C = AB$. If $A \in M_{2k}(F)$ (respectively $B, C \in M_{2k}(F)$) is diagonalizable with two distinct eigenvalues a_1, a_2 (respectively b_1, b_2, and c_1, c_2), each with multiplicity k, and when $C = AB$, all possibilities for $a_1, a_2, b_1, b_2, c_1, c_2$ are characterized. The possibilities are much more restrictive than the obvious determinant condition: $(a_1a_2b_1b_2)^k = (c_1c_2)^k$ allows. This is then used to settle the general, two eigenvalue per matrix, diagonalizable case of the Jordan form problem for $C = AB$.

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1 Introduction

We are interested in the fundamental problem of determining for each nonsingular n−by−n complex matrix C, what Jordan forms may occur for the n−by−n matrices A and B such that $AB = C$. This depends only upon the similarity class of C, and, as the problem may be posed in a variety of ways about the triple A, B, C, we call this the *three-matrix, product, Jordan form problem.*

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An important partial result was observed by Sourour [6] when C is not scalar: the spectra $\alpha_1, \alpha_2, ..., \alpha_n$ of A and $\beta_1, \beta_2, ..., \beta_n$ of B are arbitrary, subject only to the determinantal condition

$$\alpha_1\alpha_2\cdots\alpha_n\beta_1\beta_2\cdots\beta_n = \gamma_1\gamma_2\cdots\gamma_n$$

in which $\gamma_1, \gamma_2, ..., \gamma_n$ are the eigenvalues of C. No information was given about the Jordan structure of A and B (when their spectra include multiple eigenvalues), relative to that of C. In earlier work [4], see also [5], we showed that A and B could always be taken to be nonderogatory. In addition to the determinant condition, we have also shown that there is a *geometric multiplicity constraint* on the spectra of A, B and C. Let $g_X(\lambda)$ be the geometric multiplicity of the eigenvalue λ in the $n-$by$-n$ matrix X. We then have [3]:

$$g_A(\alpha) + g_B(\beta) - n \leq g_{AB}(\alpha\beta).$$

for $n-$by$-n$ matrices A and B over a field. Thus, if A and B have eigenvalues with high geometric multiplicity, the product of the two eigenvalues must appear in AB. This explains the nonscalar requirement in Sourour's theorem. It has been shown [3] that the geometric multiplicity constraint, together with the determinant condition, is necessary and sufficient for our problem when $n < 4$. Thus, $n = 4$ is the starting point for the present work, in which we find that there are additional constraints on the three matrix, Jordan form problem.

One major barrier to resolution of the $4-$by-4 case has been a key special case of the general diagonalizable case: if A, B and C each have two distinct eigenvalues of multiplicity two each ($A : a_1, a_2$; $B : b_1, b_2$; and $C : c_1, c_2$) what, if any, restrictions are there in addition to the determinant restriction:

$$(a_1a_2b_1b_2)^2 = (c_1c_2)^2$$

This question has proven surprisingly subtle.

Here, we generalize, and completely settle the above *balanced, diagonalizable* case. The general problem that we settle here may be described as follows. For which $a_1, a_2, a_1 \neq a_2, b_1, b_2, b_1 \neq b_2$ and $c_1, c_2, c_1 \neq c_2$ do their exist diagonalizable $n-$by$-n$, $n = 2k$, matrices A with eigenvalues a_1 and a_2, each with multiplicity k, B with eigenvalues b_1 and b_2, each with multiplicity k, and C with eigenvalues c_1 and c_2, each with multiplicity k, such that $AB = C$? We call this problem P_k. Of course, the determinant condition

$$(a_1a_2b_1b_2)^k = (c_1c_2)^k,$$

which is sufficient for $k = 1$, is present, but for larger k we find that this condition is far too weak. We note that when two of the three matrices have just two eigenvalues and the multiplicities are not equal (which includes the case in which n is odd), the general problem reduces to a smaller one, because of the geometric multiplicity constraint. Thus, this *balanced, even* case is central; the geometric multiplicity constraint is vacuously satisfied, leaving only the determinant constraint. In fact, the diagonalizable case of this problem tends to present the greatest difficulty. We also note that, because of diagonalizability, the question makes sense over a general field, and indeed, the present work is independent of the field in which the eigenvalues lay. Once the balanced diagonalizable case is settled (see section 5), we are able to completely understand the general, two-eigenvalue-per-matrix diagonalizable case (see section 6). The understanding of these cases with few eigenvalues of high geometric multiplicity is our primary contribution to the problem.

2 Preliminary Calculations

Let $n = 2k$ and let

$$A = \begin{bmatrix} a_1 I & 0 \\ 0 & a_2 I \end{bmatrix}, \; B = \begin{bmatrix} b_1 I & 0 \\ 0 & b_2 I \end{bmatrix}, \text{ and } C = \begin{bmatrix} c_1 I & 0 \\ 0 & c_2 I \end{bmatrix}$$

in which each I is $k-$by$-k$, $a_i, b_i, c_i \neq 0, i = 1, 2$; $a_1 \neq a_2$, $b_1 \neq b_2$ and $c_1 \neq c_2$. We wish to understand for which a_1, a_2, b_1, b_2, c_1, c_2 there exist invertible

