JOINS OF DGA MODULES AND SECTIONAL CATEGORY

LUCÍA FERNÁNDEZ SUÁREZ, PIERRE GHIENNE, THOMAS KAHL,
LUCILE VANDEMBROUCQ

Abstract. We construct an explicit semifree model for the fiber join of two fibrations $p : E \to B$ and $p' : E' \to B$ from semifree models of p and p'. Using this model, we introduce a lower bound of the sectional category of a fibration p which can be calculated from any Sullivan model of p and which is closer to the sectional category of p than the classical cohomological lower bound given by the nilpotency of the kernel of $p^* : H^*(B;\mathbb{Q}) \to H^*(E;\mathbb{Q})$. In the special case of the evaluation fibration $X^I \to X \times X$ we obtain a computable lower bound of Farber's topological complexity $\mathrm{TC}(X)$. We show that the difference between this lower bound and the classical cohomological lower bound can be arbitrarily large.

MSC 2000: 55M30, 55P62
Keywords: Lusternik-Schnirelmann category, sectional category, topological complexity, Sullivan models

1. Introduction

The *sectional category* (or *genus*) of a fibration $p : E \to B$, denoted by $\operatorname{secat} p$, is the least integer n such that the base space B can be covered by $n+1$ open subspaces on each of which p admits a section. If no such n exists one sets $\operatorname{secat} p = \infty$. This homotopy invariant of a fibration has been introduced by A.S. Schwarz [16] in the late 1950's as a generalization of the Lusternik-Schnirelmann category of a space. The Lusternik-Schnirelmann category of a space X, $\operatorname{cat} X$, is the least integer n such that X can be covered by $n+1$ open subspaces each of which is contractible in X (if no such n exists one sets $\operatorname{cat} X = \infty$). If X is a path-connected space with base point x_0 and PX is the space of paths beginning at x_0 then $\operatorname{cat} X$ is precisely the sectional category of the evaluation fibration $ev_1 : PX \to X, \omega \mapsto \omega(1)$. References on Lusternik-Schnirelmann category and sectional category are [16], [12], [13], [2].

The concept of sectional category has been used to introduce measures for the complexity of certain problems. S. Smale [15] (see also [2, sec. 9.4]) obtained results on the complexity of the root-finding problem for algebraic equations in terms of sectional category. Recently, M. Farber [3], [4] defined the *topological complexity* of a space X, $\mathrm{TC}(X)$, to be the sectional category of the evaluation fibration $ev_{0,1} : X^I \to X \times X, \omega \mapsto (\omega(0), \omega(1))$. This notion of topological complexity plays an important role in the study of the motion planning problem in robotics.

In spite of the simplicity of the definition, it is very hard to calculate the sectional category of a fibration $p : E \to B$ and therefore one will usually have to accept to work with approximations. For a surjective fibration one easily shows that $\operatorname{secat} p \leq \operatorname{cat} B$. Hence all upper bounds of $\operatorname{cat} B$, such as the dimension of B or its cone-length, are upper bounds of $\operatorname{secat} p$ as well. A classical cohomological lower bound of $\operatorname{secat} p$ is $\operatorname{nil} \ker p^*$, the nilpotency of the kernel of $p^* : H^*(B) \to H^*(E)$

(with respect to any coefficient ring), i.e., the least integer n such that any $(n+1)$-fold cup product in $\ker p^*$ is trivial (cf. [2, sec. 9.3]). There are, of course, examples where $\operatorname{nil}\ker p^* = \operatorname{secat} p$ but in general the inequality $\operatorname{nil}\ker p^* \leq \operatorname{secat} p$ is strict. As is showing the case of Lusternik-Schnirelmann category, that is, the special case where p is the evaluation fibration $ev_1 : PX \to X$, the difference between the two numbers may actually be infinite.

A far better lower bound of $\operatorname{secat} p$ than $\operatorname{nil}\ker p^*$ (at least when the coefficient ring is $\mathbb{Q}$) is the rational sectional category $\operatorname{secat}_0 p$, i.e., the sectional category of a rationalization of p. In her thesis [5], A. Fassò Velenik gave a characterization of $\operatorname{secat}_0 p$ in terms of a Sullivan model of p. Unfortunately, concrete computations based on this characterization turn out to be rather difficult due to the complexity of the algebraic manipulations involved. In the present article we introduce an approximation of $\operatorname{secat} p$ which is not as good as $\operatorname{secat}_0 p$ in general but much easier to calculate. This approximation, which we denote by $\operatorname{Msecat} p$, is still a much better lower bound of $\operatorname{secat} p$ than $\operatorname{nil}\ker p^*$, if we consider coefficients in $\mathbb{Q}$. Throughout this article we work over the field $\mathbb{Q}$.

There is a classical equivalent definition of sectional category in terms of joins which is more appropriate for our purpose than the original one. Denote by $*_B^n E$ the n-fold fiber join of the fibration $p : E \to B$ and by $j^n p : *_B^n E \to B$ the nth join map. Then $\operatorname{secat} p \leq n$ if and only if $j^n p$ has a section. We recall this fact and the join construction in section 2. Let A_{PL} denote Sullivan's functor of polynomial forms from spaces to commutative cochain algebras. Consider the morphism $A_{PL}(j^n p) : A_{PL}(B) \to A_{PL}(*_B^n E)$ as a morphism of $A_{PL}(B)$-modules. In section 5, we define the invariant $\operatorname{Msecat} p$ to be the least integer n for which $A_{PL}(j^n p) = \phi \circ i$ where ϕ is a quasi-isomorphism of $A_{PL}(B)$-modules and i is a morphism of $A_{PL}(B)$-modules which admits a retraction of $A_{PL}(B)$-modules. We show that $\operatorname{nil}\ker p^* \leq \operatorname{Msecat} p \leq \operatorname{secat} p$ (cf. 5.2). In the special case of the evaluation fibration $ev_1 : PX \to X$, Msecat coincides with the well-known invariant $\operatorname{Mcat} X$ (cf. 5.6) which in turn is known to be the rational category of X (cf. [11]). The invariant Msecat generalizes the invariant Mcat hence in the same way as secat generalizes cat. The fact that Mcat is rational category does, however, not generalize to Msecat and Msecat does not in general equal rational sectional category.

The computability of the invariant Msecat relies on an algebraic join construction which we develop in sections 3 and 4. Let (A, d) be a commutative cochain algebra. In section 3, we define the join $(M, d) *_{(A,d)} (N, d)$ of two (A, d)-semifree extensions (M, d) and (N, d) of (A, d). This is an explicitly defined semifree extension of (A, d). Moreover, if (M, d) and (N, d) are minimal semifree (A, d)-modules, so is $(M, d) *_{(A,d)} (N, d)$. Consider two fibrations $p : E \to B$ and $p' : E' \to B$ and suppose that $\alpha : (A, d) \xrightarrow{\sim} A_{PL}(B)$ is a commutative cochain algebra model of the base space B and that (M, d) and (N, d) are semifree extensions of (A, d) such that there exist quasi-isomorphisms of (A, d)-modules $(M, d) \xrightarrow{\sim} A_{PL}(E)$ and $(N, d) \xrightarrow{\sim} A_{PL}(E')$ which extend $A_{PL}(p) \circ \alpha$ and $A_{PL}(p') \circ \alpha$. We establish in section 4 that the inclusion $(A, d) \to (M, d) *_{(A,d)} (N, d)$ is an (A, d)-module model of the topological join map $E *_B E' \to B$. Iterating the join construction, we define the n-fold join $*^n_{(A,d)}(M, d)$ of (M, d) and obtain an explicit (A, d)-module model of the nth join map $j^n p : *_B^n E \to B$. The number $\operatorname{Msecat} p$ is then the least n such that the inclusion $(A, d) \to *^n_{(A,d)}(M, d)$ admits a retraction of (A, d)-modules (cf.

5.4). Through this result one obtains an effective method to compute the invariant $\mathrm{Msecat}\, p$ from a Sullivan model of p.

As an example we consider Farber's topological complexity TC. Let X be a simply connected space of finite type with Sullivan model $(\Lambda V, d)$. There is a well-known explicit minimal model of the evaluation fibration $ev_{0,1} : X^I \to X \times X$ which can be determined from $(\Lambda V, d)$. This model and the algebraic join construction permit one to calculate the invariant $\mathrm{MTC}(X) = \mathrm{Msecat}\, ev_{0,1}$ which is a lower bound of $\mathrm{TC}(X)$. Note that since $ev_{0,1}$ is the mapping path fibration associated to the diagonal map $X \to X \times X$, $ev_{0,1}^*$ can be identified with the cup product $\cup : H^*(X) \otimes H^*(X) \to H^*(X)$. If X is a formal space, i.e., a space whose rational homotopy type is entirely determined by its cohomology algebra, one has $\mathrm{MTC}(X) = \mathrm{nil}\ker \cup$. But already for the simplest example of a non-formal space, one calculates that $\mathrm{MTC}(X) = 3$ and $\mathrm{nil}\ker \cup = 2$. We show finally that the difference between the two lower bounds is unbounded.

2. Sectional category and joins

The category of spaces in which we shall work throughout this article is the category of compactly generated Hausdorff spaces. All categorical constructions (products, pullbacks etc.) are carried out in this category.

In this section we recall the link between joins and the sectional category mentioned in the introduction.

Definition 2.1. The *(fiber) join* of two maps $p : E \to B$ and $p' : E' \to B$, denoted by $E *_B E'$, is the double mapping cylinder of the projections $E \times_B E' \to E$ and $E \times_B E' \to E'$, i.e., the quotient space $((E \times_B E') \times I \amalg E \amalg E')/\sim$ where $(e, e', 0) \sim e$, $(e, e', 1) \sim e'$. The *join map* of p and p' is the map $j_{p,p'} : E *_B E' \to B$ defined by $j_{p,p'}([e, e', t]) = p(e) = p'(e')$, $j_{p,p'}([e]) = p(e)$, and $j_{p,p'}([e']) = p'(e')$. The *n-fold join* and the *nth join map* of p are iteratively defined by $*_B^0 E = E$, $*_B^n E = (*_B^{n-1} E) *_B E$, $j^0 p = p$, and $j^n p = j_{j^{n-1}p,p}$.