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \quad N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}$$

such that M_{ij}, N_{ij} are $k-$by$-k$, $i, j = 1, 2$ and

$$(M^{-1}AM)(NBN^{-1}) = C \tag{0}$$

Equivalently,

$$AMNB = MCN \tag{0'}$$

or

$$\begin{cases} a_1 b_1 (M_{11}N_{11} + M_{12}N_{21}) &= c_1 M_{11}N_{11} + c_2 M_{12}N_{21} \\ a_1 b_2 (M_{11}N_{12} + M_{12}N_{22}) &= c_1 M_{11}N_{12} + c_2 M_{12}N_{22} \\ a_2 b_1 (M_{21}N_{11} + M_{22}N_{21}) &= c_1 M_{21}N_{11} + c_2 M_{22}N_{21} \\ a_2 b_2 (M_{21}N_{12} + M_{22}N_{22}) &= c_1 M_{21}N_{12} + c_2 M_{22}N_{22} \end{cases}$$

or

$$\begin{cases} (a_1b_1 - c_1)M_{11}N_{11} &= (c_2 - a_1b_1)M_{12}N_{21} & (1) \\ (a_1b_2 - c_1)M_{11}N_{12} &= (c_2 - a_1b_2)M_{12}N_{22} & (2) \\ (a_2b_1 - c_1)M_{21}N_{11} &= (c_2 - a_2b_1)M_{22}N_{21} & (3) \\ (a_2b_2 - c_1)M_{21}N_{12} &= (c_2 - a_2b_2)M_{22}N_{22} & (4) \end{cases}$$

or

$$\begin{cases} x_1M_{11}N_{11} &= y_3M_{12}N_{21} & (1') \\ y_2M_{11}N_{12} &= x_4M_{12}N_{22} & (2') \\ y_4M_{21}N_{11} &= x_2M_{22}N_{21} & (3') \\ x_3M_{21}N_{12} &= y_1M_{22}N_{22} & (4') \end{cases}$$

with $x_1 = (a_1b_1 - c_1)$, $x_2 = (c_2 - a_2b_1)$, $x_3 = (a_2b_2 - c_1)$, $x_4 = (c_2 - a_1b_2)$, $y_1 = (c_2 - a_2b_2)$, $y_2 = (a_1b_2 - c_1)$, $y_3 = (c_2 - a_1b_1)$, $y_4 = (a_2b_1 - c_1)$.

Proposition 1. *If $x_i = 0$, then $y_j \neq 0$, for $j \neq i$ and if $y_i = 0$, then $x_j \neq 0$, for $j \neq i$.*

Proof. Without loss of generality, suppose that $x_1 = y_2 = 0$. Then it would follow that $b_1 = b_2$. This is contrary to the hypothesis that B is nonscalar. All other cases are similar. □

Proposition 2. *If $AMNB = MCN$, M, N nonsingular, then $(a_1a_2b_1b_2)^k = (c_1c_2)^k$.*

Proof. This follows from equating determinants. □

3 The Case in Which x_i, y_i are Nonzero, $i = 1, 2, 3, 4$

We first consider the case in which all 8 of x_i, y_j are nonzero and give a complete solution to our problem (P_k) in this event.

Lemma 3. *If $x_1, y_2, y_3, x_4, \neq 0$ (resp., $y_1, x_2, x_3, y_4 \neq 0$, $x_1, x_2, y_3, y_4 \neq 0$, $y_1, y_2, x_3, x_4 \neq 0$), then M_{11} and M_{12} (resp., M_{21} and M_{22}, N_{11} and N_{21}, N_{12} and N_{22}) are nonsingular.*

Proof. Consider, for example, the unparenthetical claim. The other three are similar. Suppose that M_{11} is singular and that $u \neq 0$ is a left null vector for M_{11}. Then, from $(1')$ and $(2')$ we have

$$u^T M_{12}[N_{21}\ N_{22}] = 0.$$

But, since $[N_{21}\ N_{22}]$ has full row rank because N is nonsingular, it follows that that $u^T M_{12} = 0$. However, this would mean that,

$$u^T[M_{11}\ M_{12}] = 0,$$

contradicting the assumption that $[M_{11}\ M_{12}]$ must have full row rank because M is nonsingular. We conclude that M_{11} is nonsingular. The proof for M_{12} is similar. □

Corollary 4. *If all* 8 *of* $y_i, x_i, i = 1, 2, ..., 4$, *are nonzero, then all* 8 *blocks* M_{ij}, N_{ij} *are nonsingular,* $i, j = 1, ..., 4$.

Proposition 5. *Equations* $(1') - (4')$ *may be rewritten as*

$$M_{11}[x_1 N_{11}\ y_2 N_{12}] = M_{12}[y_3 N_{21}\ x_4 N_{22}]$$

and

$$M_{21}[y_4 N_{11}\ x_3 N_{12}] = M_{22}[x_2 N_{21}\ y_1 N_{22}].$$

From the above proposition, we can see that if $M_{12}N_{21} \neq 0$, then $x_1 = 0$ implies $y_3 = 0$; if $M_{12}N_{22} \neq 0$, then $y_2 = 0$ implies $x_4 = 0$; if $M_{22}N_{21} \neq 0$, then $y_4 = 0$ implies $x_2 = 0$ and if $M_{22}N_{22} \neq 0$, then $x_3 = 0$ implies $y_1 = 0$. Since $x_1 = 0$ and $y_3 = 0$ (resp., $y_2 = 0$ and $x_4 = 0$; $y_4 = 0$ and $x_2 = 0$; $x_3 = 0$ and $y_1 = 0$) cannot occur according to proposition 1, we have

Lemma 6. *If all* 8 *blocks* M_{ij}, $N_{ij}, i, j = 1, ..., 4$, *are nonsingular, then all* 8 *parameters* x_i, y_i, $i = 1, ..., 4$, *are nonzero.*

Theorem 7. *All* 8 x_i, y_i, $i = 1, ..., 4$, *are nonzero if and only if all* 8 *blocks* M_{ij}, N_{ij} *are nonsingular, and if all* x_i, y_i *are nonzero, then*

$$x_1 x_2 x_3 x_4 = y_1 y_2 y_3 y_4$$

which is equivalent to $a_1 b_1 a_2 b_2 = c_1 c_2$.

Proof. The first part of this theorem is a consequence of lemma 3 and lemma 6. In the second part, we first show that

$$x_i \neq 0, y_i \neq 0 \Rightarrow x_1x_2x_3x_4 = y_1y_2y_3y_4.$$

In fact, eliminating M_{11} and M_{12} from $(1')$ and $(2')$ yields

$$\frac{x_1}{y_3} N_{11}N_{21}^{-1} = \frac{y_2}{x_4} N_{12}N_{22}^{-1},$$

while eliminating M_{21} and M_{22} from $(3')$ and $(4')$ yields

$$\frac{y_4}{x_2} N_{11}N_{21}^{-1} = \frac{x_3}{y_1} N_{12}N_{22}^{-1},$$

The above two equations give the desired conclusion. The rest is just a direct calculation, i.e.,

$$x_1x_2x_3x_4 = y_1y_2y_3y_4$$

$$\Leftrightarrow (a_1b_1 - c_1)(c_2 - a_2b_1)(a_2b_2 - c_1)(c_2 - a_1b_2) - (c_2 - a_2b_2)(a_1b_2 - c_1)(c_2 - a_1b_1)(a_2b_1 - c_1) = 0$$

$$\Leftrightarrow (c_2 - c_1)(b_2 - b_1)(a_2 - a_1)(a_1a_2b_1b_2 - c_1c_2) = 0.$$

According to our hypothesis $c_2 - c_1 = 0$ (resp., $b_2 - b_1 = 0$, $a_2 - a_1 = 0$) can not happen. So $x_1x_2x_3x_4 = y_1y_2y_3y_4$ is equivalent to $a_1b_1a_2b_2 = c_1c_2$. This completes the proof of the theorem. □

Note that $a_1b_1a_2b_2 = c_1c_2$ is stronger than $(a_1b_1a_2b_2)^k = (c_1c_2)^k$; $[(a_1b_1a_2b_2)^{k-1} + (a_1b_1a_2b_2)^{k-2}(c_1c_2) + \cdots + (c_1c_2)^{k-1}] = 0$ is not possible. Note also that if x_i and $y_i = 0$, then $a_1b_1a_2b_2 = c_1c_2$ also holds.

Theorem 8. *If $a_1b_1a_2b_2 = c_1c_2$, then there do exist invertible M and N satisfying (0).*

Proof. Our purpose is to find the nonsingular M and N, such that the (0) is verified. To do this we just let $M_{11} = M_{12} = M_{22} = N_{21} = N_{22} = I$. Then, by solving matrix equations $(1') - (4')$, we get that $M_{21} = \frac{x_1x_2}{y_3y_4} I$, $N_{12} = \frac{x_4}{y_2} I$ and $N_{11} = \frac{y_3}{x_1} I$. So $M = \begin{bmatrix} I & I \\ \frac{x_1x_2}{y_3y_4} I & I \end{bmatrix}$, and $N = \begin{bmatrix} \frac{y_3}{x_1} I & \frac{x_4}{y_2} I \\ I & I \end{bmatrix}$. We need prove that

$\dfrac{x_1x_2}{y_3y_4} \neq 1$, and $\dfrac{y_3}{x_1} \neq \dfrac{x_4}{y_2}$ to make sure that M and N are nonsingular. In fact, $\dfrac{x_1x_2}{y_3y_4} = 1$ means $b_1(c_2-c_1)(a_2-a_1) = 0$, and $\dfrac{y_3}{x_1} = \dfrac{x_4}{y_2}$ gives $a_1(c_2-c_1)(b_1-b_2) = 0$, which are contradictions to our hypothesis. So M and N are nonsingular and verify (0), the proof is complete. □

Now, we may state the characterization for the case of this section.

Corollary 9. *If $a_ib_j \neq c_k$ (all i, j, k), i.e, no eigenvalue of C is a product of an eigenvalue from A and from B, then there is an invertible solution M and N to problem P_k if and only if $a_1b_1a_2b_2 = c_1c_2$.*

4 The Case in Which at Least One of x_i, y_i is Zero

Suppose now that not all x_i and y_j are nonzero. Only a few combinations of $0'$s are possible. If an x_i and a y_j are 0, it can be only one of each and they must have the same index, according to proposition 1. In this event, $a_1b_1a_2b_2 = c_1c_2$, and there is a solution to P_k, as in the prior section. If just one of x_i, y_j is 0, by symmetry we may suppose it is x_1; the other seven yield the same result with a similar argument.