Theorem 2.2. *Let $p : E \to B$ be a fibration. If B is normal then $\mathrm{secat}\, p \leq n$ if and only if $j^n p$ has a section.*

Proof. The result is well-known, at least when B is paracompact. We include a short proof for the convenience of the reader.

Suppose first that $\mathrm{secat}\, p \leq n$. We show by induction that for each $0 \leq m \leq n$ there exists an open cover $U_0, \ldots, U_{n-m}$ of B such that $j^m p$ has a section on U_0 and p has a section on each of the remaining U_i. For $m = 0$ this is just the hypothesis that $\mathrm{secat}\, p \leq n$. Suppose that the assertion holds for $0 \leq m < n$. Then there exists an open cover $U_0, \ldots, U_{n-m}$ of B, a section $\sigma_0 : U_0 \to *_B^m E$ of $j^m p$, and sections $\sigma_i : U_i \to E$ of p $(1 \leq i \leq n-m)$. Since B is normal, there exist open covers $V_0, \ldots, V_{n-m}$ and $W_0, \ldots, W_{n-m}$ of B such that $\bar{V}_i \subset W_i \subset \bar{W}_i \subset U_i$. Set $A_0 = \bar{V}_0 \cap (B \backslash W_1)$, $A_1 = \bar{V}_1 \cap (B \backslash W_0)$, and $A_2 = \bar{W}_0 \cap \bar{W}_1 \cap (\bar{V}_0 \cup \bar{V}_1)$. Then A_0, A_1, and A_2 are closed subspaces of B, $A_0 \cup A_1 \cup A_2 = \bar{V}_0 \cup \bar{V}_1$, and $A_0 \cap A_1 = \emptyset$. Since B is normal, by Urysohn's Lemma, there exists a continuous map $\phi : B \to I$ such that $\phi(A_0) \subset \{0\}$ and $\phi(A_1) \subset \{1\}$. Define a section σ of $j^{m+1} p$ on $\bar{V}_0 \cup \bar{V}_1$ by

$$\sigma(x) = \begin{cases} [\sigma_0(x)], & x \in A_0, \\ [\sigma_1(x)], & x \in A_1, \\ [\sigma_0(x), \sigma_1(x), \phi(x)], & x \in A_2. \end{cases}$$

Consider the open cover $O_0, \dots O_{n-m-1}$ of B given by $O_0 = V_0 \cup V_1$ and $O_i = U_{i+1}$, $i = 1, \dots, n-m-1$. On O_0, σ is a section of $j^{m+1}p$. On each of the remaining O_i, p has a section by hypothesis. This terminates the inductive step.

Suppose now that $j^n p$ has a section $s : B \to *_B^n E$. By Lemma 2.4 below, $*_B^n E$ can be covered by $n+1$ open subspaces $U_0, \dots, U_n$ on each of which the projection $\bar{p}_n : (*_B^n E) \times_B E \to *_B^n E$ has a section. The inverse images $s^{-1}(U_i)$ form a cover of B by open subspaces on each of which p has a section. Therefore $\operatorname{secat} p \leq n$. □

Remark 2.3. If $p : E \to B$ and $p' : E' \to B$ are fibrations, so is the join map $j_{p,p'} : E *_B E' \to B$. Indeed, if $\lambda : E \times_B B^I \to E^I$ and $\lambda' : E' \times_B B^I \to E'^I$ are lifting maps for p and p' then a lifting map $\phi : (E *_B E') \times_B B^I \to (E *_B E')^I$ for $j_{p,p'}$ is given by $\phi([e, e', t], \omega)(s) = [\lambda(e, \omega)(s), \lambda'(e', \omega)(s), t]$, $\phi([e], \omega)(s) = [\lambda(e, \omega)(s)]$, and $\phi([e'], \omega)(s) = [\lambda'(e', \omega)(s)]$. Note that ϕ is continuous since we are working with compactly generated spaces. It follows that the nth join map of a fibration is again a fibration and hence that it has a section if and only if it has a homotopy section.

In the proof of Theorem 2.2 we used the following lemma. We shall need this lemma again in the proof of the inequality $\operatorname{Msecat} p \geq \operatorname{nil} \ker p^*$ (cf. 5.2).

Lemma 2.4. *Consider a fibration $p : E \to B$ and form the pullback diagram*

$$\begin{array}{ccc} (*_B^n E) \times_B E & \longrightarrow & E \\ {\scriptstyle \bar{p}_n}\downarrow & & \downarrow{\scriptstyle p} \\ *_B^n E & \xrightarrow[j^n p]{} & B. \end{array}$$

Then $\operatorname{secat} \bar{p}_n \leq n$.

Proof. We proceed by induction. For $n = 0$, the map $E \to E \times_B E$, $e \mapsto (e, e)$ is a section of $\bar{p}_n$. Suppose that $n > 0$ and that the assertion holds for $n-1$. The spaces E and $*_B^{n-1} E$ are embedded as closed subspaces in $*_B^n E$ and there are canonical projections $\pi : *_B^n E \setminus *_B^{n-1} E \to E$ and $\tilde{\pi} : *_B^n E \setminus E \to *_B^{n-1} E$. Let U_0 be the open subspace $*_B^n E \setminus *_B^{n-1} E$ of $*_B^n E$. We have $j^n p|_{U_0} = p\pi$. The inductive hypothesis implies that $*_B^{n-1} E$ can be covered by n open subspaces $V_1, \dots, V_n$ such that each restriction of the join map $j^{n-1} p|_{V_i} : V_i \to B$ factors through p. Consider the open subspaces $U_i = \tilde{\pi}^{-1}(V_i)$ of $*_B^n E$. The $n+1$ open subspaces $U_0, U_1 \dots, U_n$ of $*_B^n E$ cover $*_B^n E$. The restriction of the join map $j^n p$ to any of these open subspaces factors through p. Therefore the projection $\bar{p}_n : (*_B^n E) \times_B E \to *_B^n E$ has a section on each U_i. This shows that $\operatorname{secat} \bar{p}_n \leq n$. □

3. Joins of semifree modules

The purpose of this section is to define joins of semifree extensions of a commutative cochain algebra. Recall that we are working over $\mathbb{Q}$. All graded vector spaces we consider will be upper graded and all differential vector spaces will be cochain complexes, i.e., the differential raises the upper degree by one. The *nth suspension* $s^{-n}V$ of a graded vector space V is defined by $(s^{-n}V)^i = V^{i-n}$.

Definition 3.1. Let (A,d) be a differential algebra. A *semifree extension* of an (A,d)-module (M,d) is an (A,d)-module of the form $(M\oplus A\otimes X,d)$ where the action is the one of the direct sum, the differential on M is the differential of (M,d), and X admits a direct sum decomposition $X=\bigoplus_{i=0}^{\infty}X_i$ such that $d(X_0)\subset M$ and $d(X_n)\subset M\oplus A\otimes(\bigoplus_{i=0}^{n-1}X_i)$ for $n\geq 1$. A *semifree* (A,d)*-module* is a semifree extension of the trivial (A,d)-module 0.

For the remainder of this section we fix a commutative cochain algebra (A,d) and two semifree extensions $(M,d)=(A\oplus A\otimes X,d)$ and $(N,d)=(A\oplus A\otimes Y,d)$ of (A,d). We define the join $(M,d)*_{(A,d)}(N,d)$ of (M,d) and (N,d) which will again be a semifree extension of (A,d). Forgetting the differential, $(M,d)*_{(A,d)}(N,d)$ is the free graded A-module $A\oplus A\otimes s^{-1}X\otimes Y$. In order to define the differential of $(M,d)*_{(A,d)}(N,d)$, we decompose the differential in (M,d) of an element $m\in M$ as

$$dm=d_0m+d_+m$$

where $d_0m\in A$ and $d_+m\in A\otimes X$. Using the same notation, we decompose the differential in (N,d) of an element $n\in N$. Consider elements $x\in X$ and $y\in Y$ and write

$$d_+x=\sum_i a_i\otimes x_i,\quad\text{and}\quad d_+y=\sum_j b_j\otimes y_j.$$

The differential of the element $s^{-1}x\otimes y$ in $(M,d)*_{(A,d)}(N,d)$ is then defined by

$$\begin{aligned}d(s^{-1}x\otimes y) &= (-1)^{|x|}d_0xd_0y+\sum_i(-1)^{|a_i|+1}a_i\otimes s^{-1}x_i\otimes y\\ &\quad+\sum_j(-1)^{(|x|+1)(|b_j|+1)}b_j\otimes s^{-1}x\otimes y_j.\end{aligned}$$

We extend this differential to the whole join $(M,d)*_{(A,d)}(N,d)$ by setting

$$d(a\otimes s^{-1}x\otimes y)=da\otimes s^{-1}x\otimes y+(-1)^{|a|}a\cdot d(s^{-1}x\otimes y).$$

Proposition 3.2 below assures that d is indeed an (A,d)-module differential in $(M,d)*_{(A,d)}(N,d)$. It is an easy exercise to check that $(M,d)*_{(A,d)}(N,d)$ is a semifree extension of (A,d). Moreover, if (A,d) is augmented and (M,d) and (N,d) are minimal semifree (A,d)-modules, i.e., the differentials in $\mathbb{Q}\otimes_{(A,d)}(M,d)$ and $\mathbb{Q}\otimes_{(A,d)}(N,d)$ are zero, then $(M,d)*_{(A,d)}(N,d)$ is also minimal.

Proposition 3.2. $d^2(s^{-1}x\otimes y)=0$.