Lemma 10. *It is not possible that exactly one of $x_1, x_2, x_3, x_4, y_1, y_2, y_3$ and y_4 is 0.*

Proof. If $x_1 = 0, y_3 \neq 0$, then $M_{12}N_{21} = 0$. If $M_{12}N_{21} = 0$, at least one of them is rank deficient. But, by lemma 3, it follows from $(3')$ and $(4')$ that M_{21} and M_{22} are nonsingular, and then from a right null space argument that N_{11}, N_{21}, N_{12}, and N_{22} are nonsingular (If, say, N_{21} were singular, N_{11} and N_{21} would have the same right null space from $(3')$, because M_{21} and M_{22} are nonsingular, contradicting the nonsingularity of N). Thus $M_{12} = 0$. But, from $(2')$, M_{11} would have to be 0, as N_{12} and N_{22} are nonsingular. This would contradict the nonsingularity of M, completing the proof of the lemma. □

Lemma 11. *It is not possible that two x's or two y's are zero and all others nonzero.*

Proof. When $x_1 = 0, x_2 = 0$ or $x_1 = 0, x_4 = 0$, the argument is similar to lemma 10. When $x_1 = 0, x_3 = 0$, the equations $(1') - (4')$ become

$$\begin{cases} M_{12}N_{21} &= 0 & (5) \\ M_{22}N_{22} &= 0 & (6) \\ M_{12}N_{22} &= (\frac{y_2}{x_4})M_{11}N_{12} \\ \\ M_{22}N_{21} &= (\frac{y_4}{x_2})M_{21}N_{11} \end{cases}$$

and

$$MN = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} = \begin{bmatrix} M_{11}N_{11} & \alpha M_{11}N_{12} \\ \beta M_{21}N_{11} & M_{21}N_{12} \end{bmatrix},$$

where $\alpha = 1 + \frac{y_2}{x_4}, \beta = 1 + \frac{y_4}{x_2}$. Then $M_{11}, M_{21}, N_{11}, N_{12}$ must be nonsingular or MN would be singular. Since $M_{12}N_{22} = (\frac{y_2}{x_4})M_{11}N_{12}$ and $M_{22}N_{21} = (\frac{y_4}{x_2})M_{21}N_{11}$, then $M_{12}, M_{22}, N_{21}, N_{22}$ are also nonsingular, it would contradict equations (5) and (6). So, when two of $x's$ or $y's$ are zero, there is no solution to P_k. □

Lemma 12. *If three $x's$ or $y's$ are zero, then the fourth one must be zero.*

Proof. Without loss of generality, suppose that $x_1, x_2, x_3 = 0$. Then

$$\begin{cases} M_{12}N_{21} &= 0 & (i) \\ M_{21}N_{11} &= 0 & (ii) \\ M_{22}N_{22} &= 0 & (iii) \end{cases}$$

and

$$MN = \begin{bmatrix} M_{11}N_{11} & \alpha M_{11}N_{12} \\ M_{22}N_{21} & M_{21}N_{12} \end{bmatrix}.$$

From the first block row and last block column of MN, we see that M_{11} and N_{12} are nonsingular. Since $M_{12}N_{22} = (\frac{y_2}{x_4})M_{11}N_{12}$, then M_{12} and N_{22} are nonsingular. From (iii), N_{22} nonsingular means $M_{22} = 0$, and by (i), M_{12} nonsingular means $N_{21} = 0$. Then, the $(1, 2)$ position of MN is 0. In this event, N_{11}, M_{21} cannot be singular or MN would be singular. Since this would contradict (ii), the proof is complete. □

Lemma 13. *If the four $x's$ or the four $y's$ are zero, then each of the 8 blocks M_{ij}, N_{ij} has rank $k/2$.*

Proof. Assume that the four $x's$ are zero. Then we have that

$$\begin{cases} M_{12}N_{21} & = & 0 \\ M_{11}N_{12} & = & 0 \\ M_{21}N_{11} & = & 0 \\ M_{22}N_{22} & = & 0 \end{cases} \quad \text{that implies} \quad \begin{cases} r(M_{12})+r(N_{21}) & \leq & k & (1'') \\ r(M_{11})+r(N_{12}) & \leq & k & (2'') \\ r(M_{21})+r(N_{11}) & \leq & k & (3'') \\ r(M_{22})+r(N_{22}) & \leq & k & (4'') \end{cases}$$

From the nonsingularity of M and N, we have that

$$\begin{cases} r(M_{11})+r(M_{12}) & \geq & k & (5'') \\ r(M_{11})+r(M_{21}) & \geq & k & (6'') \\ r(M_{21})+r(M_{22}) & \geq & k & (7'') \\ r(M_{12})+r(M_{22}) & \geq & k & (8'') \end{cases} \quad \text{and} \quad \begin{cases} r(N_{11})+r(N_{12}) & \geq & k & (9'') \\ r(N_{11})+r(N_{21}) & \geq & k & (10'') \\ r(N_{21})+r(N_{22}) & \geq & k & (11'') \\ r(N_{12})+r(N_{22}) & \geq & k & (12'') \end{cases}$$

Without loss of generality, from $(5'')$, we suppose that $r(M_{11}) > k/2$, then from $(2'')$, $r(N_{12}) < k/2$. But from $(9'')$ and $(12'')$, we have $r(N_{11}), r(N_{22}) > k/2$. Then $(3'')$ and $(4'')$ oblige $r(M_{21}), r(M_{22}) < k/2$, it is contradicting to $(7'')$. So the equality must be hold. By solving linear equations $(1'')$ to $(12'')$, we have that all 8 blocks has the same rank, which is $k/2$. When the four $y's$ are zero, the arguments are similar, completing the proof. □

Note that when the four $x's$ are zero, then $c_1 = a_1b_1 = a_2b_2$, and $c_2 = a_2b_1 = a_1b_2$. So $c_1c_2 = a_1a_2b_1^2 = a_1a_2b_2^2$, which means $b_1 = -b_2$, also $c_1c_2 = a_1^2b_1b_2 = a_2^2b_1b_2$, which means $a_1 = -a_2$ and $c_1^2 = a_1a_2b_1b_2 = c_2^2$, which means $c_1 = -c_2$. Then, we have the follow theorem

Theorem 14. *When the four $x's$ (or the four $y's$) are zero, the problem P_k has a solution if and only if k is even and $a_1a_2b_1b_2 = -c_1c_2$, $a_2 = -a_1$, $b_2 = -b_1$ and $c_2 = -c_1$.*

Proof. When k is odd, lemma 13 shows that no solution is possible. If k is even, the above calculation shows that the conditions on the $a's$, $b's$ and $c's$ are necessary, while lemma 13 again shows the necessity of k even. In this event, the choice of

$$M = \begin{pmatrix} 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \qquad N = \begin{pmatrix} 0 & 0 & I & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & I & 0 & 0 \end{pmatrix}$$

verifies that there is a solution and complete the proof. ☐

Thus, all cases in which at least one of x_i, y_i is zero have been covered, and we state the general result in the next section.

5 General Balanced, Even Case

The general solution to problem P_k may now be stated as

Theorem 15. *There is a solution to problem P_k if and only if either $a_1a_2b_1b_2 = c_1c_2$, or, when k is even, $a_1a_2b_1b_2 = -c_1c_2$ and $a_2 = -a_1$, $b_2 = -b_1$ and $c_2 = -c_1$.*

Proof. A consequence of theorem 8 and theorem 14. ☐

Corollary 16. *If k is odd, then, there is a solution to problem P_k if and only if $a_1a_2b_1b_2 = c_1c_2$.*

6 The General Two-Eigenvalue, Diagonalizable Case

We now turn our attention to the general "two-eigenvalue, diagonalizable case": each of our three matrices has precisely two distinct eigenvalues and is diagonalizable. (If one had only one eigenvalue, then a certain matrix would have to be a multiple of the inverse of another, and the analysis would be straightforward.) The balanced case, analyzed above, will be crucial, as we will see that every situation may be reduced to it. Since the balanced case has been analyzed, we consider here only non-balanced situations (and apply the balanced result when appropriate). In such situations, the geometric multiplicity constraint will always apply, and, when it does, an eigenvector argument will imply the reduction.

We suppose, now, that

$$\tilde{A} = \begin{bmatrix} a_1 I_p & 0 \\ 0 & a_2 I_{n-p} \end{bmatrix}, \ \tilde{B} = \begin{bmatrix} b_1 I_q & 0 \\ 0 & b_2 I_{n-q} \end{bmatrix}, \text{ and } \tilde{C} = \begin{bmatrix} c_1 I_r & 0 \\ 0 & c_2 I_{n-r} \end{bmatrix},$$

and again ask, problem $P = P(a_1, p, a_2, b_1, q, b_2, c_1, r, c_2, n)$, when there are matrices A similar to $\tilde{A}$, B similar to $\tilde{B}$ and C similar to $\tilde{C}$ such that $AB = C$? In this event, we say that problem P is feasible. Note that, because of similarity, we may assume the numbers a_1, a_2 (resp., b_1, b_2 or c_1, c_2) are in an order of our choice. By convention, we take them so that $p \geq n-p$, $q \geq n-q$ and $r \geq n-r$. Because the problem $AB = C$ is equivalent to $C^{-1}A = B^{-1}$ or $B^T A^T = C^T$, etc, we may also suppose that p, q, and r are in any relative order we like. Often, we take them so that $p \geq q \geq r$. It is convenient to array the data of our problem as a diagram:

$$\begin{array}{ccc} A & B & C \\ a_1(p) & b_1(q) & c_1(r) \\ a_2(n-p) & b_2(n-q) & c_2(n-r) \end{array}$$

If we assume that $p \geq q \geq r \geq n-r$, with not all equalities, we must then have $p+q > n$ and also (unless $p = q$) $p + (n-q) = n + (p-q) > n$. In these events, the geometric multiplicity constraint applies, and we must have

$$a_1 b_1 = c_1 \text{ or } c_2; \quad a_1 b_2 = c_1 \text{ or } c_2.$$

If $a_1 b_1 = c_i$ and $p > q$, then $a_1 b_2 = c_{3-i}$, else $b_1 = b_2$, which is not allowed by the distinctness assumption.

Using the following lemma, when the geometric multiplicity constraint applies, the problem P may be reduced to a smaller one and the above diagram manipulated accordingly, perhaps with constraints on the data accumulated along the way.