Proof. Write $d_+x_i=\sum_k a_{ik}\otimes x_{ik}$ and $d_+y_j=\sum_l b_{jl}\otimes y_{jl}$. Since

$$\begin{aligned}0 &= d^2x\\ &= d(d_0x+\sum_i a_i\otimes x_i)\\ &= dd_0x+\sum_i da_i\otimes x_i+\sum_i(-1)^{|a_i|}a_id_0x_i+\sum_i(-1)^{|a_i|}a_id_+x_i,\end{aligned}$$

we have $dd_0x=-\sum_i(-1)^{|a_i|}a_id_0x_i=\sum_i(-1)^{|a_i|+1}a_id_0x_i$ and

$$\sum_i da_i\otimes x_i=-\sum_i(-1)^{|a_i|}a_id_+x_i=\sum_{i,k}(-1)^{|a_i|+1}a_ia_{ik}\otimes x_{ik}.$$

Similarly, $dd_0y = \sum_j (-1)^{|b_j|+1} b_j d_0 y_j$ and $\sum_j db_j \otimes y_j = \sum_{j,l} (-1)^{|b_j|+1} b_j b_{jl} \otimes y_{jl}$. Use σ to denote the isomorphism

$$A \otimes X \otimes Y \to A \otimes s^{-1}X \otimes Y, \quad a \otimes x \otimes y \mapsto (-1)^{|a|} a \otimes s^{-1}x \otimes y$$

and T to denote the isomorphism

$$A \otimes X \to X \otimes A, \quad a \otimes x \mapsto (-1)^{|a||x|} x \otimes a.$$

Applying σ to the identity $\sum_i da_i \otimes x_i \otimes y = \sum_{i,k} (-1)^{|a_i|+1} a_i a_{ik} \otimes x_{ik} \otimes y$, one obtains the identity

$$\sum_i (-1)^{|a_i|+1} da_i \otimes s^{-1}x_i \otimes y = \sum_{i,k} (-1)^{|a_{ik}|+1} a_i a_{ik} \otimes s^{-1}x_{ik} \otimes y.$$

Applying $\sigma \circ (T \otimes id_Y)$ to the identity $\sum_j x \otimes db_j \otimes y_j = \sum_{j,l} (-1)^{|b_j|+1} x \otimes b_j b_{jl} \otimes y_{jl}$, one obtains the identity

$$\sum_j (-1)^{|x|+|b_j|+|x||b_j|} db_j \otimes s^{-1}x \otimes y_j = \sum_{j,l} (-1)^{|x||b_j|+|x||b_{jl}|+|b_{jl}|} b_j b_{jl} \otimes s^{-1}x \otimes y_{jl}.$$

Using the different formulae above one easily verifies that $d^2(s^{-1}x \otimes y) = 0$. □

Definition 3.3. The *n-fold join* of (M,d) is iteratively defined by $*^0_{(A,d)}(M,d) = (M,d)$ and $*^n_{(A,d)}(M,d) = (*^{n-1}_{(A,d)}(M,d)) *_{(A,d)} (M,d)$.

Remarks 3.4. (i) Note that $*^n_{(A,d)}(M,d)$ is a semifree extension of (A,d). Moreover, if (M,d) is a minimal semifree (A,d)-module then $*^n_{(A,d)}(M,d)$ is a minimal semifree (A,d)-module as well.

(ii) We have

$$*^n_{(A,d)}(M,d) = (A \oplus A \otimes s^{-n}X^{\otimes n+1}, d).$$

Consider elements $x_0, \dots, x_n \in X$ and write $d_+x_i = \sum_{j_i} a_{ij_i} \otimes x_{ij_i}$. An easy induction shows that

$$\begin{aligned} d(s^{-n}x_0 \otimes \cdots \otimes x_n) &= (-1)^{\sum_{k=1}^{n} (k|x_{n-k}|+k-1)} d_0x_0 \cdot \cdots \cdot d_0x_n \\ &+ \sum_{i=0}^{n} \sum_{j_i} (-1)^{(|a_{ij_i}|+1)(|x_0|+\cdots+|x_{i-1}|+n)} a_{ij_i} \otimes s^{-n}x_0 \otimes \cdots \otimes x_{ij_i} \otimes \cdots \otimes x_n. \end{aligned}$$

(iii) Consider a morphism of commutative cochain algebras $(A,d) \to (B,d)$ and the (B,d)-semifree extension of (B,d) defined by $(N,d) = (B,d) \otimes_{(A,d)} (M,d)$. The formula for the differential given in (ii) shows that

$$*^n_{(B,d)}(N,d) = (B,d) \otimes_{(A,d)} *^n_{(A,d)}(M,d).$$

4. Topological versus algebraic joins

Our goal in this section is to show that the algebraic joins of the preceding section model topological joins. In this and the following sections we make frequent use of the homotopy theory of modules over a DGA and, in particular, of the following well-known result:

Theorem 4.1. *Let (A,d) be a differential algebra. The category of (A,d)-modules is a proper closed model category where weak equivalences are quasi-isomorphisms, fibrations are surjective morphisms, and cofibrations are morphims having the left lifting property with respect to surjective quasi-isomorphisms. The inclusions of semifree extensions are (particular) cofibrations.*

We refer the reader to [9] for the axioms of closed model categories. A closed model category is called *proper* if the class of weak equivalences is closed under base change along fibrations and cobase change along cofibrations. As is customary we denote weak equivalences by $\xrightarrow{\sim}$, fibrations by $\twoheadrightarrow$, and cofibrations by $\rightarrowtail$. For the convenience of the reader we include the following

Proof of Theorem 4.1. We do not use the fact that we are working over $\mathbb{Q}$ and the proof works for an arbitrary commutative ground ring. We first show that inclusions of semifree extensions are cofibrations. Consider a semifree extension $(M \oplus A \otimes (\bigoplus_{i=0}^{\infty} X_i), d)$ of an (A,d)-module (M,d) and a commutative diagram of (A,d)-modules

$$\begin{array}{ccc} (M,d) & \xrightarrow{\quad f \quad} & (P,d) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p}\ {\scriptstyle \sim} \\ (M \oplus A \otimes (\bigoplus_{i=0}^{\infty} X_i), d) & \xrightarrow[g]{} & (Q,d) \end{array}$$

where i is the inclusion. Suppose inductively that we have constructed a lifting λ for the diagram up to $(M \oplus A \otimes (\bigoplus_{i=0}^{n} X_i), d)$. Let $\mathcal{B} \subset X_{n+1}$ be a basis and $x \in \mathcal{B}$. Then $\lambda(dx)$ is defined and $d\lambda(dx) = 0$. Since p is surjective, there exists an element $\xi \in P$ such that $p(\xi) = g(x)$. Then $\lambda(dx) - d\xi$ is a cocycle in $\ker p$. Since p is a surjective quasi-isomorphism, $\ker p$ is acyclic and there exists an element $y \in \ker p$ such that $dy = \lambda(dx) - d\xi$. Set $\lambda(x) = \xi + y$. This defines λ in $(M \oplus A \otimes (\bigoplus_{i=0}^{n+1} X_i), d)$. It follows that a lifting exists and hence that i is a cofibration. The first three axioms CM1 (existence of finite limits and colimits), CM2 ("2 = 3"), and CM3 (closure under retracts) are clear. We check the factorization axiom CM5. Consider a morphism $f : (M,d) \to (N,d)$ of (A,d)-modules. Consider the acyclic semifree (A,d)-module $(A \otimes (\tilde{N} \oplus s^{-1}\tilde{N}), \delta)$ where $\tilde{N} = \bigoplus_{n \in N} \mathbb{Q} \cdot n$, $\delta n = s^{-1}n$ and $\delta s^{-1} n = 0$. Then the inclusion $i : (M,d) \to (M,d) \oplus (A \otimes (\tilde{N} \oplus s^{-1}\tilde{N}), \delta)$ is both a cofibration and a quasi-isomorphism. Let $p : (M,d) \oplus (A \otimes (\tilde{N} \oplus s^{-1}\tilde{N}) \to (N,d)$ be the morphism of (A,d)-modules defined by $p(m) = f(m)$, $p(n) = n$, and $p(s^{-1}n) = dn$. Obviously, p is surjective and $f = p \circ i$. This shows one part of CM5. In the proof of [8, 2.1(i)] it is shown that there is a factorization $f = p \circ i$ where p is a surjective quasi-isomorphism and i is the inclusion of a semifree extension. This shows the other part of CM5. We verify the lifting axiom CM4. One of the lifting properties is the definition of cofibrations. For the other one consider a commutative diagram

of (A,d)-modules

$$\begin{array}{ccc} (M,d) & \xrightarrow{f} & (P,d) \\ {\scriptstyle\sim}\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle p} \\ (N,d) & \xrightarrow[g]{} & (Q,d). \end{array}$$

Form the pullback (A,d)-module $(N,d) \times_{(Q,d)} (P,d)$. Since p is surjective, so is its base extension $\bar{p} : (N,d) \times_{(Q,d)} (P,d) \to (N,d)$. Choose a factorization of the canonical morphism $(i,f) : (M,d) \to (N,d) \times_{(Q,d)} (P,d)$ in a quasi-isomorphism $j : (M,d) \xrightarrow{\sim} (R,d)$ and a surjective morphism $r : (R,d) \to (N,d) \times_{(Q,d)} (P,d)$. The composite $\bar{p} \circ r$ is a surjective quasi-isomorphism. Since i is a cofibration, there exists a section s of $\bar{p} \circ r$ such that $s \circ i = j$. Let $\bar{g} : (N,d) \times_{(Q,d)} (P,d) \to (P,d)$ be the base extension of g. Then the composite $\bar{g} \circ r \circ s$ is a lifting for the above square. It follows that the category of (A,d)-modules is a closed model category. The proof of the fact that this closed model category is proper is an easy application of the 5-lemma. □

Consider two fibrations $p : E \to B$ and $p' : E' \to B$ where B, E, and E' are simply connected spaces of finite type. Simply connected spaces are understood to be non-empty. A space is said to be of finite type if it has finite dimensional rational homology in every dimension. Let A_{PL} be Sullivan's functor from spaces to commutative cochain algebras. Fix a commutative cochain algebra model $\alpha : (A,d) \xrightarrow{\sim} A_{PL}(B)$. For any continuous map $f : S \to B$ the morphism of commutative cochain algebras $A_{PL}(f) \circ \alpha$ induces an (A,d)-module structure on $A_{PL}(S)$ such that $A_{PL}(f) \circ \alpha$ is a morphism of (A,d)-modules. Let $(M,d) = (A \oplus A \otimes X, d)$ and $(N,d) = (A \oplus A \otimes Y, d)$ be semifree extensions of (A,d) such that there exist quasi-isomorphisms of (A,d)-modules $(M,d) \xrightarrow{\sim} A_{PL}(E)$ and $(N,d) \xrightarrow{\sim} A_{PL}(E')$ which extend $A_{PL}(p) \circ \alpha$ and $A_{PL}(p') \circ \alpha$. As in the preceding section we write $d = d_0 + d_+$ for the differentials of (M,d) and (N,d).