Lemma 17. *If the problem P is feasible for $p+q > n$ (respectively $p > q$), then one of the problems $P(a_1, p-1, a_2, b_1, q-1, b_2, c_1, r-1, c_2, n-1)$ (resp., $P(a_1, p-1, a_2, b_1, q, b_2, c_1, r-1, c_2, n-1)$) or $P(a_1, p-1, a_2, b_1, q-1, b_2, c_1, r, c_2, n-1)$ (resp., $P(a_1, p-1, a_2, b_1, q, b_2, c_1, r, c_2, n-1)$) is feasible.*

Proof. It is suffices to prove the un-parenthetical claim; the parenthetical claim is similar. The two parts of the claim only differ with regard to whether the geometric match is $a_1 b_1 = c_1$ or $a_1 b_1 = c_2$.

Since $p + q > n$ the eigenspaces for a_1 in A and b_1 in B intersect; let u be a normalized vector lying in each. As in the proof of Schur's theorem [1], let U be a unitary matrix whose first column is u. Now, let $\tilde{A} = U^* A U$, $\tilde{B} = U^* B U$, and $\tilde{C} = U^* C U$, so that from the assumption $AB = C$ in the feasibility of P, we have $\tilde{A}\tilde{B} = \tilde{C}$. But

$$\tilde{A} = \left[\begin{array}{c|c} a_1 & * \\ \hline 0 & A' \end{array}\right] \quad \text{and} \quad \tilde{B} = \left[\begin{array}{c|c} b_1 & * \\ \hline 0 & B' \end{array}\right],$$

so that

$$\tilde{C} = \left[\begin{array}{c|c} c_i & * \\ \hline 0 & C' \end{array}\right], \quad i = 1 \text{ or } 2.$$

It follows that $A'B' = C'$, that A' has eigenvalues a_1 (multiplicity $p-1$) and a_2 (multiplicity $n-p$), that B' has eigenvalues b_1 (multiplicity $q-1$) and b_2 (multiplicity $n-q$), and that C' has eigenvalues c_1 and c_2 (with one of the multiplicities decreased by one), with, via a simply Jordan form argument, each of A', B' and C' diagonalizable. Of course, by the geometric multiplicity constraint, a_1b_1 must be one of the eigenvalues of C, but, a priori, we do not know which one. The existence of A' B' and C' shows that one of the indicated smaller problems is feasible, as a consequence of the feasibility of the larger one. □

We note that the smaller problem in lemma 17 may, in general, be feasible in more ways than the larger one.

Using lemma 17, every two eigenvalue situation may be reduced to a (unique) balanced one via, perhaps several, applications of the lemma. The accumulation of restrictions en route, together with the restrictions of the resulting balanced problem, will give the totality of conditions for feasibility of the original problem. It will be helpful to determine the outcome in a few key situations before describing the general case. This also is useful for describing the method. We also note, in each situation to follow, once necessary conditions are accumulated, via reduction based upon lemma 17, their sufficiency follows from the simple fact that the conditions may be satisfied by diagonal matrices A, B and C (unlike the balanced cases). We omit the details; see examples in the next section.

It is straightforward to reduce the general case to a "semi-balanced" one, i.e., one in which $n = 2p$ is even and two of the three matrices have equal multiplicities for the two eigenvalues. In this event, we may take the data to be

A	B	C
$a_1(p)$	$b_1(p)$	$c_1(r)$
$a_2(p)$	$b_2(p)$	$c_2(n-r)$

in which $n = 2p$ and $r > p$. Since $r + p > n$, the geometric multiplicity constraint gives that c_1 must be the product of one of the $a_i's$ and one of the $b_i's$. By symmetry, we may (by renumbering, if necessary) assume $c_1 = a_1b_1$. Then, applying the reduction $r - p$ times we obtain the diagram:

$$\begin{array}{ccc} A & B & C \\ a_1(2p-r) & b_1(2p-r) & c_1(p) \\ a_2(p) & b_2(p) & c_2(2p-r) \end{array}.$$

Now, the geometric multiplicity constraint implies that $a_2b_2 = c_1$, as well and application of the reduction another $r-p$ times yields the balanced, diagram:

$$\begin{array}{ccc} A & B & C \\ a_1(2p-r) & b_1(2p-r) & c_1(2p-r) \\ a_2(2p-r) & b_2(2p-r) & c_2(2p-r) \end{array}.$$

Now, we may apply theorem 15 with $k = 2p-r$, and, since the parity of $2p-r$ is that of r, we use r as the parameter. According to the theorem, if r is odd, we must have

$$a_1a_2b_1b_2 = c_1c_2.$$

As we also have $a_1b_1 = a_2b_2 = c_1$, we conclude that $c_1 = c_2$, which is not allowed. Thus, for r odd no semi-balanced problem is feasible. However, if r is even, there is the additional possibility that

$$a_1a_2b_1b_2 = -c_1c_2, \text{and } a_1 = -a_2, b_1 = -b_2, c_1 = -c_2.$$

This is consistent with the accumulated conditions $a_1b_1 = c_1 = a_2b_2$ and gives the following lemma in the semi-balanced case.

Lemma 18. *If $r > p$ and*

$$\tilde{A} = \begin{bmatrix} a_1I_p & 0 \\ 0 & a_2I_p \end{bmatrix}, \ \tilde{B} = \begin{bmatrix} b_1I_p & 0 \\ 0 & b_2I_p \end{bmatrix}, \text{ and } \tilde{C} = \begin{bmatrix} c_1I_r & 0 \\ 0 & c_2I_{2p-r} \end{bmatrix},$$

then there are matrices A similar to $\tilde{A}$, B similar to $\tilde{B}$ and C similar to $\tilde{C}$ such that $AB = C$ if and only if r is even and $a_1 = -a_2, b_1 = -b_2, c_1 = -c_2$ and $c_1 = a_ib_j$ for some $i \in \{1,2\}$ and some $j \in \{1,2\}$.