Theorem 4.2. *(i) The morphism of (A,d)-modules $A_{PL}(j_{p,p'}) \circ \alpha$ extends to a quasi-isomorphism of (A,d)-modules $(M,d) *_{(A,d)} (N,d) \xrightarrow{\sim} A_{PL}(E *_B E')$.*

(ii) The morphism of (A,d)-modules $A_{PL}(j^n p) \circ \alpha$ extends to a quasi-isomorphism of (A,d)-modules $*^n_{(A,d)}(M,d) \xrightarrow{\sim} A_{PL}(*^n_B E)$.

Proof. The second part follows from the first by induction. The proof of (i) is divided in 3 steps.

Step 1: A model of the pullback. Choose Sullivan models $\psi : (A \otimes \Lambda V, d) \xrightarrow{\sim} A_{PL}(E)$ and $\psi' : (A \otimes \Lambda V', d) \xrightarrow{\sim} A_{PL}(E')$ of $A_{PL}(p) \circ \alpha$ and $A_{PL}(p') \circ \alpha$. Since the inclusions $(A,d) \to (M,d)$ and $(A,d) \to (N,d)$ are cofibrations, by the lifting lemma [1, II.1.11], there exist quasi-isomorphisms of (A,d)-modules $h : (M,d) \xrightarrow{\sim} (A \otimes \Lambda V, d)$ and $h' : (N,d) \xrightarrow{\sim} (A \otimes \Lambda V', d)$ which extend the inclusions of (A,d). Form the pushout of commutative cochain algebras

$$\begin{array}{ccc} (A,d) & \longrightarrow & (A \otimes \Lambda V, d) \\ \downarrow & & \downarrow \\ (A \otimes \Lambda V', d) & \longrightarrow & (A \otimes \Lambda V, d) \otimes_{(A,d)} (A \otimes \Lambda V', d). \end{array}$$

It follows from [10, 15(c)] that the morphisms $A_{PL}(pr_E) \circ \psi$ and $A_{PL}(pr'_E) \circ \psi'$, where $pr_E : E \times_B E' \to E$ and $pr_{E'} : E \times_B E' \to E'$ are the projections, induce a quasi-isomorphism of commutative cochain algebras $(A \otimes \Lambda V, d) \otimes_{(A,d)} (A \otimes \Lambda V', d) \to A_{PL}(E \times_B E')$. By [10, 6.7], since h and h' are quasi-isomorphisms between semifree (A, d)-modules, the morphism

$$h \otimes_A h' : (M, d) \otimes_{(A,d)} (N, d) \to (A \otimes \Lambda V, d) \otimes_{(A,d)} (A \otimes \Lambda V', d)$$

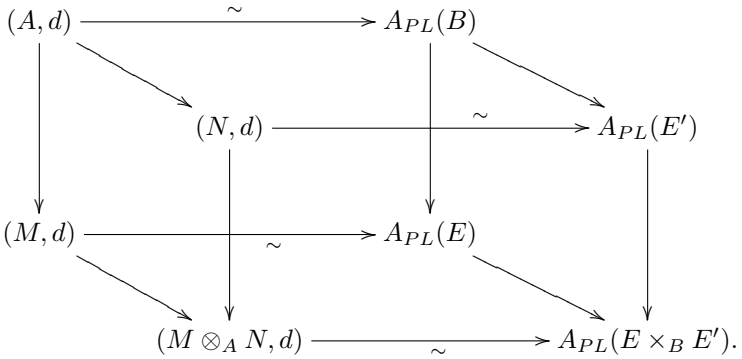
is a quasi-isomorphism. Since A is commutative, $(M, d) \otimes_{(A,d)} (N, d)$ is an (A, d)-module and $h \otimes_A h'$ a quasi-isomorphism of (A, d)-modules. Note that

$$(M, d) \otimes_{(A,d)} (N, d) = (M \otimes_A N, d) = (A \oplus A \otimes (X \oplus Y \oplus X \otimes Y), d)$$

contains both (M, d) and (N, d) as sub (A, d)-modules. Note also that if $d_+x = \sum_i a_i \otimes x_i$ and $d_+y = \sum_j b_j \otimes y_j$ then the differential of $x \otimes y$ in $(M, d) \otimes_{(A,d)} (N, d)$ is given by

$$\begin{aligned} d(x \otimes y) &= d_0 x \otimes y + \sum_i a_i \otimes x_i \otimes y + (-1)^{|x||y|} d_0 y \otimes x \\ &\quad + \sum_j (-1)^{|x|(|b_j|+1)} b_j \otimes x \otimes y_j. \end{aligned}$$

We have obtained the following commutative diagram of (A, d)-modules:



Step 2: A model of the join map. Consider the mapping cylinder factorization of the projection $pr'_E : E \times_B E' \to E'$ in a cofibration $\iota : E \times_B E' \to Z$ and a homotopy equivalence $\rho : Z \to E'$. We have the following pushout:

$$\begin{array}{ccc} E \times_B E' & \xrightarrow{\iota} & Z \\ {\scriptstyle pr_E}\downarrow & & \downarrow \\ E & \longrightarrow & E *_B E'. \end{array}$$

Let ν be the inclusion $(N, d) \to (M \otimes_A N, d)$. Consider the (A, d)-module

$$(Q, D) = (M \otimes_A N \oplus N \oplus s^{-1}M \otimes_A N, D)$$

where the action on $M \otimes_A N \oplus N$ is the one of the direct sum, $a \cdot s^{-1}w = (-1)^{|a|} s^{-1} aw$, and the differential is given by

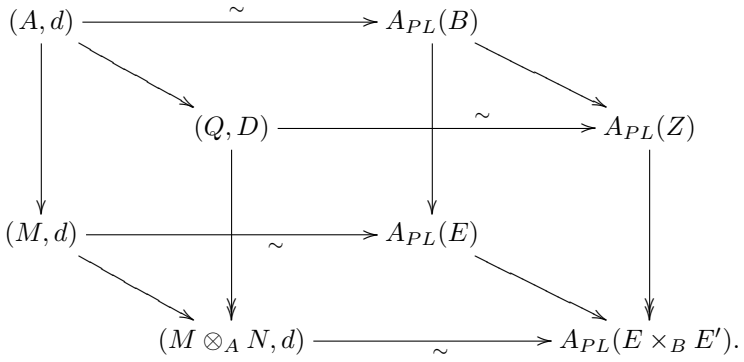
$$\begin{aligned} D(m \otimes_A n) &= d(m \otimes_A n) + s^{-1}m \otimes_A n, \\ Dn &= dn + s^{-1}\nu(n), \\ Ds^{-1}w &= -s^{-1}dw. \end{aligned}$$

Let $i : (N, d) \to (Q, D)$ be the injection defined by $i(n) = \nu(n) - n$. One easily checks that this is a morphism of (A, d)-modules. We show that i is both a cofibration and a quasi-isomorphism. Set $U = \mathbb{Q} \oplus X \oplus Y \oplus X \otimes Y$ where the elements of $\mathbb{Q}$ have degree 0. Then $M \otimes_A N$ is the free graded A-module $A \otimes U$. Consider the acyclic semifree (A, d)-module $(A \otimes (U \oplus s^{-1}U), \delta)$ where $\delta u = s^{-1}u$ and $\delta s^{-1}u = 0$. Then the inclusion $(N, d) \to (N, d) \oplus (A \otimes (U \oplus s^{-1}U), \delta)$ is both a cofibration and a quasi-isomorphism. Consider the isomorphism of (A, d)-modules $\Phi : (N, d) \oplus (A \otimes (U \oplus s^{-1}U), \delta) \to (Q, D)$ defined by $\Phi(n) = \nu(n) - n$, $\Phi(u) = u$, and $\Phi(s^{-1}u) = du + s^{-1}u$. Since i is the restriction of Φ to (N, d), it is both a cofibration and a quasi-isomorphism. Let $\pi : (Q, D) \to (M \otimes_A N, d)$ be the obvious projection. Then π is a surjective morphism of (A, d)-modules and $\pi \circ i = \nu$.