Using lemma 18 and additional manipulation of appropriate diagrams with lemma 17 and the geometric multiplicity constraint, we may now give three theorems that along with theorem 15, cover all possible situations in the two-eigenvalue, diagonalizable case (because of the fact that p, q, r and $n-r$ may be arranged as needed).

If, for example, all three are equal ($p = q = r$) and we are unbalanced ($r > n-r$), the diagram is:

$$\begin{array}{ccc} A & B & C \\ a_1(p) & b_1(p) & c_1(p) \\ a_2(n-p) & b_2(n-p) & c_2(n-p) \end{array}$$

and we get $a_1b_1 = c_1$ from the geometric multiplicity constraint and $2p - n$ applications of the reduction gives the (balanced) diagram:

$$\begin{array}{ccc} A & B & C \\ a_1(n-p) & b_1(n-p) & c_1(n-p) \\ a_2(n-p) & b_2(n-p) & c_2(n-p) \end{array}$$

Now, application of theorem 15 yields the conditions in this case.

Theorem 19. *If $p > n - p$ and*

$$\tilde{A} = \begin{bmatrix} a_1 I_p & 0 \\ 0 & a_2 I_{n-p} \end{bmatrix}, \ \tilde{B} = \begin{bmatrix} b_1 I_p & 0 \\ 0 & b_2 I_{n-p} \end{bmatrix}, \text{ and } \tilde{C} = \begin{bmatrix} c_1 I_p & 0 \\ 0 & c_2 I_{n-p} \end{bmatrix},$$

then there are matrices A similar to $\tilde{A}$, B similar to $\tilde{B}$ and C similar to $\tilde{C}$ such that $AB = C$ if and only if either (a) $a_1b_1 = c_1$ and $a_2b_2 = c_2$ or (b) $n-p$ is even and $a_1b_1 = c_1, a_2 = -a_1, b_2 = -b_1$ and $c_2 = -c_1$.

Now, if $p = q > r \geq n - r$, the diagram

$$\begin{array}{ccc} A & B & C \\ a_1(p) & b_1(p) & c_1(r) \\ a_2(n-p) & b_2(n-p) & c_2(n-r) \end{array}$$

may, upon $2p - n$ applications of reduction be manipulated to

$$\begin{array}{ccc} A & B & C \\ a_1(n-p) & b_1(n-p) & c_1(n+r-2p) \\ a_2(n-p) & b_2(n-p) & c_2(n-r) \end{array}$$

if $a_1b_1 = c_1$ and $n + r \geq 2p$, or to

$$\begin{array}{ccc} A & B & C \\ a_1(n-p) & b_1(n-p) & c_1(r) \\ a_2(n-p) & b_2(n-p) & c_2(2(n-p)-r) \end{array}$$

if $a_1b_1 = c_2$ and $2(n-p) \geq r$. Upon application of lemma 18 this gives

Theorem 20. *If $p > r \geq n - r$ and*

$$\tilde{A} = \begin{bmatrix} a_1 I_p & 0 \\ 0 & a_2 I_{n-p} \end{bmatrix}, \ \tilde{B} = \begin{bmatrix} b_1 I_p & 0 \\ 0 & b_2 I_{n-p} \end{bmatrix}, \text{ and } \tilde{C} = \begin{bmatrix} c_1 I_r & 0 \\ 0 & c_2 I_{n-r} \end{bmatrix},$$

then there are matrices A similar to $\tilde{A}$, B similar to $\tilde{B}$ and C similar to $\tilde{C}$ such that $AB = C$ if and only if $a_2 = -a_1, b_2 = -b_1, c_2 = -c_1$ and either i) $c_1 = a_1 b_1$, $n + r \geq 2p$ and $n - r$ is even or ii) $c_1 = a_2 b_1$, $2(n-p) \geq r$, and r is even.

We finally turn our attention to the generic two eigenvalue case of P in which no equalities occur: with $p > q > r \geq n - r$ and the diagram is

$$\begin{array}{ccc} A & B & C \\ a_1(p) & b_1(q) & c_1(r) \\ a_2(n-p) & b_2(n-q) & c_2(n-r) \end{array}$$

For the first step of reduction there are now two possibilities (for satisfaction of the geometric multiplicity constraint), depending upon whether $a_1 b_1 = c_1$ or c_2. In the former case, the diagram reduces to

$$\begin{array}{ccc} A & B & C \\ a_1(n-q) & b_1(n-p) & c_1(n+r-(p+q) \\ a_2(n-p) & b_2(n-q) & c_2(n-r) \end{array}$$

This means that we must have $n + r \geq p + q$, and, if this inequality is satisfied with equality, we would revert to a case of one eigenvalue in C (so that the new B would be a multiple of A^{-1}). Since $2(n-q) > n - q + n - p$, the geometric multiplicity constraint now applies to a_1 and b_2, whose product must be c_2 (if it were c_1, we would conclude that $b_2 = b_1$, contradicting distinctness). We may then apply equivalence $p - q$ times to arrive at the semi-balanced case:

$$\begin{array}{ccc} A & B & C \\ a_1(n-p) & b_1(n-p) & c_1(n+r-(p+q)) \\ a_2(n-p) & b_2(n-p) & c_2(n+q-(p+r)). \end{array}$$

Application of lemma 18 in this case now requires that $n + q - (p + r)(> n + r - (p + q))$ is even, that $a_2 = -a_1, b_2 = -b_1$, and $c_2 = -c_1$. We already have $a_1 b_1 = c_1$ (and $a_1 b_2 = c_2$, which is implied) and $n + r \geq p + q$.