Form the following commutative diagram of (A, d)-modules:

$$\begin{array}{ccccc}
(N,d) & \xrightarrow[\sim]{} & A_{PL}(E') & \xrightarrow[\sim]{A_{PL}(\rho)} & A_{PL}(Z) \\
{\scriptstyle\sim}\downarrow{\scriptstyle i} & & & & \downarrow{\scriptstyle A_{PL}(\iota)} \\
(Q,D) & \xrightarrow[\pi]{} & (M \otimes_A N, d) & \xrightarrow{\sim} & A_{PL}(E \times_B E').
\end{array}$$

Since $A_{PL}(\iota)$ is surjective, there exists a lifting $\lambda : (Q, D) \to A_{PL}(Z)$ making the diagram commutative. Note that λ is automatically a quasi-isomorphism. Consider the following commutative cube of (A, d)-modules:



Form the pullback (A, d)-module $(J, D) = (M, d) \times_{(M \otimes_A N, d)} (Q, D)$ and the pullback cochain algebra $A_{PL}(E) \times_{A_{PL}(E \times_B E')} A_{PL}(Z)$. By the dual of the gluing lemma [1, II.1.2], [9, 8.13], the horizontal quasi-isomorphisms in the above cube induce a quasi-isomorphism of (A, d)-modules

$$(J, D) \xrightarrow{\sim} A_{PL}(E) \times_{A_{PL}(E \times_B E')} A_{PL}(Z).$$

By [10, 13.5], the canonical morphism

$$A_{PL}(E *_B E') \to A_{PL}(E) \times_{A_{PL}(E \times_B E')} A_{PL}(Z)$$

is a quasi-isomorphism and we obtain the following commutative diagram of (A, d)-modules:

$$\begin{array}{ccccc}
(A,d) & \xrightarrow{\sim} & A_{PL}(B) & \xleftarrow{=} & A_{PL}(B) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle A_{PL}(j_{p,p'})} \\
(J,D) & \xrightarrow[\sim]{} & A_{PL}(E) \times_{A_{PL}(E \times_B E')} A_{PL}(Z) & \xleftarrow[\sim]{} & A_{PL}(E *_B E').
\end{array}$$

Step 3: A quasi-isomorphism $(J,D) \xrightarrow{\sim} (M,d) *_{(A,d)} (N,d)$. We have

$$(J,D) = (M \oplus N \oplus s^{-1}M \otimes_A N, D).$$

The action on $M \oplus N$ is the one of the direct sum, $a \cdot s^{-1}w = (-1)^{|a|}s^{-1}aw$, and the differential is given by $Dm = dm + s^{-1}\gamma(m)$, $Dn = dn + s^{-1}\nu(n)$, and $Ds^{-1}w = -s^{-1}dw$. Here, $\gamma : M \to M \otimes_A N$ is the inclusion. Let $j : (A,d) \to (J,D)$ be the canonical morphism. If we write a_M for the elements of J which lie in the copy of A coming from M and a_N for the elements of J which lie in the copy of A coming from N then j is given by $j(a) = a_M - a_N$. Since the join $(M,d) *_{(A,d)} (N,d)$ is a semifree extension of (A,d), by the lifting lemma [1, II.1.11], in order to finish the proof it is enough to construct a quasi-isomorphism of (A,d)-modules

$$f : (J,D) \to (M,d) *_{(A,d)} (N,d) = (A \oplus A \otimes s^{-1}X \otimes Y, d)$$

such that the composite of $f \circ j$ is the inclusion $(A,d) \to (M,d) *_{(A,d)} (N,d)$. We define the map f by $f(a_M) = \frac{1}{2}a$, $f(a \otimes x) = 0$, $f(a_N) = -\frac{1}{2}a$, $f(a \otimes y) = 0$, $f(s^{-1}a) = 0$, $f(s^{-1}a \otimes x) = -\frac{1}{2}(-1)^{|a|}ad_0x$, $f(s^{-1}a \otimes y) = \frac{1}{2}(-1)^{|a|}ad_0y$, and $f(s^{-1}a \otimes x \otimes y) = (-1)^{|a|}a \otimes s^{-1}x \otimes y$. It is straightforward to check that f is A-linear and obvious that $f \circ j$ is the inclusion. Consider an element $x \in X$ and write $d_+x = \sum_i a_i \otimes x_i$. As we have shown at the beginning of the proof of 3.2, $dd_0x = -\sum_i (-1)^{|a_i|}a_i d_0 x_i$. Using this identity and a corresponding one for $y \in Y$, it is straightforward to check that f commutes with the differentials. It remains to show that f is a quasi-isomorphism. Consider the pushout (A,d)-module $(R,d) = (A \oplus A \otimes X \oplus A \otimes Y, d)$ of the inclusions $(A,d) \to (M,d)$ and $(A,d) \to (N,d)$ and form the acyclic differential vector space

$$(R \oplus s^{-1}R, D) = (A \oplus A \otimes X \oplus A \otimes Y \oplus s^{-1}A \oplus s^{-1}A \otimes X \oplus s^{-1}A \otimes Y, D)$$

where $Dr = dr + s^{-1}r$ and $Ds^{-1}r = -s^{-1}dr$. Define a map $g : R \oplus s^{-1}R \to J$ by $ga = \frac{1}{2}a_M + \frac{1}{2}a_N$, $g(a \otimes x) = a \otimes x$, $g(a \otimes y) = a \otimes y$, $g(s^{-1}a) = s^{-1}a$,

$$g(s^{-1}a \otimes x) = s^{-1}a \otimes x + \frac{1}{2}(-1)^{|a|}(ad_0x)_M - \frac{1}{2}(-1)^{|a|}(ad_0x)_N,$$

and

$$g(s^{-1}a \otimes y) = s^{-1}a \otimes y - \frac{1}{2}(-1)^{|a|}(ad_0y)_M + \frac{1}{2}(-1)^{|a|}(ad_0y)_N.$$

One easily checks that $f \circ g = 0$. Write A_M to denote the copy of A in J coming from M. Then

$$J = A_M \oplus \mathrm{im} g \oplus s^{-1}A \otimes X \otimes Y.$$

Therefore g is an isomorphism onto $\ker f$. Using once more the identity $dd_0x = -\sum_i (-1)^{|a_i|}a_i d_0 x_i$, one checks that g commutes with the differentials. Since $(R \oplus s^{-1}R, D)$ is acyclic, this implies that f is a quasi-isomorphism. □

5. The invariant Msecat

Definition 5.1. Let $p : E \to B$ be fibration. We define $\operatorname{Msecat} p$ to be the least integer n such that there exists a commutative diagram of $A_{PL}(B)$-modules

$$\begin{array}{ccc} A_{PL}(B) & = & A_{PL}(B) \\ {\scriptstyle A_{PL}(j^n p)}\downarrow & \searrow & \uparrow \\ A_{PL}(*_B^n E) & \xleftarrow[\sim]{} & (P,d). \end{array}$$

If no such n exists we set $\operatorname{Msecat} p = \infty$.

We first show that $\operatorname{Msecat} p$ is a lower bound of $\operatorname{secat} p$ which is closer to the sectional category than the classical lower bound $\operatorname{nil}\ker p^*$:

Theorem 5.2. *For any fibration $p : E \to B$, $\operatorname{nil}\ker p^* \leq \operatorname{Msecat} p$. If B is normal then $\operatorname{Msecat} p \leq \operatorname{secat} p$.*

Proof. Suppose that $\operatorname{Msecat} p \leq n$. We show that $\operatorname{nil}\ker p^* \leq n$. Form the pullback

$$\begin{array}{ccc} *_B^n E \times_B E & \longrightarrow & E \\ {\scriptstyle \bar{p}_n}\downarrow & & \downarrow{\scriptstyle p} \\ *_B^n E & \xrightarrow[j^n p]{} & B. \end{array}$$

By Lemma 2.4, $\operatorname{secat} \bar{p}_n \leq n$. Therefore $\operatorname{nil}\ker \bar{p}_n^* \leq n$. Since $\operatorname{Msecat} p \leq n$, the join map $j^n p$ is injective in cohomology. Now consider elements $\alpha_0, \dots, \alpha_n \in \ker p^*$. Since $\operatorname{nil}\ker \bar{p}_n^* \leq n$, we have $(j_n p)^*(\alpha_0 \cup \dots \cup \alpha_n) = 0$. Since $(j_n p)^*$ is injective, $\alpha_0 \cup \dots \cup \alpha_n = 0$. This shows that $\operatorname{nil}\ker p^* \leq n$.

Suppose now that B is normal and that $\operatorname{secat} p \leq n$. Then there exists a section $s : B \to *_B^n E$ of the join map $j^n p$. We therefore have the following commutative diagram of $A_{PL}(B)$-modules:

$$\begin{array}{ccc} A_{PL}(B) & = & A_{PL}(B) \\ {\scriptstyle A_{PL}(j^n p)}\downarrow & \searrow & \uparrow{\scriptstyle A_{PL}(s)} \\ A_{PL}(*_B^n E) & = & A_{PL}(*_B^n E). \end{array}$$

It follows that $\operatorname{Msecat} p \leq n$. □

The number $\operatorname{Msecat} p$ can be calculated using the algebraic join construction of the previous sections. For the proof of this fact we need the following lemma:

Lemma 5.3. *Let $p : E \to B$ be a fibration, $\alpha : (A,d) \xrightarrow{\sim} A_{PL}(B)$ be a commutative cochain algebra model, $i : (A,d) \to (Q,d)$ be a cofibration of (A,d)-modules, and $\phi : (Q,d) \to A_{PL}(*_B^n E)$ be a morphism of (A,d)-modules such that $\phi \circ i = A_{PL}(j^n p) \circ \alpha$. If $\operatorname{Msecat} p \leq n$ then i admits a retraction of (A,d)-modules.*

Proof. By definition, there is a commutative diagram of $A_{PL}(B)$-modules

$$\begin{array}{ccc} A_{PL}(B) & = & A_{PL}(B) \\ {\scriptstyle A_{PL}(j^n p)}\downarrow & \overset{j}{\searrow} & \uparrow{\scriptstyle r} \\ A_{PL}(*_B^n E) & \underset{\psi}{\xleftarrow{\sim}} & (P,d). \end{array}$$

This is automatically a commutative diagram of (A,d)-modules. Form the following commutative diagram of (A,d)-modules:

$$\begin{array}{ccc} (A,d) & \xrightarrow{j\circ\alpha} & (P,d) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \sim\,\psi} \\ (Q,d) & \xrightarrow[\phi]{} & A_{PL}(*^n_B E). \end{array}$$

By the lifting lemma [1, II.1.11], there exists a morphism of (A,d)-modules $\lambda : (Q,d) \to (P,d)$ such that $\lambda\circ i = j\circ\alpha$. We have obtained the following commutative diagram of (A,d)-modules:

$$\begin{array}{ccc} (A,d) & = & (A,d) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \alpha\,\sim} \\ (Q,d) & \xrightarrow[r\circ\lambda]{} & A_{PL}(B). \end{array}$$

The lifting lemma [1, II.1.11] yields the required retraction of i. □

Theorem 5.4. *Let $p : E \to B$ be a fibration between simply connected spaces of finite type, $\alpha : (A,d) \xrightarrow{\sim} A_{PL}(B)$ be a commutative cochain algebra model, and $(M,d) = (A \otimes (\mathbb{Q} \oplus X), d)$ be a semifree extension of (A,d) such that there exists a quasi-isomorphism of (A,d)-modules $(M,d) \xrightarrow{\sim} A_{PL}(E)$ extending $A_{PL}(p)\circ\alpha$. Then* $\mathrm{Msecat}\, p \leq n$ *if and only if the inclusion $(A,d) \to *^n_{(A,d)}(M,d)$ admits a retraction of (A,d)-modules.*

Proof. Suppose first that $\mathrm{Msecat}\, p \leq n$. By Theorem 4.2, the morphism of (A,d)-modules $A_{PL}(j^n p)\circ\alpha$ extends to a quasi-isomorphism of (A,d)-modules $*^n_{(A,d)}(M,d) \xrightarrow{\sim} A_{PL}(*^n_B E)$. By Lemma 5.3, the inclusion $(A,d) \to *^n_{(A,d)}(M,d)$ admits a retraction of (A,d)-modules.

Suppose now that the inclusion $(A,d) \to *^n_{(A,d)}(M,d)$ admits a retraction ρ of (A,d)-modules. Then the morphism of $A_{PL}(B)$-modules $A_{PL}(B) \otimes_{(A,d)} \rho$ is a retraction of the morphism of $A_{PL}(B)$-modules

$$A_{PL}(B) = A_{PL}(B) \otimes_{(A,d)} (A,d) \to A_{PL}(B) \otimes_{(A,d)} *^n_{(A,d)}(M,d).$$

By 4.2, the morphism of (A,d)-modules $A_{PL}(j^n p)\circ\alpha$ extends to a quasi-isomorphism of (A,d)-modules $\psi : *^n_{(A,d)}(M,d) \xrightarrow{\sim} A_{PL}(*^n_B E)$. Consider the following commutative diagram of $A_{PL}(B)$-modules

$$\begin{array}{ccc} A_{PL}(B) \otimes_{A_{PL}(B)} A_{PL}(B) & \xrightarrow{A_{PL}(B)\otimes_\alpha \alpha} & A_{PL}(B) \otimes_{(A,d)} (A,d) \\ \downarrow{\scriptstyle A_{PL}(B)\otimes_{A_{PL}(B)} A_{PL}(j^n p)} & & \downarrow \\ A_{PL}(B) \otimes_{A_{PL}(B)} A_{PL}(*^n_B E) & \xleftarrow[A_{PL}(B)\otimes_\alpha \psi]{} & A_{PL}(B) \otimes_{(A,d)} *^n_{(A,d)}(M,d). \end{array}$$

Using [10, 6.10] one sees that $A_{PL}(B) \otimes_\alpha \psi$ is a quasi-isomorphism. The left hand vertical morphism is precisely $A_{PL}(j^n p)$. We have obtained the following

commutative diagram of $A_{PL}(B)$-modules:

$$\begin{array}{ccc} A_{PL}(B) & = & A_{PL}(B) \\ {\scriptstyle A_{PL}(j^n p)}\downarrow & \searrow & \uparrow{\scriptstyle A_{PL}(B)\otimes_{(A,d)}\rho} \\ A_{PL}(*^n_B E) & \xleftarrow[A_{PL}(B)\otimes_\alpha \psi]{\sim} & A_{PL}(B)\otimes_{(A,d)} *^n_{(A,d)}(M,d). \end{array}$$

This shows that $\mathrm{Msecat}\, p \leq n$. □

Note that we have not yet shown that Msecat is a homotopy invariant. This is contained in the following proposition.

Proposition 5.5. *Consider a commutative diagram*

$$\begin{array}{ccc} E & \xrightarrow{g} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow[f]{} & B' \end{array}$$

in which p and p' are fibrations.

(a) If f is a homotopy equivalence then $\mathrm{Msecat}\, p' \leq \mathrm{Msecat}\, p$.

(b) If f and g are homotopy equivalences then $\mathrm{Msecat}\, p' = \mathrm{Msecat}\, p$.

(c) If the diagram is a pullback and all spaces are simply connected and of finite type then $\mathrm{Msecat}\, p \leq \mathrm{Msecat}\, p'$.

Proof. (a) Suppose that $\mathrm{Msecat}\, p \leq n$. Choose a factorization $A_{PL}(j^n p') = \psi \circ i$ where $i : A_{PL}(B') \to (Q,d)$ is a cofibration of $A_{PL}(B')$-modules and $\psi : (Q,d) \to A_{PL}(*^n_{B'} E')$ is a quasi-isomorphism of $A_{PL}(B')$-modules. Then $A_{PL}(*^n_f g) \circ \psi \circ i = A_{PL}(j^n p) \circ A_{PL}(f)$. By Lemma 5.3, i admits a retraction of $A_{PL}(B')$-modules. This shows that $\mathrm{Msecat}\, p' \leq n$.

(b) This is a formal consequence of (a). Indeed, by (a), $\mathrm{Msecat}\, p' \leq \mathrm{Msecat}\, p$. But if f and g are homotopy equivalences then the homotopy inverses can be used to construct a commutative square

$$\begin{array}{ccc} E' & \xrightarrow{\simeq} & E \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow[\simeq]{} & B \end{array}$$

showing $\mathrm{Msecat}\, p \leq \mathrm{Msecat}\, p'$.

(c) Applying the functor A_{PL} to the given square we obtain the following commutative diagram of commutative cochain algebras:

$$\begin{array}{ccc} A_{PL}(B') & \xrightarrow{A_{PL}(f)} & A_{PL}(B) \\ {\scriptstyle A_{PL}(p')}\downarrow & & \downarrow{\scriptstyle A_{PL}(p)} \\ A_{PL}(E') & \xrightarrow[A_{PL}(g)]{} & A_{PL}(E). \end{array}$$

Let $\alpha : (A',d) \xrightarrow{\sim} A_{PL}(B')$ be a Sullivan model. Choose factorizations $A_{PL}(f)\circ\alpha = \psi\circ i$ and $A_{PL}(p')\circ\alpha = \phi\circ j$ such that $i : (A',d) \to (A,d)$ and $j : (A',d) \to (M',d)$ are

inclusions of relative Sullivan algebras and $\psi : (A,d) \to A_{PL}(B)$ and $\phi : (M',d) \to A_{PL}(E')$ are quasi-isomorphisms. Then, by [10, 15(c)], the induced morphism of cochain algebras

$$(M,d) = (A,d) \otimes_{(A',d)} (M',d) \to A_{PL}(E)$$

is a quasi-isomorphism. Note that (M',d) is a semifree extension of (A',d) and (M,d) is a semifree extension of (A,d). Suppose that $\operatorname{Msecat} p' \leq n$. By the preceding theorem, the inclusion $(A',d) \to *^n_{(A',d)}(M',d)$ admits a retraction ρ of (A',d)-modules. As remarked in 3.4(iii), $*^n_{(A,d)}(M,d) = (A,d) \otimes_{(A',d)} *^n_{(A',d)}(M',d)$. The morphism of (A,d)-modules

$$A \otimes_{A'} \rho : (A,d) \otimes_{(A',d)} *^n_{(A',d)}(M',d) \to (A,d) \otimes_{(A',d)} (A',d) = (A,d)$$

is a retraction of the inclusion $(A,d) \to *^n_{(A,d)}(M,d)$. By the preceding theorem, this implies that $\operatorname{Msecat} p \leq n$. □

The next proposition shows that the invariant Msecat is a generalization of the well-known invariant Mcat of spaces. Let B be a simply connected space of finite type with Sullivan model $(\Lambda V,d)$. By definition, $\operatorname{Mcat} B$ is the least integer n such that for some (equivalently: any) Sullivan model $(\Lambda V \otimes \Lambda W,d) \xrightarrow{\sim} (\Lambda V/\Lambda^{>n}V,d)$ of the projection $(\Lambda V,d) \to (\Lambda V/\Lambda^{>n}V,d)$, the inclusion $(\Lambda V,d) \to (\Lambda V \otimes \Lambda W,d)$ admits a retraction of $(\Lambda V,d)$-modules. If no such n exists, $\operatorname{Mcat} B = \infty$.

Proposition 5.6. *Let B be a simply connected pointed space of finite type. Consider the evaluation fibration $ev_1 : PB \to B$, $\omega \mapsto \omega(1)$. Then* $\operatorname{Msecat} ev_1 = \operatorname{Mcat} B$.

Proof. Let $\alpha : (\Lambda V,d) \xrightarrow{\sim} A_{PL}(B)$ be a Sullivan model of B. Denote the projection $(\Lambda V,d) \to (\Lambda V/\Lambda^{>n}V,d)$ by q_n and choose a factorization $q_n = \phi \circ i$ where $i : (\Lambda V,d) \to (\Lambda V \otimes \Lambda W,d)$ is the inclusion of a relative Sullivan algebra and $\phi : (\Lambda V \otimes \Lambda W,d) \to (\Lambda V/\Lambda^{>n}V,d)$ is a quasi-isomorphism. Choose a factorization $A_{PL}(j^n ev_1) \circ \alpha = \psi \circ j$ where $j : (\Lambda V,d) \to (\Lambda V \otimes \Lambda X,d)$ is the inclusion of a relative Sullivan algebra and $\psi : (\Lambda V \otimes \Lambda X,d) \to A_{PL}(*^n_B PB)$ is a quasi-isomorphism. It follows from [6] that there exist morphisms of commutative cochain algebras $\sigma : (\Lambda V \otimes \Lambda X,d) \to (\Lambda V \otimes \Lambda W,d)$ and $\rho : (\Lambda V \otimes \Lambda W,d) \to (\Lambda V \otimes \Lambda X,d)$ such that $\sigma \circ j = i$ and $\rho \circ i = j$. This implies that $\operatorname{Mcat} B \leq n$ if and only if j admits a retraction of $(\Lambda V,d)$-modules. Let (M,d) be a semifree extension of $(\Lambda V,d)$ such that there exists a quasi-isomorphism of $(\Lambda V,d)$-modules $(M,d) \xrightarrow{\sim} A_{PL}(PB)$ which extends $A_{PL}(ev_1) \circ \alpha$. By Theorem 4.2, the morphism of $(\Lambda V,d)$-modules $A_{PL}(j^n ev_1) \circ \alpha$ extends to a quasi-isomorphism of $(\Lambda V,d)$-modules $*^n_{(\Lambda V,d)}(M,d) \xrightarrow{\sim} A_{PL}(*^n_B PB)$. Use the lifting lemma [1, II.1.11] to construct quasi-isomorphisms of $(\Lambda V,d)$-modules $\beta : *^n_{(\Lambda V,d)}(M,d) \xrightarrow{\sim} (\Lambda V \otimes \Lambda X,d)$ and $\gamma : (\Lambda V \otimes \Lambda X,d) \xrightarrow{\sim} *^n_{(\Lambda V,d)}(M,d)$ such that β extends j and $\gamma \circ j$ is the inclusion of $(\Lambda V,d)$. We obtain that $\operatorname{Mcat} B \leq n$ if and only if the inclusion $(\Lambda V,d) \to *^n_{(\Lambda V,d)}(M,d)$ admits a retraction of $(\Lambda V,d)$-modules. By 5.4, this is the case if and only if $\operatorname{Msecat} ev_1 \leq n$. □

Recall that an upper bound for the sectional category of a surjective fibration is given by the Lusternik-Schnirelmann category of the base space. The following is the analogous result for Msecat and Mcat.