One the other hand, if $a_1 b_1 = c_2$, the original diagram similarly reduces to the semi-balanced one:

$$\begin{array}{ccc} A & B & C \\ a_1(n-p) & b_1(n-p) & c_1(r-p+q) \\ a_2(n-p) & b_2(n-p) & c_2(2n-r-p-q). \end{array}$$

The accumulated conditions are: $c_2 = a_1b_1, c_1 = a_1b_2$ and $2n \geq p+q+r$. From lemma 18, we obtain that $r-p+q(> 2n-r-p-q)$ must be even and that $a_2 = -a_1, b_2 = -b_1$, and $c_2 = -c_1$, and we already have $c_2 = a_1b_1$ (and $c_1 = a_1b_2$, which is implied) and $2n \geq p+q+r$. Note that the parity requirements in the two cases are the same if and only if n is even, and note that the second inequality requirement is more stringent than the first.

The two cases may be combined to give the general result in this situation.

Theorem 21. *If $p > q > r \geq n-r$ and*

$$\tilde{A} = \begin{bmatrix} a_1 I_p & 0 \\ 0 & a_2 I_{n-p} \end{bmatrix}, \ \tilde{B} = \begin{bmatrix} b_1 I_q & 0 \\ 0 & b_2 I_{n-q} \end{bmatrix}, \text{ and } \tilde{C} = \begin{bmatrix} c_1 I_r & 0 \\ 0 & c_2 I_{n-r} \end{bmatrix},$$

then there are matrices A similar to $\tilde{A}$, B similar to $\tilde{B}$ and C similar to $\tilde{C}$ such that $AB = C$ if and only if $a_2 = -a_1, b_2 = -b_1, c_2 = -c_1$ and either

i) $n+q-(p+r)$ is even, $n+r \geq p+q$ and $c_1 = a_1b_1$ or

ii) $r-p+q$ is even, $2n \geq p+q+r$ and $c_2 = a_1b_1$.

7 Some Indicative Examples

It is clear that in some balanced cases, the matrices A, B and C cannot all be diagonal (of course, one can be, as simultaneous similarity leaves our problem unchanged). For example, when $n = 4$, the numbers $a_1 = 1$, $a_2 = 2$, $b_1 = 2$, $b_2 = 4$, and $c_1 = 1$, $c_2 = 16$ satisfy theorem 15. Thus, there is a solution, A is similar to $\tilde{A}$ and B is similar to $\tilde{B}$ such that

$$A = \begin{bmatrix} 3/5 & 0 & -7/5 & 0 \\ 0 & 3/5 & 0 & -7/5 \\ 2/5 & 0 & 12/5 & 0 \\ 0 & 2/5 & 0 & 12/5 \end{bmatrix}, \ B = \begin{bmatrix} 6/5 & 0 & 56/5 & 0 \\ 0 & 6/5 & 0 & 56/5 \\ -1/5 & 0 & 24/5 & 0 \\ 0 & -1/5 & 0 & 24/5 \end{bmatrix},$$

and $AB = \tilde{C}$, but in no solution can all three matrices be diagonal, as there is no pairing of the a_i and b_i to give the c_i as products.

However, all non-balanced cases that are feasible face more stringent requirements than balanced cases. These requirements mean that a solution may be taken to be such that all matrices are diagonal. The inequality constraints on p, q, r and n that occur insure that there will be sufficiently many a_i and b_i to match the c_1 and c_2 that are present. For example, a modification of the situation above leaves it feasible, but, satisfaction of the requirements is sufficiently more demanding (theorem 21) that a diagonal solution now exist when the conditions of theorem 21 are met. Suppose $n = 17, p = 12, q = 10$, $r = 9$. Then $n + q - (p + r) = 6$ is even and $n + r > p + q$. Without loss of generality, we may suppose $a_1 = b_1 = 1$, and then $c_1 = 1$ and $a_2 = b_2 = c_2 = -1$, according to theorem 21. Thus, we may suppose that

$$\tilde{A} = \begin{bmatrix} I_{12} & 0 \\ 0 & -I_5 \end{bmatrix}, \ \tilde{B} = \begin{bmatrix} I_{10} & 0 \\ 0 & -I_7 \end{bmatrix}, \text{ and } \tilde{C} = \begin{bmatrix} I_9 & 0 \\ 0 & -I_8 \end{bmatrix}.$$

But, then $A = \begin{bmatrix} I_7 & 0 & 0 & 0 \\ 0 & -I_2 & 0 & 0 \\ 0 & 0 & I_5 & 0 \\ 0 & 0 & 0 & -I_3 \end{bmatrix}$ is similar to $\tilde{A}$, $B = \begin{bmatrix} I_7 & 0 & 0 & 0 \\ 0 & -I_2 & 0 & 0 \\ 0 & 0 & -I_5 & 0 \\ 0 & 0 & 0 & I_3 \end{bmatrix}$ is similar to $\tilde{B}$, and they satisfy $AB = \tilde{C}$.

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