Proposition 5.7. *Let B be a simply connected space of finite type. For any surjective fibration $p : E \to B$,* $\operatorname{Msecat} p \leq \operatorname{Mcat} B$.

Proof. Recall that for us simply connected spaces are non-empty. Fix any base point in B and consider the evaluation fibration $ev_1 : PB \to B$. Since p is surjective, $E \neq \emptyset$. Since PB is contractible, there exists a continuous map $\lambda : PB \to E$ such that $p \circ \lambda = ev_1$. By 5.5(a) and 5.6, $\operatorname{Msecat} p \leq \operatorname{Msecat} ev_1 = \operatorname{Mcat} B$. □

6. Topological complexity

In [3] and [4], M. Farber defined the *topological complexity* of a space X, $\operatorname{TC}(X)$, to be the sectional category of the evaluation fibration $ev_{0,1} : X^I \to X \times X, \omega \mapsto (\omega(0), \omega(1))$. This invariant has proved to be very useful in the study of the motion planning problem in robotics. Note that Farber's definition of TC differs by 1 from the one given here. In this section we study the invariant

$$\operatorname{MTC}(X) = \operatorname{Msecat}(ev_{0,1} : X^I \to X \times X).$$

In order to simplify the presentation we restrict our attention to simply connected spaces of finite type having the homotopy type of CW complexes.

The evaluation fibration $ev_{0,1} : X^I \to X \times X$ is the mapping path fibration associated to the diagonal map $\Delta : X \to X \times X$. We may therefore identify the map $ev_{0,1}^* : H^*(X \times X) \to H^*(X^I)$ with the cup product $\cup : H^*(X) \otimes H^*(X) \to H^*(X)$.

Proposition 6.1. *We have* $\operatorname{nil} \ker \cup \leq \operatorname{MTC}(X) \leq \operatorname{TC}(X)$ *and* $\operatorname{Mcat} X \leq \operatorname{MTC}(X) \leq 2\operatorname{Mcat} X$.

Proof. The first inequalities follow from 5.2. By 5.7 and [10, 30.2], $\operatorname{MTC}(X) \leq \operatorname{Mcat}(X \times X) = 2\operatorname{Mcat} X$. For the remaining inequality consider the map $f : X \to X \times X$, $x \mapsto (*, x)$ where $* \in X$ is any base point and form the following pullback diagram:

$$\begin{array}{ccc} PX & \longrightarrow & X^I \\ {\scriptstyle ev_1}\downarrow & & \downarrow{\scriptstyle ev_{0,1}} \\ X & \xrightarrow[f]{} & X \times X. \end{array}$$

By 5.6 and 5.5(c), $\operatorname{Mcat} X = \operatorname{Msecat} ev_1 \leq \operatorname{Msecat} ev_{0,1} = \operatorname{MTC}(X)$. □

Consider a space X with Sullivan model $(\Lambda V, d)$. A Sullivan model of the product space $X \times X$ is then given by $(\Lambda(V \oplus V'), d) = (\Lambda V, d) \otimes (\Lambda V', d)$ where $(\Lambda V', d)$ is second copy of $(\Lambda V, d)$. As is shown in [10, p.206/207], a model of the evaluation fibration (and the diagonal map) is given by the inclusion

$$(\Lambda(V \oplus V'), d) \to (\Lambda(V \oplus V') \otimes \Lambda \overline{V}, d)$$

where

$$d(\bar{v}) = v' - v - \sum_{i=1}^{\infty} \frac{(\zeta d)^i}{i!}(v).$$

Here, ζ is the derivation of degree -1 defined by $\zeta(v) = \zeta(v') = \bar{v}$ and $\zeta(\bar{v}) = 0$. Using this explicit semifree extension of $(\Lambda(V \oplus V'), d)$ and the formula for the differential of the iterated join given in 3.4(ii), one can calculate the invariant $\operatorname{MTC}(X)$ from $(\Lambda V, d)$. We remark that $d_0 \bar{v} = v' - v$ and that $d_0 x = 0$ for

$x \in \Lambda^{>1}\overline{V}$. Note also that if $(\Lambda V, d)$ is the minimal Sullivan model of X then $(\Lambda(V \oplus V') \otimes \Lambda\overline{V}, d)$ is a minimal semifree $(\Lambda(V \oplus V'), d)$-module.

The following proposition provides an upper bound for MTC.

Proposition 6.2. *Let (A, d) be a commutative cochain algebra model of X with multiplication μ. Then* $\mathrm{MTC}(X) \leq \operatorname{nil}\ker\mu$.

Proof. Suppose that $\operatorname{nil}\ker\mu \leq n$. We show that the inclusion

$$i : (\Lambda(V \oplus V'), d) \to *^n_{(\Lambda(V\oplus V'),d)}(\Lambda(V \oplus V') \otimes \Lambda\overline{V}, d)$$

admits a retraction of $(\Lambda(V \oplus V'), d)$-modules. Choose a quasi-isomorphism of commutative cochain algebras $\alpha : (\Lambda V, d) \xrightarrow{\sim} (A, d)$. Consider the tensor product algebra $(A, d)\otimes(A, d) = (A\otimes A, d)$ and the $(A\otimes A, d)$-semifree extension of $(A\otimes A, d)$ defined by

$$(M, d) = (A \otimes A, d) \otimes_{(\Lambda(V\oplus V'),d)} (\Lambda(V \oplus V') \otimes \Lambda\overline{V}, d) = (A \otimes A \otimes \Lambda\overline{V}, d).$$

We have

$$*^n_{(A\otimes A,d)}(M, d) = (A \otimes A \oplus A \otimes A \otimes s^{-n}(\Lambda^+\overline{V})^{\otimes n+1}, d).$$

Consider an element $s^{-n}x_0 \otimes \cdots \otimes x_n \in s^{-n}(\Lambda^+\overline{V})^{\otimes n+1}$. If one of the x_i lies in $\Lambda^{>1}\overline{V}$ then $d(s^{-n}x_0 \otimes \cdots \otimes x_n)$ has no term in $A \otimes A$. Since $\operatorname{nil}\ker\mu \leq n$, this also holds if all $x_i \in \overline{V}$. We can thus define an $(A \otimes A, d)$-module retraction r of the inclusion $j : (A \otimes A, d) \to *^n_{(A\otimes A,d)}(M, d)$ by sending $A \otimes A \otimes s^{-n}(\Lambda^+\overline{V})^{\otimes n+1}$ to 0. By 3.4(iii), the map j is obtained by applying the functor $(A \otimes A, d) \otimes_{(\Lambda(V\oplus V'),d)} -$ to the inclusion i. Consider the following commutative diagram of $(\Lambda(V \oplus V'), d)$-modules:

$$\begin{array}{ccc} (\Lambda(V \oplus V'), d) & = & (\Lambda(V \oplus V'), d) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \alpha\otimes\alpha}\sim \\ *^n_{(\Lambda(V\oplus V'),d)}(\Lambda(V \oplus V') \otimes \Lambda\overline{V}, d) & \xrightarrow[r\circ((\alpha\otimes\alpha)\otimes_{id} id)]{} & (A \otimes A, d). \end{array}$$

The lifting lemma [1, II.1.11] yields the required retraction of i. □

Note that the number $\operatorname{nil}\ker\mu$ is in general not the same for different commutative cochain algebra models of X. Given a commutative graded algebra A with multiplication μ, the number $\operatorname{nil}\ker\mu$ can be determined using the following lemma:

Lemma 6.3. *Let ΛW be a commutative graded algebra and $I \subset \Lambda W$ be an ideal such that $A = \Lambda W/I$. Let $\Lambda W'$ be a second copy of ΛW and $J \subset \Lambda(W \oplus W') = \Lambda W \otimes \Lambda W'$ be the ideal $I\otimes\Lambda W' + \Lambda W \otimes I'$ where I' is the ideal of $\Lambda W'$ corresponding to I. Let finally $\mathcal{B}$ be a basis of the graded vector space W. Then* $\operatorname{nil}\ker\mu \leq n$ *if and only if, for all $w_0, \ldots, w_n \in \mathcal{B}$, $(w'_0 - w_0)\cdots(w'_n - w_n) \equiv 0 \mod J$.*

Proof. Denote the multiplication $\Lambda(W \oplus W') = \Lambda W \otimes \Lambda W' \to \Lambda W$ by m. We have $A\otimes A = \Lambda(W\oplus W')/J$ and $\ker\mu = \ker m/(J\cap\ker m)$. It suffices to show that $\ker m$ is the ideal of $\Lambda(W \oplus W')$ generated by the elements $w' - w$, $w \in \mathcal{B}$. Denote this ideal by K. Obviously, $K \subset \ker m$. Note also that $K \cap \Lambda W = 0$. In order to show the equality $K = \ker m$ we show that $\Lambda W \oplus K = \Lambda(W \oplus W')$. For this it is enough to show that for each $n \geq 1$, $\Lambda W \otimes \Lambda^n W' \subset \Lambda W \oplus K$. We proceed by induction. Consider $w \in \mathcal{B}$ and $\xi \in \Lambda W$. We have $\xi w' = \xi w + \xi(w' - w) \in \Lambda W \oplus K$. Suppose the assertion holds for some $n \geq 1$. Consider $\zeta \in \Lambda W \otimes \Lambda^n W'$ and $w \in \mathcal{B}$. By the

inductive hypothesis, $\zeta \in \Lambda W \oplus K$. Write $\zeta = \theta + k$ with $\theta \in \Lambda W$ and $k \in K$. By the inductive hypothesis, $\theta w' \in \Lambda W \oplus K$. Since K is an ideal, $kw' \in K$. It follows that $\zeta w' \in \Lambda W \oplus K$. This closes the induction and the result follows. □

Recall that a space X is called *formal* if $H^*(X)$ is a commutative cochain algebra model of X. Proposition 6.2 immediately implies

Proposition 6.4. *If X is formal then* $\mathrm{MTC}(X) = \operatorname{nil}\ker\cup$.

Example 6.5. The simplest example of a non-formal space is the space

$$X = S^3_a \vee S^3_b \cup e^8 \cup e^8$$

where the 8-cells are attached by means of the iterated Whitehead products $[S^3_a, [S^3_a, S^3_b]]$ and $[S^3_b, [S^3_a, S^3_b]]$. We show that this space satisfies $\mathrm{MTC}(X) = 3$ and $\operatorname{nil}\ker\cup = 2$. For degree reasons, the space X has the same cohomology algebra as the wedge of spheres $S^3 \vee S^3 \vee S^8 \vee S^8$. Therefore X satisfies $\operatorname{nil}\ker\cup = 2$. Indeed, since $\mathrm{TC} \leq 2\mathrm{cat}$, the topological complexity of a wedge of spheres is ≤ 2. On the other hand, it is an easy exercise to show that any space with at least two cohomology generators satisfies $\operatorname{nil}\ker\cup \geq 2$. The minimal Sullivan model of X is the algebra $(\Lambda(V), d)$ where the graded $\mathbb{Q}$-vector space V is generated by cocycles a and b of degree 3, an element u of degree 5 with $du = ab$, and elements of degree > 8. Consider the d-stable ideal $I = (\Lambda V)^{\geq 9}$ and form the quotient algebra $(A, d) = (\Lambda V/I, d)$. Since (I, d) is acyclic, the projection $(\Lambda V, d) \to (\Lambda V/I, d)$ is a quasi-isomorphism. Consider the ideal $J = I \otimes \Lambda V' + \Lambda V \otimes I' \subset \Lambda(V \oplus V')$ as in 6.3. We have $(a' - a)(b' - b)(u' - u) \not\equiv 0 \mod J$. Since a, b, u are of odd degree, any longer non-zero product of the form $(v'_0 - v_0) \cdots (v'_n - v_n)$ must contain at least one factor $v'_i - v_i$ with $|v_i| \geq 9$. For $n \geq 3$ any such product is therefore an element of J. By 6.2 and 6.3, this implies that $\mathrm{MTC}(X) \leq 3$. We show that $\mathrm{MTC}(X) > 2$. The differential of the generators $\bar{a}$, $\bar{b}$, and $\bar{u}$ of the model $(\Lambda(V \oplus V') \otimes \Lambda\overline{V}, d)$ of X^I is given by $d\bar{a} = a' - a$, $d\bar{b} = b' - b$, and $d\bar{u} = u' - u + \alpha \otimes \bar{a} + \beta \otimes \bar{b}$ where $\alpha, \beta \in \Lambda(V \oplus V')$ are some elements of degree 3. A straightforward calculation shows that $\alpha = -\frac{1}{2}(b + b')$ and $\beta = \frac{1}{2}(a + a')$. This information is, however, not needed for the calculations. It suffices to show that the inclusion

$$\begin{aligned}(\Lambda(V \oplus V'), d) &\to *^2_{(\Lambda(V\oplus V'),d)}(\Lambda(V \oplus V') \otimes \Lambda\overline{V}, d)\\ &= (\Lambda(V \oplus V') \oplus \Lambda(V \oplus V') \otimes s^{-2}\Lambda^+\overline{V} \otimes \Lambda^+\overline{V} \otimes \Lambda^+\overline{V}, d)\end{aligned}$$

is not injective in cohomology. The element $z = (a' - a)(b' - b)(u' - u)$ is a cocycle of degree 11 in $\Lambda(V \oplus V')$ which is not a coboundary. In the 2-fold join, however, we have

$$d(s^{-2}\bar{a}\otimes\bar{b}\otimes\bar{u} + s^{-2}\bar{b}\otimes\bar{u}\otimes\bar{a} + s^{-2}\bar{u}\otimes\bar{a}\otimes\bar{b} - s^{-2}\bar{a}\otimes\bar{u}\otimes\bar{b} - s^{-2}\bar{b}\otimes\bar{a}\otimes\bar{u} - s^{-2}\bar{u}\otimes\bar{b}\otimes\bar{a}) = -6z$$

so that $[z] = 0 \in H^{11}(*^2_{(\Lambda(V\oplus V'),d)}(\Lambda(V \oplus V') \otimes \Lambda\overline{V}, d))$.

Our last result is the fact that the difference $\mathrm{MTC}(X) - \operatorname{nil}\ker\cup$ can be arbitrarily large:

Proposition 6.6. *(i) For any $n \in \mathbb{N}$ there exists a finite CW-complex X such that* $\mathrm{MTC}(X) - \operatorname{nil}\ker\cup \geq n$.

(ii) There exists a space X such that $\mathrm{MTC}(X) = \infty$ *and* $\operatorname{nil}\ker\cup < \infty$.

Proof. (i) Let Z be a simply connected finite CW-complex having the same cohomology algebra as a wedge of spheres Y and satisfying $\mathrm{Mcat}\, Z = 3$. Such a space has for instance been constructed in [14]. Let X be the n-fold product of the space Z, $X = Z^n$. Then X is a finite CW-complex which has the same cohomology algebra as Y^n and satisfies $\mathrm{nil}\ker\cup \leq \mathrm{TC}(Y^n) \leq 2\mathrm{cat}\,(Y^n) \leq 2n\mathrm{cat}\, Y = 2n$. On the other hand, by 6.1 and [10, 30.2], $\mathrm{MTC}(X) \geq \mathrm{Mcat}\,(X) = n\mathrm{Mcat}\,(Z) = 3n$.

(ii) It suffices to take a space X such that $\mathrm{Mcat}\, X = \infty$ and $\mathrm{nil}\, H^+(X) < \infty$. Such a space has been constructed in [7]. □

References

[1] H.J. Baues: *Algebraic Homotopy*, Cambridge University Press (1989).

[2] O. Cornea, G. Lupton, J. Oprea, and D. Tanré: *Lusternik-Schnirelmann category*, AMS Mathematical Surveys and Monographs 103 (2003).

[3] M. Farber: Topological Complexity of Motion Planning, *Discrete Comput. Geom.* 29 (2003), 211-221.

[4] M. Farber: Instabilities of Robot Motion, *Topology Appl.* 140 (2004), 245-266.

[5] A. Fassò Velenik: *Relative homotopy invariants of the type of the Lusternik-Schnirelmann category*, Ph.D. thesis, FU Berlin (2002).

[6] Y. Félix and S. Halperin: Rational LS category and its applications, *Trans. Amer. Math. Soc.* 273 (1982), 1-37.

[7] Y. Félix, S. Halperin, and J.-C. Thomas: L.S. catégorie et suite spectrale de Milnor-Moore (une nuit dans le train). *Bull. S.M.F.* 111 (1983), 89-96.

[8] Y. Félix, S. Halperin, and J.-C. Thomas: Differential graded algebras in topology, *Handbook of Algebraic Topology*, Elsevier (1995), 829-865.

[9] P.G. Goerss and J.F. Jardine: *Simplicial Homotopy Theory*, PM 174, Birkhäuser (1999).

[10] Y. Félix, S. Halperin, and J.-C. Thomas: *Rational homotopy theory*, GTM 200, Springer-Velag (2001).

[11] K. Hess: A proof of Ganea's conjecture for rational spaces, *Topology* 30 (1991), 205-214.

[12] I.M. James: On category, in the sense of Lusternik-Schnirelmann, *Topology* **17** (1978), 331-348.

[13] I.M. James: Lusternik-Schnirelmann category, *Handbook of Algebraic Topology*, Elsevier (1995), 1293-1310.

[14] T. Kahl and L. Vandembroucq: Gaps in the Milnor-Moore spectral sequence, *Bull. Belg. Math. Soc.* 9 (2002), 265-277.

[15] S. Smale: On the topology of algorithms I, *J. of Complexity* 3 (1987), 81-89.

[16] A.S. Schwarz: The genus of a fiber space, *Amer. Math. Soc. Transl.* 55 (1966), 49-140.

Lucía Fernández Suárez, Universidade do Minho, Centro de Matemática, Campus de Gualtar, 4710-057 Braga, Portugal. *E-mail:* lfernandez@math.uminho.pt

Pierre Ghienne, Université d'Artois, Faculté des Sciences Jean Perrin, Laboratoire de Mathématiques, Rue Jean Souvraz - S.P. 18, 62307 Lens cedex, France. *E-mail:* ghienne@euler.univ-artois.fr

Thomas Kahl, Universidade do Minho, Centro de Matemática, Campus de Gualtar, 4710-057 Braga, Portugal. *E-mail:* kahl@math.uminho.pt

Lucile Vandembroucq, Universidade do Minho, Centro de Matemática, Campus de Gualtar, 4710-057 Braga, Portugal. *E-mail:* lucile@math.uminho.pt